Portfolios that Contain Risky Assets:
8.1. IID Models for Markets and Portfolios

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Portfolios that Contain Risky Assets Part II: Probabilistic Models

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8. Independent, Identically-Distributed Models for Portfolios

IID Models for Markets and Portfolios

IID Models for Markets (Market Return Histories)

We now consider a market with N risky assets. Let $\{s_i(d)\}_{d=0}^{D}$ be the share price history of asset i. The associated return history is $\{r_i(d)\}_{d=1}^{D}$ where

$$r_i(d) = \frac{s_i(d)}{s_i(d-1)} - 1 \,.$$

Because each $s_i(d)$ is positive, we see that each $r_i(d)$ is in $(-1,\infty)$. Let $\mathbf{r}(d)$ be the N-vector

$$\mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix} .$$

Then the market return history can be expressed compactly as $\{\mathbf{r}(d)\}_{d=1}^{D}$.

An IID model for the market return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ draws D random vectors $\{\mathbf{R}_d\}_{d=1}^{D}$ from a fixed probablity density $q(\mathbf{R})$ over $(-1,\infty)^N$.

IID Models for Markets (Mean and Variance)

In this model the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Xi}$ of $\mathbf{R}$ are given by

$$\begin{aligned} \boldsymbol{\mu} &= \mathrm{Ex}(\mathbf{R}) = \int \mathbf{R}\, q(\mathbf{R})\, \mathrm{d}\mathbf{R}\,, \\ \boldsymbol{\Xi} &= \mathrm{Ex}\Big((\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}}\Big) = \int (\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}} q(\mathbf{R})\, \mathrm{d}\mathbf{R}\,. \end{aligned} \tag{1.1}$$

Notice that $\boldsymbol{\mu} \in (-1,\infty)^N \subset \mathbb{R}^N$ and that $\boldsymbol{\Xi} \in \mathbb{R}^{N\times N}$ is a symmetric, positive definite matrix. Indeed, for every nonzero $\mathbf{y} \in \mathbb{R}^N$ we have

$$\begin{aligned} \mathbf{y}^{\mathrm{T}}\boldsymbol{\Xi}\,\mathbf{y} &= \mathbf{y}^{\mathrm{T}} \left[\int (\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}} q(\mathbf{R})\, \mathrm{d}\mathbf{R}\right] \mathbf{y} \\ &= \int \mathbf{y}^{\mathrm{T}}(\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{y}\, q(\mathbf{R})\, \mathrm{d}\mathbf{R} \\ &= \int \Big(\mathbf{y}^{\mathrm{T}}(\mathbf{R}-\boldsymbol{\mu})\Big)^2 q(\mathbf{R})\, \mathrm{d}\mathbf{R} > 0\,. \end{aligned}$$

IID Models for Markets (Unbiased Estimators)

Unbiased estimators for $\boldsymbol{\mu}$ and $\boldsymbol{\Xi}$ are given as follows.

Fact 1. Given any IID sample $\{\mathbf{R}_d\}_{d=1}^{D}$ drawn from $q(\mathbf{R})$ and any positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Xi}$ given by (1.1) have the unbiased estimators

$$\hat{\boldsymbol{\mu}} = \sum_{d=1}^{D} w_d \, \mathbf{R}_d \,, \tag{1.2a}$$

$$\hat{\boldsymbol{\Xi}} = \frac{1}{1-\bar{w}} \sum_{d=1}^{D} w_d \left(\mathbf{R}_d - \hat{\boldsymbol{\mu}}\right)\left(\mathbf{R}_d - \hat{\boldsymbol{\mu}}\right)^{\mathrm{T}} \,, \tag{1.2b}$$

where

$$\bar{w} = \sum_{d=1}^{D} w_d^2 \,. \tag{1.2c}$$

IID Models for Markets (**Fact 1** Proof)

Proof. The proof of **Fact 1** has three steps.

Step 1. Prove that $\hat{\boldsymbol{\mu}}$ defined by (1.2a) satisfies

$$\mathrm{Ex}(\hat{\boldsymbol{\mu}}) = \boldsymbol{\mu}\,. \tag{1.3a}$$

Step 2. Prove that $\hat{\boldsymbol{\mu}}$ defined by (1.2a) satisfies

$$\mathrm{Vr}(\hat{\boldsymbol{\mu}}) = \bar{w}\,\boldsymbol{\Xi}\,. \tag{1.3b}$$

Step 3. Prove that $\hat{\boldsymbol{\Xi}}$ defined by (1.2b) satisfies

$$\mathrm{Ex}\Big(\hat{\boldsymbol{\Xi}}\Big) = \boldsymbol{\Xi}\,. \tag{1.3c}$$

Fact 1 is proved after these steps have been completed. □

Remark. These steps are analogs of the steps that we took to prove the analogous facts about IID models for single assets. As was the case earlier, **Step 2** is the key to **Step 3**.

IID Models for Markets (**Step 1** Proof)

Proof of Step 1. Defintion (1.2a) of $\hat{\boldsymbol{\mu}}$, the fact that $\{\mathbf{R}_d\}_{d=1}^{D}$ is an IID sample with mean $\boldsymbol{\mu}$, and the fact that the $\{w_d\}_{d=1}^{D}$ sum to 1 give

$$\mathrm{Ex}(\hat{\boldsymbol{\mu}}) = \mathrm{Ex}\left(\sum_{d=1}^{D} w_d\, \mathbf{R}_d\right) = \sum_{d=1}^{D} w_d\, \mathrm{Ex}(\mathbf{R}_d) = \sum_{d=1}^{D} w_d\, \boldsymbol{\mu} = \boldsymbol{\mu}\,.$$

This proves **Step 1**. □

Remark. Because $\mathrm{Ex}(\hat{\boldsymbol{\mu}}) = \boldsymbol{\mu}$, we see that $\mathrm{Vr}(\hat{\boldsymbol{\mu}})$ is given by

$$\mathrm{Vr}(\hat{\boldsymbol{\mu}}) = \mathrm{Ex}\Big((\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\,(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}}\Big)\,. \tag{1.4}$$

The proof of **Step 2** will use this fact.

IID Models for Markets (**Step 2** Proof)

Proof of Step 2. Set $\tilde{\mathbf{R}}_d = \mathbf{R}_d - \boldsymbol{\mu}$. Then defintion (1.2a) of $\hat{\boldsymbol{\mu}}$, the fact that $\{\mathbf{R}_d\}_{d=1}^{D}$ is an IID sample with mean $\boldsymbol{\mu}$, and the fact that the $\{w_d\}_{d=1}^{D}$ sum to 1 give

$$\hat{\boldsymbol{\mu}} - \boldsymbol{\mu} = \sum_{d=1}^{D} w_d \left(\mathbf{R}_d - \boldsymbol{\mu}\right) = \sum_{d=1}^{D} w_d \, \tilde{\mathbf{R}}_d \,,$$

whereby

$$(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\,(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}} = \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} w_{d_1} w_{d_1} \, \tilde{\mathbf{R}}_{d_1} \, \tilde{\mathbf{R}}_{d_2}^{\mathrm{T}} \,.$$

We thereby see that

$$\mathrm{Ex}\Big((\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\,(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}}\Big) = \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} w_{d_1} w_{d_1} \, \mathrm{Ex}\Big(\tilde{\mathbf{R}}_{d_1} \, \tilde{\mathbf{R}}_{d_2}^{\mathrm{T}}\Big) \,. \tag{1.5}$$

IID Models for Markets (**Step 2** Proof)

Because $\{\tilde{\mathbf{R}}_d\}_{d=1}^{D}$ is an IID sample with mean $\mathbf{0}$ and variance $\boldsymbol{\Xi}$ we have

$$\mathrm{Ex}\Big(\tilde{\mathbf{R}}_{d_1}\,\tilde{\mathbf{R}}_{d_2}^{\mathrm{T}}\Big) = \delta_{d_1 d_2}\,\boldsymbol{\Xi}\,,$$

where $\delta_{d_1 d_2}$ is the Kronecker delta. Then (1.4), (1.5) and definition (1.2c) of $\bar{w}$, yield

$$\begin{aligned}\mathrm{Vr}(\hat{\boldsymbol{\mu}}) &= \mathrm{Ex}\Big((\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\,(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}}\Big) \\ &= \sum_{d_1=1}^{D}\sum_{d_2=1}^{D} w_{d_1}\,w_{d_1}\,\delta_{d_1 d_2}\,\boldsymbol{\Xi} = \sum_{d=1}^{D} w_d^2\,\boldsymbol{\Xi} = \bar{w}\,\boldsymbol{\Xi}\,. \end{aligned} \tag{1.6}$$

This proves **Step 2**. □

IID Models for Markets (**Step 3** Proof)

Proof of Step 3. We begin with the identity

$$\begin{aligned}\sum_{d=1}^{D} w_d \, (\mathbf{R}_d - \hat{\boldsymbol{\mu}}) \, (\mathbf{R}_d - \hat{\boldsymbol{\mu}})^{\mathrm{T}} = & \sum_{d=1}^{D} w_d \, \tilde{\mathbf{R}}_d \, \tilde{\mathbf{R}}_d^{\mathrm{T}} \\ & - \sum_{d=1}^{D} w_d \, (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu}) \, (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}} \, .\end{aligned} \tag{1.7}$$

Because $\{\tilde{\mathbf{R}}_d\}_{d=1}^{D}$ is an IID sample with mean $\mathbf{0}$ and variance $\boldsymbol{\Xi}$ and because the $\{w_d\}_{d=1}^{D}$ sum to 1, we have

$$\mathrm{Ex}\left(\sum_{d=1}^{D} w_d \, \tilde{\mathbf{R}}_d \, \tilde{\mathbf{R}}_d^{\mathrm{T}} \right) = \sum_{d=1}^{D} w_d \, \mathrm{Ex}\left(\tilde{\mathbf{R}}_d \, \tilde{\mathbf{R}}_d^{\mathrm{T}} \right) = \sum_{d=1}^{D} w_d \, \boldsymbol{\Xi} = \boldsymbol{\Xi} \, . \tag{1.8}$$

IID Models for Markets (**Step 3** Proof)

By using (1.6) from **Step 2** and the fact that the $\{w_d\}_{d=1}^D$ sum to 1, we obtain

$$\begin{aligned} \mathrm{Ex}\left(\sum_{d=1}^{D} w_d \, (\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\,(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}}\right) &= \sum_{d=1}^{D} w_d \, \mathrm{Ex}\Big((\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})\,(\hat{\boldsymbol{\mu}} - \boldsymbol{\mu})^{\mathrm{T}}\Big) \\ &= \sum_{d=1}^{D} w_d \, \bar{w}\, \boldsymbol{\Xi} = \bar{w}\, \boldsymbol{\Xi} \,. \end{aligned} \tag{1.9}$$

We see from (1.7) (1.8) and (1.9) that

$$\mathrm{Ex}\left(\sum_{d=1}^{D} w_d \, (\mathbf{R}_d - \hat{\boldsymbol{\mu}})\,(\mathbf{R}_d - \hat{\boldsymbol{\mu}})^{\mathrm{T}}\right) = \boldsymbol{\Xi} - \bar{w}\,\boldsymbol{\Xi} = (1 - \bar{w})\,\boldsymbol{\Xi} \,.$$

It follows that $\hat{\boldsymbol{\Xi}}$ defined by (1.2b) satisfies (1.3c). This proves **Step 3** and thereby completes the proof of **Fact 1**.

IID Models for Markets (Calibration)

In order to use an IID model for the market, we must gather statistical information about the return probability density $q(\mathbf{R})$. **Fact 1** says that given any IID sample $\{\mathbf{R}_d\}_{d=1}^{D}$ drawn from $q(\mathbf{R})$ and any positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, the mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Xi}$ given by (1.1) have the unbiased estimators (1.2) that are

$$\hat{\boldsymbol{\mu}} = \sum_{d=1}^{D} w_d \, \mathbf{R}_d \,, \qquad \hat{\boldsymbol{\Xi}} = \frac{1}{1-\bar{w}} \sum_{d=1}^{D} w_d \, (\mathbf{R}_d - \hat{\boldsymbol{\mu}}) \, (\mathbf{R}_d - \hat{\boldsymbol{\mu}})^{\mathrm{T}} \,.$$

If we assume that such a sample is given by the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ then these estimators are given in terms of the vector $\mathbf{m}$ and matrix $\mathbf{V}$ by

$$\hat{\boldsymbol{\mu}} = \mathbf{m} \,, \qquad \hat{\boldsymbol{\Xi}} = \frac{1}{1-\bar{w}} \mathbf{V} \,. \tag{1.10}$$

IID Models for Markets (Assessing IID)

An IID model is reasonable when the points $\{(d, \mathbf{r}(d))\}_{d=1}^{D}$ are distributed uniformly in d. Graphical assessments when N is small are not hard to imagine. For example, a graphical assessment based on pairs of assets can be carried out by plotting the points $\{(d, r_i(d), r_j(d))\}_{d=1}^{D}$ in $\mathbb{R}^3$ with an interactive 3D graphics package. However, things become harder to visualize when N is not small.

You might think that a necessary condition for the entire market to have an IID model is that each asset has an IID model. This can be assessed for each asset by either the graphical or quantitative methods from the previous chapter. Such assessments often show that funds behave more like IID models than individual stocks or bonds. This means that portfolio balancing strategies based on IID models might work better for portfolios composed largely of funds. This is one reason why some investors prefer investing in funds over investing in individual stocks and bonds.

IID Models for Markets (Diverse Portfolios)

A better lesson to be drawn from the observation in the previous paragraph is that every sufficiently diverse portfolio of assets in a market will behave more like an IID model than many of the individual assets in that market. In other words, IID models for a market can be used to develop portfolio balancing strategies when the portfolios considered are sufficiently diverse, even when the behavior of individual assets in that market may not be described well by the model. This is another reason to prefer holding diverse, broad-based portfolios.

More importantly, this suggests that it is better to apply graphical or quantitative assessments to representative portfolios rather than to each individual asset in the market. In order to do this we must first show how an IID model for market return histories leads to an IID model for the return history of every Markowitz portfolio.

IID Models for Portfolios (Introduction)

Now suppose that we are using an IID model with return probability density $q(\mathbf{R})$ for a market of N risky assets with return history $\{\mathbf{r}(d)\}_{d-1}^{D}$. Here we show that there is a related IID model for the return history $\{r(d)\}_{d=1}^{D}$ of any Markowitz portfolio. Recall the following.

- For any allocation $\mathbf{f} \in \mathcal{M}$ we have

$$r(d) = \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,. \tag{2.11a}$$

- For any allocation $(\mathbf{f}, f_{\mathrm{rf}}) \in \mathcal{M}_1$ we have

$$r(d) = \mu_{\mathrm{rf}} f_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,. \tag{2.11b}$$

- For any allocation $(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathcal{M}_2$ we have

$$r(d) = \mu_{\mathrm{si}} f_{\mathrm{si}} + \mu_{\mathrm{cl}} f_{\mathrm{cl}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,. \tag{2.11c}$$

IID Models for Portfolios (Return Histories)

All of the expressions in (2.11) can be put into the general form

$$r(d) = r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,, \tag{2.12}$$

where r_{rf} is the *risk-free return of the portfolio* given by

$$r_{\mathrm{rf}} = \begin{cases} 0 & \text{when } \mathbf{f} \in \mathcal{M}\,, \\ \mu_{\mathrm{rf}} f_{\mathrm{rf}} & \text{when } (\mathbf{f}, f_{\mathrm{rf}}) \in \mathcal{M}_1\,, \\ \mu_{\mathrm{si}} f_{\mathrm{si}} + \mu_{\mathrm{cl}} f_{\mathrm{cl}} & \text{when } (\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathcal{M}_2\,. \end{cases}$$

Therefore any Markowitz portfolio with risk-free return r_{rf} and risky asset allocation $\mathbf{f}$ has its return history $\{r(d)\}_{d=1}^{D}$ given in terms of the market return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ by (2.12).

IID Models for Portfolios (Return Probability Density)

Because the market return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ is IID modeled by $\{\mathbf{R}_d\}_{d=1}^{D}$ drawn from the probability density $q(\mathbf{R})$, we see from (2.12) that the return history $\{r(d)\}_{d=1}^{D}$ of any Markowitz portfolio with a risk-free return r_{rf} and a risky asset allocation $\mathbf{f}$ is IID modeled by $\{R_d\}_{d=1}^{D}$ given by

$$R_d = r_{\mathrm{rf}} + \mathbf{R}_d^{\mathrm{T}}\mathbf{f} \,, \tag{2.13a}$$

drawn from the return probability density $q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$ given by

$$q_{(r_{\mathrm{rf}},\mathbf{f})}(R) = \int \delta\Big(R - r_{\mathrm{rf}} - \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big)\; q(\mathbf{R})\,\mathrm{d}\mathbf{R} \,. \tag{2.13b}$$

Here $\delta(\,\cdot\,)$ denotes the *Dirac delta distribution*, which can be defined by the property that for every sufficiently nice function $\psi(R)$

$$\int \psi(R)\,\delta\Big(R - r_{\mathrm{rf}} - \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big)\,\mathrm{d}R = \psi\Big(r_{\mathrm{rf}} + \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big) \,. \tag{2.14}$$

IID Models for Portfolios (Expected Values)

Hence, by combining formula (2.13b) for $q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$ with the defining property (2.14) of the Dirac delta distribution, we see that for every sufficiently nice function $\psi(R)$ we have the formula

$$
\begin{aligned}
\mathrm{Ex}(\psi(R)) &= \int \psi(R)\, q_{(r_{\mathrm{rf}},\mathbf{f})}(R)\, \mathrm{d}R \\
&= \int \psi(R) \left[\int \delta\Big(R - r_{\mathrm{rf}} - \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big)\, q(\mathbf{R})\, \mathrm{d}\mathbf{R} \right] \mathrm{d}R \\
&= \int \left[\int \psi(R)\, \delta\Big(R - r_{\mathrm{rf}} - \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big)\, \mathrm{d}R \right] q(\mathbf{R})\, \mathrm{d}\mathbf{R} \\
&= \int \psi\Big(r_{\mathrm{rf}} + \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big)\, q(\mathbf{R})\, \mathrm{d}\mathbf{R}\,.
\end{aligned}
\tag{2.15}
$$

Here $\psi(R)$ will be "sufficiently nice" if it is continous and the integral in the last line makes sense and has a finite value.

IID Models for Portfolios ($q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$)

Those unfamiliar with the Dirac delta distribution can view $q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$ as being defined by (2.15). It says for every sufficiently nice $\psi(R)$ we have

$$\int \psi(R)\, q_{(r_{\mathrm{rf}},\mathbf{f})}(R)\,\mathrm{d}R = \int \psi\Big(r_{\mathrm{rf}} + \mathbf{R}^{\mathrm{T}}\mathbf{f}\Big)\; q(\mathbf{R})\,\mathrm{d}\mathbf{R}\,.$$

This shows that because $q(\mathbf{R})$ is a probability density, $q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$ is too.

- Because $q(\mathbf{R}) \geq 0$, this shows that

  $$\int \psi(R)\, q_{(r_{\mathrm{rf}},\mathbf{f})}(R)\,\mathrm{d}R \geq 0 \qquad \text{for every } \psi(R) \geq 0\,.$$

  But this implies that $q_{(r_{\mathrm{rf}},\mathbf{f})}(R) \geq 0$.
- Because $q(\mathbf{R})$ integrates to 1, by setting $\psi(R) = 1$ this shows that

  $$\int q_{(r_{\mathrm{rf}},\mathbf{f})}(R)\,\mathrm{d}R = \int q(\mathbf{R})\,\mathrm{d}\mathbf{R} = 1\,.$$

Because $q_{(r_{\mathrm{rf}},\mathbf{f})}(R) \geq 0$ and integrates to 1, it is a probability density.

IID Models for Portfolios (Return Means)

By taking $\psi(R) = R$ in (2.15), we can compute the return mean μ as

$$\begin{aligned} \mu = \mathrm{Ex}(R) &= \int \left(r_{\mathrm{rf}} + \mathbf{R}^{\mathrm{T}}\mathbf{f}\right) q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\ &= r_{\mathrm{rf}} \int q(\mathbf{R})\,\mathrm{d}\mathbf{R} + \left(\int \mathbf{R}\, q(\mathbf{R})\,\mathrm{d}\mathbf{R}\right)^{\mathrm{T}} \mathbf{f} \\ &= r_{\mathrm{rf}} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{f}\,, \end{aligned} \tag{2.16}$$

where in the last step we have used that facts that

$$\int q(\mathbf{R})\,\mathrm{d}\mathbf{R} = 1\,, \qquad \int \mathbf{R}\, q(\mathbf{R})\,\mathrm{d}\mathbf{R} = \boldsymbol{\mu}\,.$$

The first fact follows because $q(\mathbf{R})$ is a probability density while the second fact comes from (1.1).

IID Models for Portfolios (Return Variance)

By taking $\psi(R) = (R - \mu)^2$ in (2.15) and using formula (2.16) for μ, we can then compute the return variance ξ as

$$\begin{aligned}
\xi = \mathrm{Ex}\Big((R-\mu)^2\Big) &= \int \Big(r_{\mathrm{rf}} + \mathbf{R}^{\mathrm{T}}\mathbf{f} - \mu\Big)^2 q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= \int \Big(\mathbf{R}^{\mathrm{T}}\mathbf{f} - \boldsymbol{\mu}^{\mathrm{T}}\mathbf{f}\Big)^2 q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= \int \mathbf{f}^{\mathrm{T}}(\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}}\mathbf{f}\,q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= \mathbf{f}^{\mathrm{T}}\left(\int (\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}} q(\mathbf{R})\,\mathrm{d}\mathbf{R}\right)\mathbf{f} \\
&= \mathbf{f}^{\mathrm{T}}\boldsymbol{\Xi}\,\mathbf{f}\,,
\end{aligned} \tag{2.17}$$

where in the last step we have used the fact from (1.1) that

$$\int (\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}} q(\mathbf{R})\,\mathrm{d}\mathbf{R} = \boldsymbol{\Xi}\,.$$

IID Models for Portfolios (Estimators)

If the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ is modeled by IID samples drawn from a probability density $q(\mathbf{R})$ then the associated return mean $\boldsymbol{\mu}$ and return variance $\boldsymbol{\Xi}$ have the unbiased estimators (1.10) given by

$$\hat{\boldsymbol{\mu}} = \mathbf{m}\,, \qquad \hat{\boldsymbol{\Xi}} = \frac{1}{1-\bar{w}}\,\mathbf{V}\,. \tag{2.18}$$

The Markowitz portfolio with a risk-free return r_{rf} and a risky asset allocation $\mathbf{f}$ has the return history $\{r(d)\}_{d=1}^{D}$ where

$$r(d) = r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,.$$

This history is modeled by IID samples drawn from the probability density $q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$. Formulas (2.16), (2.17) and (2.18) show that the return mean μ and return variance ξ of this portfolio have the unbiased estimators

$$\hat{\mu} = r_{\mathrm{rf}} + \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \hat{\xi} = \frac{1}{1-\bar{w}}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,. \tag{2.19}$$

IID Portfolio Metrics (Introduction)

Previously we have applied several IID related metrics to individual assets. For the i^{th} asset we have the mean and variance estimator metrics for assessing our certainty in the calibration of the IID model

$$\omega_i^{\hat{\mu}}\,, \qquad \omega_i^{\hat{\xi}}\,, \tag{3.20a}$$

the mean, variance, Kolmogorov-Smirnov and Kuiper comparison metrics for assessing indentical distribution

$$\omega_i^{m}\,, \qquad \omega_i^{v}\,, \qquad \omega_i^{\mathrm{KS}}\,, \qquad \omega_i^{\mathrm{Ku}}\,, \tag{3.20b}$$

and the autoregression and autocovariance metrics for assessing indepedence

$$\omega_i^{\mathrm{ar}}\,, \qquad \omega_i^{\mathrm{ac}}\,. \tag{3.20c}$$

IID Portfolio Metrics (Two Individual Assets)

Not all individual assets yield useful metrics. Some that do are:

- funds that track a large capitalization equity index like the S&P 500 or the Russell 1000,
- funds that track a broad-based bond index.

Typically a portfolio will have just one of each of these types of funds. Each of these have 8 metrics given by (3.20), giving the 16 metrics:

$$\begin{array}{cccccccc} \omega_{\mathrm{EI}}^{\hat{\mu}}\,, & \omega_{\mathrm{EI}}^{\hat{\xi}}\,, & \omega_{\mathrm{EI}}^{m}\,, & \omega_{\mathrm{EI}}^{v}\,, & \omega_{\mathrm{EI}}^{\mathrm{KS}}\,, & \omega_{\mathrm{EI}}^{\mathrm{Ku}}\,, & \omega_{\mathrm{EI}}^{\mathrm{ar}}\,, & \omega_{\mathrm{EI}}^{\mathrm{ac}}\,, \\ \omega_{\mathrm{BI}}^{\hat{\mu}}\,, & \omega_{\mathrm{BI}}^{\hat{\xi}}\,, & \omega_{\mathrm{BI}}^{m}\,, & \omega_{\mathrm{BI}}^{v}\,, & \omega_{\mathrm{BI}}^{\mathrm{KS}}\,, & \omega_{\mathrm{BI}}^{\mathrm{Ku}}\,, & \omega_{\mathrm{BI}}^{\mathrm{ar}}\,, & \omega_{\mathrm{BI}}^{\mathrm{ac}}\,. \end{array}$$

Because an IID model for markets yields and IID model for each Markowitz portfolio, these metrics can now be applied to portfolios.

IID Portfolio Metrics (Three Portfolios)

Portfolios that are natural candidates for the application of these metrics are the safe tangent portfolio, the credit tangent portfolio, and the efficient long tangent portfolio.

- The safe tangent portfolio exists whenever $\mu_{\mathrm{si}} \neq \mu_{\mathrm{mv}}$. In that case simply set $r_{\mathrm{st}}(d) = \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}_{\mathrm{st}}$ and apply each metric to the return history $\{r_{\mathrm{st}}(d)\}_{d=1}^{D}$. If $\mu_{\mathrm{si}} = \mu_{\mathrm{mv}}$ then simply set all the metrics to 1.
- The credit tangent portfolio exists whenever $\mu_{\mathrm{cl}} \neq \mu_{\mathrm{mv}}$. In that case simply set $r_{\mathrm{cl}}(d) = \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}_{\mathrm{st}}$ and apply each metric to the return history $\{r_{\mathrm{cl}}(d)\}_{d=1}^{D}$. If $\mu_{\mathrm{cl}} = \mu_{\mathrm{mv}}$ then simply set all the metrics to 1.
- The efficient long tangent portfolio exists whenever $\mu_{\mathrm{si}} < \mu_{\mathrm{mx}}$. In that case simply set $r_{\mathrm{elt}}(d) = \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}_{\mathrm{elt}}$ and apply each metric to the return history $\{r_{\mathrm{elt}}(d)\}_{d=1}^{D}$. If $\mu_{\mathrm{si}} \geq \mu_{\mathrm{mx}}$ then simply set all the metrics to 1.

IID Portfolio Metrics (Summary)

This gives six calibration metrics

$$\omega^{\hat{\mu}}_{\mathrm{st}}\,, \qquad \omega^{\hat{\mu}}_{\mathrm{ct}}\,, \qquad \omega^{\hat{\mu}}_{\mathrm{elt}}\,, \qquad \omega^{\hat{\xi}}_{\mathrm{st}}\,, \qquad \omega^{\hat{\xi}}_{\mathrm{ct}}\,, \qquad \omega^{\hat{\xi}}_{\mathrm{elt}}\,, \tag{3.21a}$$

twelve indentical distribution metrics

$$\begin{aligned} &\omega^{m}_{\mathrm{st}}\,, \qquad \omega^{m}_{\mathrm{ct}}\,, \qquad \omega^{m}_{\mathrm{elt}}\,, \qquad \omega^{v}_{\mathrm{st}}\,, \qquad \omega^{v}_{\mathrm{ct}}\,, \qquad \omega^{v}_{\mathrm{elt}}\,, \\ &\omega^{\mathrm{KS}}_{\mathrm{st}}\,, \qquad \omega^{\mathrm{KS}}_{\mathrm{ct}}\,, \qquad \omega^{\mathrm{KS}}_{\mathrm{elt}}\,, \qquad \omega^{\mathrm{Ku}}_{\mathrm{st}}\,, \qquad \omega^{\mathrm{Ku}}_{\mathrm{ct}}\,, \qquad \omega^{\mathrm{Ku}}_{\mathrm{elt}}\,, \end{aligned} \tag{3.21b}$$

and six independence metrics

$$\omega^{\mathrm{ar}}_{\mathrm{st}}\,, \qquad \omega^{\mathrm{ar}}_{\mathrm{ct}}\,, \qquad \omega^{\mathrm{ar}}_{\mathrm{elt}}\,, \qquad \omega^{\mathrm{ac}}_{\mathrm{st}}\,, \qquad \omega^{\mathrm{ac}}_{\mathrm{ct}}\,, \qquad \omega^{\mathrm{ac}}_{\mathrm{elt}}\,. \tag{3.21c}$$

These metrics are usually more useful that those for most individual assets because the portfolios used are the building blocks of efficient frontiers.

Application to CAPM (Introduction)

The *Capital Asset Pricing Model (CAPM)* was introduced in Section 3.4. There we took two steps.

Step 1. In the setting of Markowitz portfolios with a one-rate model of risk-free assets, we added some very strong assumptions that allowed us to identify the tangent portfolio with the market capitalization portfolio. This step is the heart of CAPM.

Step 2. We developed some relations between the return means and volatilities of individual assets with those of the tangent portfolio. These did not require the very strong assumptions of the first step. We then saw what they said given the conclusion of **Step 1**.

We are now ready for our last step in its development.

Step 3. We use the IID model for markets to relate the returns of individual assets to those of the market capitalization portfolio.

Application to CAPM (Assumptions)

We start by recalling the five assumptions upon which CAPM is built. Here we restate them in the setting of an IID model for the market.

1. The market consists of N risky assets and risk-free assets with a common return rate μ_{rf}.
2. There are K investors, each of which holds a solvent Markowitz portfolio governed by the one-rate model for risk-free assets.
3. The market capitalization of each asset is equal to the sum of its value of that asset held in each portfolio.
4. The density $q(\mathbf{R})$ of daily return vectors is known and is stationary. Let $\boldsymbol{\mu}$ and $\boldsymbol{\Xi}$ be its known mean vector and covariance matrix. Then μ_{rf}, $\boldsymbol{\mu}$ and $\boldsymbol{\Xi}$ are the same for all investors and are constant in time.
5. Each investor holds a portfolio on the efficient Tobin frontier.

Application to CAPM (Conclusions so Far)

Next we recall the conclusions from **Step 1** and **Step 2**.

- The tangent portfolio exists and is efficient $(\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}})$.
- The tangent allocation $\mathbf{f}_{\mathrm{tg}}$ is the market capitalization allocation $\mathbf{f}_{\mathrm{M}}$, and is thereby long.
- The market capitalization allocation $\mathbf{f}_{\mathrm{M}}$ does not depend on time.
- The return mean μ_i of the i^{th} asset satisfies

$$\mu_i - \mu_{\mathrm{rf}} = \beta_i \left(\mu_{\mathrm{M}} - \mu_{\mathrm{rf}} \right) , \tag{4.22a}$$

where

$$\mu_{\mathrm{M}} = \boldsymbol{\mu}^{\mathrm{T}} \mathbf{f}_{\mathrm{M}} \, , \qquad \beta_i = \frac{1}{\sigma_{\mathrm{M}}^2} \, \mathbf{e}_i^{\mathrm{T}} \boldsymbol{\Xi} \, \mathbf{f}_{\mathrm{M}} \, , \qquad \sigma_{\mathrm{M}}^2 = \mathbf{f}_{\mathrm{M}}^{\mathrm{T}} \boldsymbol{\Xi} \, \mathbf{f}_{\mathrm{M}} \, . \tag{4.22b}$$

- The variance ${\sigma_i}^2$ of the i^{th} asset has the decomposition

$${\sigma_i}^2 = {\beta_i}^2 \sigma_{\mathrm{M}}^2 + {\eta_i}^2 \, , \tag{4.22c}$$

where $\beta_i \, \sigma_{\mathrm{M}}$ is the systemic risk and η_i is the diversifiable risk.

Application to CAPM (**Step 3** Conclusion)

The conclusion of **Step 3** will be that the random variable $\mathbf{R} \in (-1, \infty)^N$ has the decomposition

$$\mathbf{R} = \boldsymbol{\mu} + \boldsymbol{\beta}\,(R_{\mathrm{M}} - \mu_{\mathrm{M}}) + \mathbf{Z}\,, \tag{4.23a}$$

where

$$R_{\mathrm{M}} = \mathbf{R}^{\mathrm{T}} \mathbf{f}_{\mathrm{M}}\,, \qquad \boldsymbol{\beta} = \frac{1}{\sigma_{\mathrm{M}}^2}\,\boldsymbol{\Xi}\,\mathbf{f}_{\mathrm{M}}\,, \tag{4.23b}$$

with μ_{M} and σ_{M} given by (4.22b),

$$\boldsymbol{\mu} - \mu_{\mathrm{rf}}\mathbf{1} = \boldsymbol{\beta}\,(\mu_{\mathrm{M}} - \mu_{\mathrm{rf}})\,, \tag{4.23c}$$

and where the random variable $\mathbf{Z} \in \mathbb{R}^N$ satisfies

$$\mathrm{Ex}(\mathbf{Z}) = \mathbf{0}\,, \qquad \mathrm{Ex}(\mathbf{Z}\,(R_{\mathrm{M}} - \mu_{\mathrm{M}})) = \mathbf{0}\,, \tag{4.23d}$$

$$\boldsymbol{\Xi} = \sigma_{\mathrm{M}}^2\,\boldsymbol{\beta}\,\boldsymbol{\beta}^{\mathrm{T}} + \mathrm{Ex}\left(\mathbf{Z}\,\mathbf{Z}^{\mathrm{T}}\right)\,, \tag{4.23e}$$

and $\mathrm{Ex}(|\mathbf{Z}|^2)$ is the minimum over all similar decompositions.

Application to CAPM (Relation to Earlier Results)

Because μ_i and β_i appearing in equation (4.22a) are related to $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$ appearing in equation (4.23c) by

$$\mu_i = \mathbf{e}_i^{\mathrm{T}} \boldsymbol{\mu} \,, \qquad \beta_i = \mathbf{e}_i^{\mathrm{T}} \boldsymbol{\beta} \,,$$

we see that equation (4.22a) is equation (4.23c) multiplied on the left by $\mathbf{e}_i^{\mathrm{T}}$. Therefore equation (4.23c) is just a restatement of equation (4.22a).

Because σ_i and η_i appearing in decomposition (4.22c) are related to $\boldsymbol{\Xi}$ and $\mathrm{Ex}\left(\mathbf{Z}\,\mathbf{Z}^{\mathrm{T}}\right)$ appearing in decomposition (4.23e) by

$$\sigma_i{}^2 = \mathbf{e}_i^{\mathrm{T}} \boldsymbol{\Xi}\, \mathbf{e}_i \,, \qquad \eta_i{}^2 = \mathbf{e}_i^{\mathrm{T}} \mathrm{Ex}\left(\mathbf{Z}\,\mathbf{Z}^{\mathrm{T}}\right) \mathbf{e}_i \,,$$

we see that decomposition (4.22c) is decomposition (4.23e) multiplied on the left by $\mathbf{e}_i^{\mathrm{T}}$ and on the right by $\mathbf{e}_i$. Therefore the diagonal entries of decomposition (4.23e) are restatements of the decompositions (4.22c), but its off-diagonal entries make it a stronger statement about covariances.

Application to CAPM (Outline of Development)

Recall that the conclusions of **Step 2** are (4.22). These followed from some general facts about tangent portfolios and the conclusion of **Step 1** that the tangent portfolio has the market capitalization allocation $\mathbf{f}_{\mathrm{M}}$.

In a similar way the conclusions of **Step 3** given by (4.23) will follow from:

- some general facts about any reference portfolio,
- some general facts when the reference portfolio is a tangent portfolio,
- the conclusion of **Step 1** that the tangent portfolio has the market capitalization allocation $\mathbf{f}_{\mathrm{M}}$.

Application to CAPM (Least Squares Problem)

Let $\mathbf{f}_{\mathrm{R}} \in \mathcal{M}$ be the allocation of any reference portfolio. For every random variable $\mathbf{R} \in (-1, \infty)^N$ we want to find $\boldsymbol{\alpha} \in \mathbb{R}^N$ and $\boldsymbol{\beta} \in \mathbb{R}^N$ such that the decomposition

$$\mathbf{R} = \boldsymbol{\alpha} + \boldsymbol{\beta}\, R_{\mathrm{R}} + \mathbf{Z}\,, \tag{4.24a}$$

makes $\mathrm{Ex}(|\mathbf{Z}|^2)$ as small as possible, where

$$R_{\mathrm{R}} = \mathbf{R}^{\mathrm{T}} \mathbf{f}_{\mathrm{R}}\,. \tag{4.24b}$$

This is a *least squares problem*.

It simplifies the calculation to replace $\boldsymbol{\alpha}$ with $\tilde{\boldsymbol{\alpha}}$ defined by the relation

$$\boldsymbol{\alpha} = \tilde{\boldsymbol{\alpha}} + \boldsymbol{\mu} - \boldsymbol{\beta}\, \mu_{\mathrm{R}}\,, \tag{4.25a}$$

where

$$\mu_{\mathrm{R}} = \boldsymbol{\mu}^{\mathrm{T}} \mathbf{f}_{\mathrm{R}}\,. \tag{4.25b}$$

Application to CAPM ($|\mathbf{Z}|^2$)

Therefore we want to find $\widetilde{\boldsymbol{\alpha}} \in \mathbb{R}^N$ and $\boldsymbol{\beta} \in \mathbb{R}^N$ that minimizes $\mathrm{Ex}(|\mathbf{Z}|^2)$ where

$$\mathbf{R} = \widetilde{\boldsymbol{\alpha}} + \boldsymbol{\mu} + \boldsymbol{\beta}\,(R_{\mathrm{R}} - \mu_{\mathrm{R}}) + \mathbf{Z}\,. \tag{4.26a}$$

Because

$$\mathbf{Z} = (\mathbf{R} - \boldsymbol{\mu}) - \widetilde{\boldsymbol{\alpha}} - \boldsymbol{\beta}\,(R_{\mathrm{R}} - \mu_{\mathrm{R}})\ , \tag{4.26b}$$

we see that

$$\begin{aligned} |\mathbf{Z}|^2 = {} & |\mathbf{R} - \boldsymbol{\mu}|^2 + |\widetilde{\boldsymbol{\alpha}}|^2 + |\boldsymbol{\beta}|^2\,(R_{\mathrm{R}} - \mu_{\mathrm{R}})^2 \\ & - 2\widetilde{\boldsymbol{\alpha}}^{\mathrm{T}}(\mathbf{R} - \boldsymbol{\mu}) - 2\boldsymbol{\beta}^{\mathrm{T}}(\mathbf{R} - \boldsymbol{\mu})\,(R_{\mathrm{R}} - \mu_{\mathrm{R}}) \\ & + 2\widetilde{\boldsymbol{\alpha}}^{\mathrm{T}}\boldsymbol{\beta}\,(R_{\mathrm{R}} - \mu_{\mathrm{R}})\ . \end{aligned} \tag{4.27}$$

Application to CAPM $\left(\mathrm{Ex}\left(|\mathbf{Z}|^2\right)\right)$

Because $\boldsymbol{\mu}$ and $\boldsymbol{\Xi}$ were defined by

$$\mathrm{Ex}(\mathbf{R}) = \boldsymbol{\mu}\,, \qquad \mathrm{Ex}\Big((\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}}\Big) = \boldsymbol{\Xi}\,, \tag{4.28}$$

while R_{R} and μ_{R} are given by (4.24b) and (4.25b), we see that

$$\begin{gathered}\mathrm{Ex}(\mathbf{R}-\boldsymbol{\mu}) = \mathbf{0}\,, \qquad \mathrm{Ex}(R_{\mathrm{R}}-\mu_{\mathrm{R}}) = 0\,, \\ \mathrm{Ex}\Big(|\mathbf{R}-\boldsymbol{\mu}|^2\Big) = \mathrm{tr}(\boldsymbol{\Xi})\,, \qquad \mathrm{Ex}((\mathbf{R}-\boldsymbol{\mu})\,(R_{\mathrm{R}}-\mu_{\mathrm{R}})) = \boldsymbol{\Xi}\,\mathbf{f}_{\mathrm{R}}\,, \\ \mathrm{Ex}\Big((R_{\mathrm{R}}-\mu_{\mathrm{R}})^2\Big) = \mathbf{f}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{\Xi}\,\mathbf{f}_{\mathrm{R}}\,.\end{gathered} \tag{4.29}$$

We thereby see from (4.27) that

$$\mathrm{Ex}\Big(|\mathbf{Z}|^2\Big) = \mathrm{tr}(\boldsymbol{\Xi}) + |\tilde{\boldsymbol{\alpha}}|^2 + |\boldsymbol{\beta}|^2\sigma_{\mathrm{R}}^2 - 2\boldsymbol{\beta}^{\mathrm{T}}\boldsymbol{\Xi}\,\mathbf{f}_{\mathrm{R}}\,, \tag{4.30a}$$

where $\sigma_{\mathrm{R}} > 0$ is determined from

$$\sigma_{\mathrm{R}}^2 = \mathbf{f}_{\mathrm{R}}^{\mathrm{T}}\boldsymbol{\Xi}\,\mathbf{f}_{\mathrm{R}}\,. \tag{4.30b}$$

Application to CAPM (Minimizer)

The minimizer of $\mathrm{Ex}(|\mathbf{Z}|^2)$ given by (4.27) is

$$\tilde{\boldsymbol{\alpha}} = \mathbf{0}\,, \qquad \boldsymbol{\beta} = \frac{1}{\sigma_{\mathrm{R}}^2}\,\boldsymbol{\Xi}\,\mathbf{f}_{\mathrm{R}}\,.$$

It then follows from (4.26b) and (4.29) that the random variable $\mathbf{Z}$ satisfies

$$\mathrm{Ex}(\mathbf{Z}) = \mathbf{0}\,, \qquad \mathrm{Ex}(\mathbf{Z}\,(R_{\mathrm{R}} - \mu_{\mathrm{R}})) = \mathbf{0}\,. \tag{4.31a}$$

Moreover from (4.26a) we see that

$$\begin{aligned}(\mathbf{R}-\boldsymbol{\mu})\,(\mathbf{R}-\boldsymbol{\mu})^{\mathrm{T}} = {} & (R_{\mathrm{R}} - \mu_{\mathrm{R}})^2\,\boldsymbol{\beta}\,\boldsymbol{\beta}^{\mathrm{T}} + \mathbf{Z}\,\mathbf{Z}^{\mathrm{T}} \\ & + (R_{\mathrm{R}} - \mu_{\mathrm{R}})\left(\boldsymbol{\beta}\,\mathbf{Z}^{\mathrm{T}} + \mathbf{Z}\,\boldsymbol{\beta}^{\mathrm{T}}\right)\,.\end{aligned}$$

We thereby see from (4.28), (4.29), (4.30b), and (4.31a) that

$$\boldsymbol{\Xi} = \sigma_{\mathrm{R}}^2\,\boldsymbol{\beta}\,\boldsymbol{\beta}^{\mathrm{T}} + \mathrm{Ex}\left(\mathbf{Z}\,\mathbf{Z}^{\mathrm{T}}\right)\,. \tag{4.31b}$$

Application to CAPM (Reference is Tangent)

When the reference portfolio is the tangent portfolio then $\mathbf{f}_{\mathrm{R}} = \mathbf{f}_{\mathrm{tg}}$ and by (4.26a) the random variable $\mathbf{R} \in (-1, \infty)^N$ has the decomposition

$$\mathbf{R} = \boldsymbol{\mu} + \boldsymbol{\beta} \left(R_{\mathrm{tg}} - \mu_{\mathrm{tg}} \right) + \mathbf{Z} \,, \tag{4.32a}$$

where

$$R_{\mathrm{tg}} = \mathbf{R}^{\mathrm{T}} \mathbf{f}_{\mathrm{tg}} \,, \qquad \boldsymbol{\beta} = \frac{1}{\sigma_{\mathrm{tg}}^2} \, \boldsymbol{\Xi} \, \mathbf{f}_{\mathrm{tg}} \,, \tag{4.32b}$$

while by the general conclusion of **Step 2** we have

$$\boldsymbol{\mu} - \mu_{\mathrm{rf}} \mathbf{1} = \boldsymbol{\beta} \left(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}} \right) \,, \tag{4.32c}$$

and by (4.31) the random variable $\mathbf{Z} \in \mathbb{R}^N$ satisfies

$$\mathrm{Ex}(\mathbf{Z}) = \mathbf{0} \,, \qquad \mathrm{Ex}\Big(\mathbf{Z} \left(R_{\mathrm{tg}} - \mu_{\mathrm{tg}} \right) \Big) = \mathbf{0} \,, \tag{4.32d}$$

$$\boldsymbol{\Xi} = \sigma_{\mathrm{tg}}^2 \, \boldsymbol{\beta} \, \boldsymbol{\beta}^{\mathrm{T}} + \mathrm{Ex}\Big(\mathbf{Z} \, \mathbf{Z}^{\mathrm{T}} \Big) \,. \tag{4.32e}$$

Application to CAPM (Tangent is Market Capitalization)

Finally, recall the conclusions from **Step 1** of our CAPM development.

- The tangent portfolio exists and is efficient $(\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}})$.
- The tangent allocation $\mathbf{f}_{\mathrm{tg}}$ is the market capitalization allocation $\mathbf{f}_{\mathrm{M}}$, and is thereby long.
- The market capitalization allocation $\mathbf{f}_{\mathrm{M}}$ does not depend on time.

Therefore the conclusions (4.23) of **Step 3** of our CAPM development follow from (4.32) by setting $\mathbf{f}_{\mathrm{tg}} = \mathbf{f}_{\mathrm{M}}$. □

Remark. The main critique of CAPM that was given in Section 3.4 was that because it assumes perfect knowledge by all investors that does not require any data collection, it is an overly simple, purely probabilistic model that leads to predictions that are not supported by observations. However, because its predictions were approximately right often enough, it had a huge impact on the finance industry and led to the development of more flexible models that are still used today.