# Lecture 4: The Cheeger Constant and the Spectral Gap


**Radu Balan**

Department of Mathematics, AMSC and NWC
University of Maryland, College Park, MD


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# Eigenvalues of Laplacians

## $\Delta, L, \tilde{\Delta}$

Today we discuss the spectral theory of graphs. Recall the Laplacian matrices:

$$\Delta = D - A \quad , \quad \Delta_{ij} = \begin{cases} d_i & \textit{if} \quad i = j \\ -1 & \textit{if} \quad (i,j) \in \mathcal{E} \\ 0 & \quad \textit{otherwise} \end{cases}$$

$$L = D^{\#}\Delta \quad , \quad L_{i,j} = \begin{cases} 1 & \textit{if} \quad i = j \text{ and } d_i > 0 \\ -\frac{1}{d(i)} & \textit{if} \quad (i,j) \in \mathcal{E} \\ 0 & \quad \textit{otherwise} \end{cases}$$

$$\tilde{\Delta} = D^{\#/2}\Delta D^{\#/2} \quad , \quad \tilde{\Delta}_{i,j} = \begin{cases} 1 & \textit{if} \quad i = j \text{ and } d_i > 0 \\ -\frac{1}{\sqrt{d(i)d(j)}} & \textit{if} \quad (i,j) \in \mathcal{E} \\ 0 & \quad \textit{otherwise} \end{cases}$$

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Remark: $D^{\#}, D^{\#/2}$ denote the pseudoinverses of $D$ and $D^{1/2}$ respectively.

# Eigenvalues of Laplacians

## $\Delta, L, \tilde{\Delta}$

What do we know about the set of eigenvalues of these matrices for a graph $G$ with $n$ vertices?

1. $\Delta = \Delta^T \geq 0$ and hence its eigenvalues are non-negative real numbers.
2. $eigs(\tilde{\Delta}) = eigs(L) \subset [0, 2]$.
3. 0 is always an eigenvalue and its multiplicity equals the number of connected components of $G$,

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Let $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{n-1}$ be the eigenvalues of $\tilde{\Delta}$. Denote $\lambda(G) = \max_{1 \leq i \leq n-1} |1 - \lambda_i|$. Note $\sum_{i=1}^{n-1} \lambda_i = trace(\tilde{\Delta}) = n$. Hence the average eigenvalue is about 1. $1 - \lambda(G)$ is called *the absolute gap*. $\lambda G$ is called the *relative gap* and measures the spread of eigenvalues away from 1.

# The absolute spectral gap

## $\lambda(G)$

The main result in [8] says that for connected graphs w/h.p.:

$$\lambda_1 \geq 1 - \frac{C}{\sqrt{\text{Average Degree}}} = 1 - \frac{C}{\sqrt{p(n-1)}} = 1 - C\sqrt{\frac{n}{2m}}.$$

### Theorem (For class $\mathcal{G}_{n,p}$)

*Fix $\delta > 0$ and let $p > (\frac{1}{2} + \delta)log(n)/n$. Let $d = p(n-1)$ denote the expected degree of a vertex. Let $\tilde{G}$ be the giant component of the Erdös-Rényi graph. For every fixed $\varepsilon > 0$, there is a constant $C = C(\delta, \varepsilon)$, so that for non-zero eigenvalues of $\tilde{\Delta}$,*

$$\lambda(\tilde{G}) := \max_{\lambda_k > 0}(|1 - \lambda_k|) \leq \frac{C}{\sqrt{d}} = C\sqrt{\frac{n}{2m}}$$

*with probability at least $1 - Cn\exp(-(2-\varepsilon)d) - C\exp(-d^{1/4}log(n))$.*

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### Theorem (For class $\Gamma^{n,m}$)

*Fix $\delta > 0$ and let $m > \frac{1}{2}(\frac{1}{2} + \delta) n \log(n)$. Let $d = \frac{2m}{n}$ denote the expected degree of a vertex. Let $\tilde{G}$ denote the giant component of the Erdös-Rényi graph. For every fixed $\varepsilon > 0$, there is a constant $C = C(\delta, \varepsilon)$, so that for non-zero eigenvalues of $\tilde{\Delta}$,*

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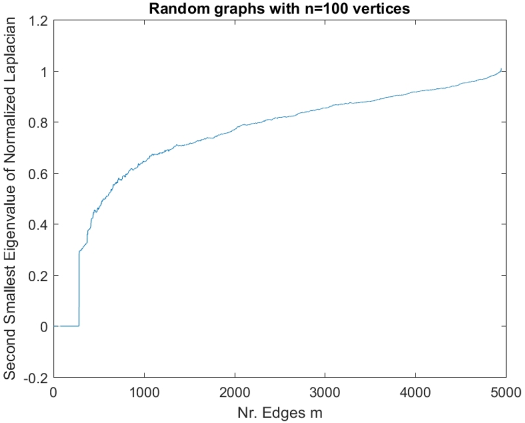
# Random graphs

## $\lambda_1$ for random graphs

Results for $n = 100$ vertices: $\lambda_1(\tilde{G}) \approx 1 - \frac{C}{\sqrt{m}}$.



# Random graphs

## $1 - \lambda_1$ for random graphs

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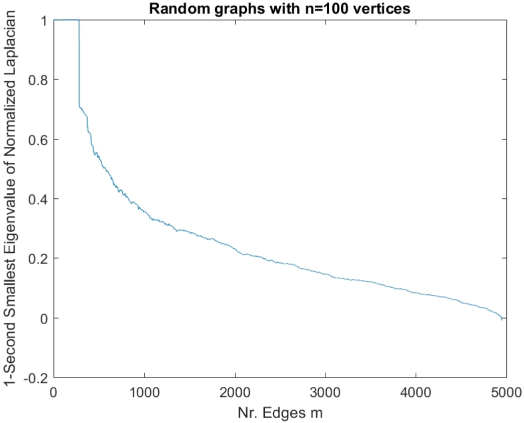
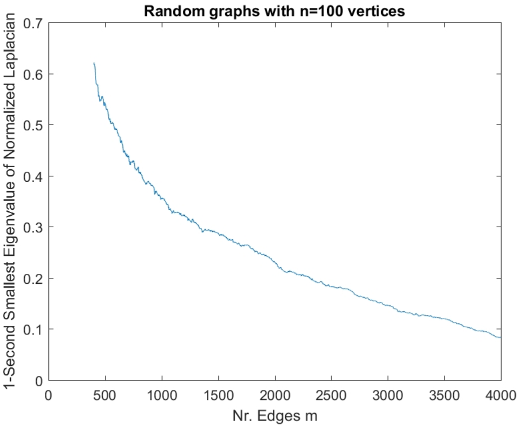


# Random graphs

## $1-\lambda_1$ for random graphs

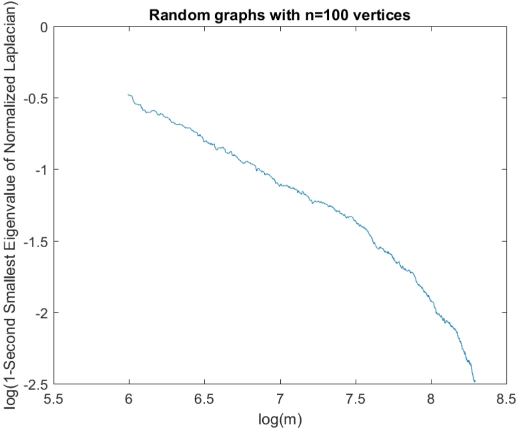
Results for $n=100$ vertices: $1-\lambda_1(\tilde{G}) \approx \frac{C}{\sqrt{m}}$. Detail.



# Random graphs

## $log(1-\lambda_1)$ vs. $log(m)$ for random graphs

Results for $n = 100$ vertices: $log(1-\lambda_1(\tilde{G})) \approx b_0 - \frac{1}{2}log(m)$.



# The absolute spectral gap

## Proof

How to obtain such estimates? Following [4]:
First note: $\lambda_i = 1 - \lambda_i(D^{-1/2}AD^{-1/2})$. Thus

$$\lambda(G) = \max_{1 \leq i \leq n-1} |1 - \lambda_i| = \|D^{-1/2}AD^{-1/2}\| = \sqrt{\lambda_{max}((D^{-1/2}AD^{-1/2})^2)}$$

Ideas:

1. For $X = D^{-1/2}AD^{-1/2}$, and any positive integer $k > 0$,

$$\lambda_{max}(X^2) = \left(\lambda_{max}(X^{2k})\right)^{1/k} \leq \left(trace(X^{2k})\right)^{1/k}$$

2. (Markov's inequality)

$$Prob\{\lambda(G) > t\} = Prob\{\lambda(G)^{2k} > t^{2k}\} \leq \frac{1}{t^{2k}}\mathbb{E}[trace(X^{2k})].$$

# The absolute spectral gap

## Proof (2)

Consider the easier case when $D = dI$ (all vertices have the same degree):

$$\mathbb{E}[trace(X^{2k})] = \frac{1}{d^{2k}} \mathbb{E}[trace(A^{2k})].$$

The expectation turns into numbers of $2k$-cycles and loops. Combinatorial kicks in ...

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### Remark

*Bernstein's "trick" (Chernoff bound) for $X \geq 0$, (the "Laplace method")*

$$Prob\{X \leq t\} = Prob\{e^{-sX} \geq e^{-st}\} \leq \inf_{s \geq 0} \frac{\mathbb{E}[e^{-sX}]}{e^{-st}} = \inf_{s \geq 0} e^{st} \int_0^\infty e^{-sx} p_X(x) dx$$

*If $P\{X < t\} > 0$ then the infimum is achieved, hence it becomes a minimum.*
*Such bounds give exponential decay instead of $\frac{1}{t}$ or $\frac{1}{t^2}$.*

# The Cheeger constant

## Partitions

Fix a graph $G = (\mathcal{V}, \mathcal{E})$ with $n$ vertices and $m$ edges. We try to find an optimal partition $\mathcal{V} = A \cup B$ that minimizes a certain quantity.
Here are the concepts:

1. For two disjoint sets of vertices $A$ and $B$, $E(A, B)$ denotes the set of edges that connect vertices in $A$ with vertices in $B$:

$$E(A, B) = \{(x, y) \in \mathcal{E} \ \ , \ \ x \in A \ , \ y \in B\}.$$

2. The *volume* of a set of vertices is the sum of its degrees:

$$vol(A) = \sum_{x \in A} d_x .$$

3. For a set of vertices $A$, denote $\bar{A} = \mathcal{V} \setminus A$ its complement.

# The Cheeger constant

## $h_G$

The *Cheeger constant* $h_G$ is defined as

$$h_G = \min_{S\subset\mathcal{V}} \frac{|E(S,\bar{S})|}{min(vol(S),vol(\bar{S}))}.$$

### Remark

*It is a min edge-cut problem: This means, find the minimum number of edges that need to be cut so that the graph becomes disconnected, while the two connected components are not too small.*
*There is a similar min vertex-cut problem, where $E(S,\bar{S})$ is replaced by $\delta(S)$, the set of boundary points of $S$ (the constant is denoted by $g_G$).*

### Remark

*The graph is connected iff $h_G > 0$.*

# The Cheeger inequalities

## $h_G$ and $\lambda_1$

See [2](ch.2):

**Theorem**

*For a connected graph*

$$2h_G \geq \lambda_1 > 1 - \sqrt{1 - h_G^2} > \frac{h_G^2}{2}.$$

*Equivalently:*

$$\sqrt{2\lambda_1} > \sqrt{1 - (1 - \lambda_1)^2} > h_G \geq \frac{\lambda_1}{2}.$$

Why is it interesting: finding the exact $h_G$ is a NP-hard problem.

# The Cheeger inequalities

## Proof of upper bound

Why the upper bound: $2h_G \geq \lambda_1$?
All starts from understanding what $\lambda_1$ is:

$$\Delta 1 = 0 \rightarrow \tilde{\Delta} D^{1/2} 1 = 0$$

Hence an eigenvector associated to $\lambda_0 = 0$ is

$$g^0 = (\sqrt{d_1}, \sqrt{d_2}, \cdots, \sqrt{d_n})^T.$$

The eigenpair $(\lambda_1, g^1)$ is given by a solution of the following optimization problem:

$$\lambda_1 = \min_{h \perp g^0} \frac{\langle \tilde{\Delta} h, h \rangle}{\langle h, h \rangle}$$

In particular any $h$ so that $\langle h, g^0 \rangle = \sum_{k=1}^n h_k \sqrt{d_k} = 0$ satisfies

$$\langle \tilde{\Delta} h, h \rangle \geq \lambda_1 \|h\|^2.$$

# The Cheeger inequalities

## Proof of upper bound (2)

Assume that we found the optimal partition $(A = S, B = \bar{S})$ of $\mathcal{V}$ that minimizes the edge-cut.
Define the following particular $n$-vector:

$$h_k = \begin{cases} \frac{\sqrt{d_k}}{vol(A)} & if \quad k \in A = S \\ -\frac{\sqrt{d_k}}{vol(B)} & if \quad k \in B = \mathcal{V} \setminus S \end{cases}$$

One checks that $\sum_{k=1}^{n} h_k \sqrt{d_k} = 1 - 1 = 0$, and $\|h\|^2 = \frac{1}{vol(A)} + \frac{1}{vol(B)}$.
But:

$$\langle \tilde{\Delta} h, h \rangle = \sum_{(i,j):A_{i,j}=1} \left(\frac{h_i}{\sqrt{d_i}} - \frac{h_j}{\sqrt{d_j}}\right)^2 = |E(A,B)| \left(\frac{1}{vol(A)} + \frac{1}{vol(B)}\right)^2.$$

Thus:

$$2h_G = \frac{2|E(A,B)|}{min(vol(A), vol(B))} \geq |E(A,B)| \left(\frac{1}{vol(A)} + \frac{1}{vol(B)}\right) \geq \lambda_1.$$

# Min-cut Problems

## Initialization

The proof of the upper bound in Cheeger inequality reveals a "good" initial guess of the optimal partition:

1. Compute an eigenpair $(\lambda_1, g^1)$ associated to the second smallest eigenvalue;
2. Form the partition:

$$S = \{k \in \mathcal{V} \ , \ \ g_k^1 \geq 0\} \ , \ \ \bar{S} = \{k \in \mathcal{V} \ , \ \ g_k^1 < 0\}$$

# Min-cut Problems

## Weighted Graphs

The Cheeger inequality holds true for weighted graphs, $G = (\mathcal{V}, \mathcal{E}, W)$.

- $\Delta = D - W$, $D = diag(w_i)_{1 \leq i \leq n}$, $w_i = \sum_{j \neq i} w_{i,j}$
- $\tilde{\Delta} = D^{\#/2} \Delta D^{\#/2} = I - D^{-1/2} W D^{-1/2}$
- $eigs(\tilde{\Delta}) \subset [0, 2]$
- $h_G = \min_S \frac{\sum_{x \in S, y \in \bar{S}} W_{x,y}}{min(\sum_{x \in S} D_{x,x}, \sum_{y \in \bar{S}} D_{y,y})}$; $D = diag(W \cdot 1)$.
- $2h_G \geq \lambda_1 \geq 1 - \sqrt{1 - h_G^2}$
- Good initial guess for optimal partition: Compute the eigenpair $(\lambda_1, g^1)$ associated to the second smallest eigenvalue of $\tilde{\Delta}$; set:

$$S = \{k \in \mathcal{V} \ , \ g_k^1 \geq 0\} \ , \ \bar{S} = \{k \in \mathcal{V} \ , \ g_k^1 < 0\}$$

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