Lectures 2: Random Graphs

Radu Balan

Department of Mathematics, AMSC and NWC
University of Maryland, College Park, MD

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The Erdös-Rényi class $\mathcal{G}_{n,p}$

Definition

Today we discuss about random graphs. The *Erdös-Rényi class* $\mathcal{G}_{n,p}$ of random graphs is defined as follows.

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Today we discuss about random graphs. The *Erdös-Rényi class* $\mathcal{G}_{n,p}$ of random graphs is defined as follows.

Let $\mathcal{V}$ denote the set of n vertices, $\mathcal{V} = \{1, 2, \cdots, n\}$, and let $\mathcal{G}$ denote the set of all graphs with vertices $\mathcal{V}$. There are exactly $2^{\binom{n}{2}}$ such graphs.

The probability mass function on $\mathcal{G}$, $P : \mathcal{G} \to [0,1]$, is obtained by assuming that, as random variables, edges are independent from one another, and each edge occurs with probability $p \in [0,1]$. Thus a graph $G \in \mathcal{G}$ with m edges will have probability $P(G)$ given by

$$P(G) = p^m (1-p)^{\binom{n}{2} - m}.$$

(explain why)

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Probability space

Formally, $\mathcal{G}_{n,p}$ stands for the the probability space $(\mathcal{G}, P)$ composed of the set $\mathcal{G}$ of all graphs with n vertices, and the probability mass function P defined above.

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A reformulation of P: Let $G = (\mathcal{V}, \mathcal{E})$ be a graph with n vertices and m edges and let A be its adjacency matrix. Then:

$$P(G) = \prod_{(i,j)\in\mathcal{E}} Prob((i,j) \ is \ an \ edge) \prod_{(i,j)\notin\mathcal{E}} Prob((i,j) \ is \ not \ an \ edge) =$$

$$= \prod_{1\leq i<j\leq n} p^{A_{i,j}}(1-p)^{1-A_{i,j}}$$

where the product is over all ordered pairs (i,j) with $1 \leq i < j \leq n$. Note:

$$|\{(i,j)\,,\ 1\leq i<j\leq n\}| = \begin{pmatrix} n \\ 2 \end{pmatrix} \ \& \ |\{(i,j)\in\mathcal{E}\}| = |\mathcal{E}| = m = \sum_{1\leq i<j\leq n} A_{i,j}.$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Computations in $\mathcal{G}_{n,p}$

How to compute the expected number of edges of a graph in $\mathcal{G}_{n,p}$?

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Computations in $\mathcal{G}_{n,p}$

How to compute the expected number of edges of a graph in $\mathcal{G}_{n,p}$?

Let $X_2 : \mathcal{G} \to \{0, 1, \cdots, \binom{n}{2}\}$ be the random variable of *number of edges of a graph G*.

$$X_2 = \sum_{1 \leq i < j \leq n} 1_{(i,j)} \quad , \quad 1_{(i,j)}(G) = \begin{cases} 1 & \text{if} \quad (i,j) \text{ is edge in } G \\ 0 & \text{if} \quad \text{otherwise} \end{cases}$$

Use linearity and the fact that $\mathbb{E}[1_{(i,j)}] = Prob((i,j) \in \mathcal{E}) = p$ to obtain:

$$\mathbb{E}[Number\ \ of\ \ Edges] = \binom{n}{2} p = \frac{n(n-1)}{2} p$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

MLE of p

Given a realization G of a graph with n vertices and m edges, how to estimate the most likely p that explains the graph.
Concept: The Maximum Likelihood Estimator (MLE).
In statistics: The MLE of a parameter θ given an observation x of a random variable $X \sim p_X(x;\theta)$ is the value θ that maximizes the probability $P_X(x;\theta)$:

$$\theta_{MLE} = argmax_\theta P_X(x;\theta).$$

In our case: our observation G has m edges. We know

$$P(G;p) = p^m (1-p)^{\binom{n}{2} - m}.$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

MLE of p

Lemma

Given a random graph with n vertices and m edges, the MLE estimator of p is

$$p_{MLE} = \frac{m}{\binom{n}{2}} = \frac{2m}{n(n-1)}.$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

MLE of p

Lemma

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Why

Note $log(P(G;p)) = mlog(p) + (\binom{n}{2} - m)log(1-p)$ and solve for p:

$$\frac{dlog(P)}{dp} = \frac{m}{p} - \frac{\binom{n}{2} - m}{1-p} = 0.$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Method of Moments Estimator for p

An alternative parameter estimation method is the moment matching method. Given a likelihood function for observed data $p(x;\theta)$ and a sequence of observations $(x_1, x_2, \cdots, x_N)$, the moment matching method computes the parameters $\theta \in \mathbb{R}^d$ by solving the system of equations:

$$\mathbb{E}[X] = \frac{1}{N}\sum_{t=1}^{N} x_t \quad \cdots \quad \mathbb{E}[X^d] = \frac{1}{N}\sum_{t=1}^{N} x_t^d$$

(or unbiased estimates of the moments). In particular, for the Erdös-Rényi class, we match the first moment with the observation: $\frac{n(n-1)}{2}p = m$. Hence

$$p_{MM} = \frac{2m}{n(n-1)},$$

same as the MLE estimator.

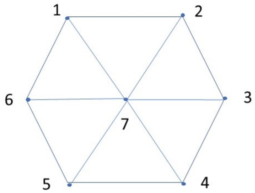
Cliques

q-cliques

Definition

Given a graph $G = (\mathcal{V}, \mathcal{E})$, a subset of q vertices $S \subset \mathcal{V}$ is called a *q-clique* if the subgraph $(S, \mathcal{E}|_{S\times S})$ is complete.

In other words, S is a q-clique if for every $i \neq j \in S$, $(i,j) \in \mathcal{E}$ (or $(j,i) \in \mathcal{E}$), that is, (i,j) is an edge in G.



- Each edge is a 2-clique.

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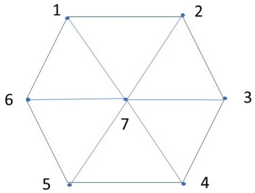

- Each edge is a 2-clique.
- $\{1,2,7\}$ is a 3-clique. And so are $\{2,3,7\}, \{3,4,7\}, \{4,5,7\}, \{5,6,7\}, \{1,6,7\}$

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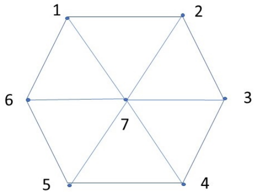

- Each edge is a 2-clique.
- $\{1,2,7\}$ is a 3-clique. And so are $\{2,3,7\}, \{3,4,7\}, \{4,5,7\}, \{5,6,7\}, \{1,6,7\}$
- There is no k-clique, with $k \geq 4$.

Finding the largest clique is a NP-hard problem, see for instance: *https://en.wikipedia.org/wiki/Clique_problem*

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Computations in $\mathcal{G}_{n,p}$: q-cliques

How to compute the expected number of q-cliques?

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Computations in $\mathcal{G}_{n,p}$: q-cliques

How to compute the expected number of q-cliques?
For $k = 2$ we computed earlier the number of edges, which is also the number of 2-cliques.
We shall compute now the number of 3-cliques: triangles, or 3-cycles.
Let $X_3 : \mathcal{G} \to \mathbb{N}$ be the random variable of number of 3-cliques. Note the maximum number of 3-cliques is $\begin{pmatrix} n \\ 3 \end{pmatrix}$.
Let S_3 denote the set of all distinct 3-cliques of the complete graph with n vertices, $S_3 = \{(i,j,k)\ ,\ 1 \leq i < j < k \leq n\}$.
Let

$$1_{(i,j,k)}(G) = \begin{cases} 1 & if \quad (i,j,k) \ is \ a \ 3-clique \ in \ G \\ 0 & if \quad otherwise \end{cases}$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Expectation of the number of 3-cliques

Note: $X_3 = \sum_{(i,j,k)\in S_3} 1_{(i,j,k)}$. Thus

$$\mathbb{E}[X_3] = \sum_{(i,j,k)\in S_3} \mathbb{E}[1_{(i,j,k)}] = \sum_{(i,j,k)\in S_3} Prob((i,j,k)\ is\ a\ clique).$$

Since $Prob((i,j,k)\ is\ a\ clique) = p^3$ we obtain:

$$\mathbb{E}[Number\ of\ 3-cliques] = \binom{n}{3} p^3 = \frac{n(n-1)(n-2)}{6} p^3.$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Number of 3 cliques

Assume we observe a graph G with n vertices and m edges. What would be the expected number N_3 of 3-cliques?

$$\mathbb{E}[X_3|X_2 = m] = \frac{1}{L}\sum_{k=1}^{L} X_3(G_k)$$

where L denotes the numbe of graphs with m edges and n vertices, and $G_1, \cdots, G_L$ is an enumeration of these graphs.

The Erdös-Rényi class $\mathcal{G}_{n,p}$

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We approximate:

$$\mathbb{E}[X_3|X_2 = m] \approx \mathbb{E}[X_3; p = p_{MLE}(m)]$$

and obtain:

$$E[X_3|X_2 = m] \approx \frac{4(n-2)}{3n^2(n-1)^2}m^3.$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Expectation of the number of q-cliques

Let $X_q : \mathcal{G} \to \mathbb{N}$ be the random variable of number of q-cliques. Note the maximum number of q-cliques is $\begin{pmatrix} n \\ q \end{pmatrix}$.

Let S_q denote the set of all distinct q-cliques of the complete graph with n vertices, $S_q = \{(i_1, i_2, \cdots, i_q)\ ,\ 1 \leq i_1 < i_2 < \cdots < i_q \leq n\}$. Note $|S_q| = \begin{pmatrix} n \\ q \end{pmatrix}$.

Let

$$1_{(i_1,i_2,\cdots,i_q)}(G) = \begin{cases} 1 & if \quad (i_1, i_2, \cdots, i_q) \ is\ a\ q-clique\ in\ G \\ 0 & if \quad otherwise \end{cases}$$

The Erdös-Rényi class $\mathcal{G}_{n,p}$

Expectation of the number of q-cliques

Since $X_q = \sum_{(i_1,\cdots,i_q)\in S_q} 1_{i_1,\cdots,i_q}$ and

$Prob((i_1,\cdots,i_q)\ is\ a\ clique) = p^{\binom{q}{2}}$ we obtain:

$$\mathbb{E}[Number\ of\ q-cliques] = \binom{n}{q} p^{q(q-1)/2}.$$

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Expectation of the number of q-cliques

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$$\mathbb{E}[Number\ of\ q-cliques] = \binom{n}{q} p^{q(q-1)/2}.$$

Using a similar argument as before, if G has m edges, then

$$\mathbb{E}[X_q|X_2=m] \approx \binom{n}{q}\left(\frac{2m}{n(n-1)}\right)^{q(q-1)/2}.$$

The Stochastic Block Model

The *Stochastic Block Model* (SBM), a.k.a. *Planted Partition Model*, was introduced in mathematial sociology by Holland, Laskey and Leinhardt in 1983 and by Wang and Wong in 1987. Here we follow Abbe (2017).

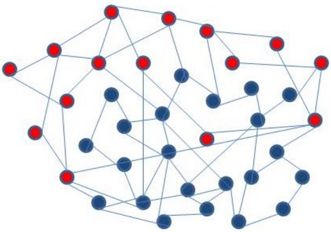


Figure: Example of a SBM

A Stochastic Block Model with $k = 2$ classes ('red' and 'blue') over $n = 15+22 = 37$ nodes. Number of edges: $m_{rr} = 21$, $m_{rb} = 6$, $m_{bb} = 35$.

The Stochastic Block Model

The general SBM

Data. Let n be a positive integer (the number of vertices), k be a positive integer (the number of communities), $\mathfrak{p} = (p_1, p_2, \cdots, p_k)$ be a probability vector on $[k] := \{1, 2, \cdots, k\}$ (the prior on the k communities), and Q be a $k \times k$ symmetric matrix with entries in $[0, 1]$ (the connectivity probabilities).

Definition

The pair (Z, G) is drawn under $SBM(n, \mathfrak{p}, Q)$ if Z is an n-dimensional random vector with i.i.d. components distributed under $\mathfrak{p}$, and G is an n-vertex graph where vertices i and j are connected with probability Q_{Z_i,Z_j}, independently of other pairs of vertices.

The *community sets* are defined by $\Omega_i = \Omega_i(Z) = \{v \in [n]\ ,\ Z_v = i\}$, $1 \leq i \leq k$.

The Stochastic Block Model

The Symmetric SBM (SSBM)

Definition

The pair (Z,G) is drawn under $SSBM(n,k,a,b)$ if Z is an n-dimensional random vector with i.i.d. components uniformly distributed over $[k]=\{1,2,\cdots,k\}$, and G is an n-vertex graph where distinct vertices i and j are connected with probability a if $Z_i=Z_j$ and probability b if $Z_i \neq Z_j$, independently of other pairs of vertices.

Data:
- the number of vertices: n;
- the number of communities: k;
- prior on k communities: $\mathfrak{p} = (\frac{1}{k},\frac{1}{k},\cdots,\frac{1}{k})$ on $[k]:=\{1,2,\cdots,k\}$;
- connectivity probabilities: Q

$$Q=\begin{bmatrix} a & b & \cdots & b \\ b & a & \cdots & b \\ \vdots & \vdots & \ddots & \vdots \\ b & b & \cdots & a \end{bmatrix}.$$

The Erdös-Rényi random graph is obtained when $a=b=p$.

The Binary Symmetric Stochastic Block Model

Distributions (1)

Consider a realization (Z, G) drawn randomly under $SSBM(n, 2, a, b)$ that models two communities. This means every node belongs with equal probability to either community, 1 or 2: $Z = (Z_1, Z_2, \cdots, z_n)$, where $Z_i \in \{1, 2\}$, $P(Z_i = 1) = P(Z_i = 2) = \frac{1}{2}$. The graph G of n nodes has adjacency matrix A. The conditional probability of realization A given the vector Z:

$$P(A|Z) = \prod_{1 \leq u < v \leq n} Q_{Z_u, Z_v}^{A_{u,v}} (1 - Q_{Z_u, Z_v})^{1 - A_{u,v}} =$$

$$= a^{m_{11} + m_{22}} b^{m_{12}} (1 - a)^{m_{11}^c + m_{22}^c} (1 - b)^{m_{12}^c}$$

where m_{11}, m_{22} are the number of edges inside community 1, respectively 2, m_{12} is the number of edges between the two communities, and m_{11}^c, m_{22}^c, m_{12}^c are the number of missing edges inside each community/between the two communities.

The Binary Symmetric Stochastic Block Model

Distributions (2)

Explicitely these numbers are given by:

$$m_{11} = \#\text{Edges inside community } 1 = \sum_{\substack{i<j \\ i,j\in\Omega_1}} A_{i,j}$$

$$m_{11}^c = \binom{n_1}{2} - m_{11} \quad n_1 = |\Omega_1|$$

$$m_{22} = \#\text{Edges inside community } 2 = \sum_{\substack{i<j \\ i,j\in\Omega_2}} A_{i,j}$$

$$m_{22}^c = \binom{n_2}{2} - m_{22} \quad n_2 = |\Omega_2|$$

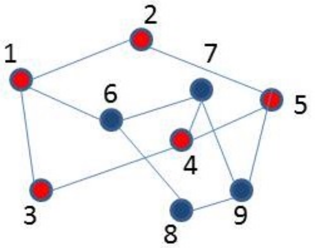
The Binary Symmetric Stochastic Block Model

Distributions (3)

$$m_{12} = \#\text{Edges between community 1 and 2} = \sum_{\substack{i \in \Omega_1 \\ j \in \Omega_2}} A_{i,j}$$

$$m_{12}^c = n_1 n_2 - m_{12}$$

Example:



$n = 9$, $\Omega_1 = \{1,2,3,4,5\}$, $\Omega_2 = \{6,7,8,9\}$.

$$m_{11} = 5 \ , \ m_{11}^c = 5$$

$$m_{22} = 4 \ , \ m_{22}^c = 2$$

$$m_{12} = 3 \ , \ m_{11}^c = 17$$

The Stochastic Block Model

Community Detection

The main problem: Community Detection.
This means a partition of the set of vertices $\mathcal{V} = \{1, 2, \cdots, n\}$ compatible with the observed graph G for a given connectivity probability matrix W.
To formulate mathematically we need to define the *agreement* between two community vectors.

Definition

The agreement between two community vectors $x, y \in [k]^n$ is obtained by maximizing the number of common components of these two vectors over all possible relabelling (i.e., permutations):

$$Agr(x, y) = \max_{\pi \in S_k} \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}(x_i = \pi(y_i))$$

where S_k denotes the group of permutations.

The Binary Symmetric Stochastic Block Model

Model Calibration: Supervised Learning

How to estimate parameters a, b in the 2-community symmetric stochastic block model $SSBM(n, 2, a, b)$. Use the Maximum Likelihood Estimator (MLE):

$$(a_{MLE}, b_{MLE}) = argmax_{a,b} Prob(G|Z, a, b)$$

Setup: Assume we have access to a training (i.e., labelled) data set (Z, G). Then for parameters a, b maximize:

$$a^{m_{11}+m_{22}}(1-a)^{m_{11}^c+m_{22}^c} b^{m_{12}}(1-b)^{m_{12}^c}$$

Take the logarithm and obtain:

$$a_{MLE} = \frac{m_{11}+m_{22}}{\binom{n_1}{2}+\binom{n_2}{2}} = \frac{2(m_{11}+m_{22})}{n_1(n_1-1)+n_2(n_2-1)}$$

$$b_{MLE} = \frac{m_{12}}{n_1 n_2}$$

The Binary Symmetric Stochastic Block Model

Model Calibration: Unsupervised Learning

Assume we have access to only one realization $G = (\mathcal{V}, A)$ of the random graph drawn from a binary symmetric SBM $SSBM(n, 2, a, b)$. The MLE is hard to solve. Instead we use the Method of Moment Matching. Since there are two parameters to estimate, a and b, we need to equations. We choose to match the numbers of 2-cliques (edges) and the number of 3-cliques. The expectations are computed by conditioning first on $n_1 = |\Omega_1|$ the size of partition, with $n_2 = n - n_1$:

$$\mathbb{E}[X_2|n_1] = \binom{n_1}{2} a + n_1 n_2 b + \binom{n_2}{2} a$$

$$\mathbb{E}[X_3|n_1] = \binom{n_1}{3} a^3 + \left[\binom{n_1}{2} n_2 + n_1 \binom{n_2}{2} \right] ab^2 + \binom{n_2}{3} a^3$$

$$\mathbb{E}[X_2|n_1] = \binom{n_1}{2} a + n_1 n_2 b + \binom{n_2}{2} a =$$
$$= \frac{n_1(n_1-1)+(n-n_1)(n-n_1-1)}{2} a + n_1(n-n_1)b$$
$$= \frac{n_1^2 - n_1 + n^2 - 2nn_1 + n_1^2 - n + n_1}{2} a + (nn_1 - n_1^2)b$$
$$= \left(n_1^2 - nn_1 + \frac{n(n-1)}{2}\right) a + (nn_1 - n_1^2)b$$

Next compute the expectation of the number of edges by double expectation. To do so we need

$$\mathbb{E}[n_1] = \mathbb{E}\left[\sum_{v=1}^{n} 1_{Z_v=1}\right] = \frac{n}{2}$$
$$\mathbb{E}[n_1^2] = \mathbb{E}\left[\left(\sum_{v=1}^{n} 1_{Z_v=1}\right)^2\right] = n\frac{1}{2} + 2\frac{n(n-1)}{2}\frac{1}{4} = \frac{n(n+1)}{4}$$

Thus

$$\mathbb{E}[X_2] = \mathbb{E}[\mathbb{E}[X_2|n_1]] = (\frac{n^2+n}{4} - \frac{n^2}{2} + \frac{n^2-n}{2})a + (\frac{n^2}{2} - \frac{n^2+n}{4})b =$$

$$= \frac{n^2-n}{4}(a+b)$$

Similarly,

$$\mathbb{E}[X_3|n_1] = \left(\begin{array}{c} n_1 \\ 3 \end{array}\right) a^3 + \left[\left(\begin{array}{c} n_1 \\ 2 \end{array}\right) n_2 + n_1 \left(\begin{array}{c} n_2 \\ 2 \end{array}\right)\right] ab^2 + \left(\begin{array}{c} n_2 \\ 3 \end{array}\right) a^3$$

$$= \frac{n_1(n_1-1)(n_1-2) + n_2(n_2-1)(n_2-2)}{6} a^3 + \frac{n_1 n_2 (n_1 - 1 + n_2 - 1)}{2} ab^2$$

$$= \frac{n_1^3 + n_2^3 - 3(n_1^2+n_2^2) + 2(n_1+n_2)}{6} a^3 + \frac{(nn_1 - n_1^2)(n-2)}{2} ab^2$$

$$= \frac{(n_1+n_2)(n_1^2 - n_1 n_2 + n_2^2) - 3(n_1^2+n_2^2) + 2n}{6} a^3 + \frac{(nn_1 - n_1^2)(n-2)}{2} ab^2$$

$$= \frac{(n-3)(n^2-2nn_1+2n_1^2) - nn_1(n-n_1) + 2n}{6} a^3 + \frac{(nn_1-n_1^2)(n-2)}{2} ab^2$$

$$= \frac{n^3-3n^2+2n+(3n-6)n_1^2-(3n^2-6n)n_1}{6} a^3 + \frac{(nn_1-n_1^2)(n-2)}{2} ab^2$$

Substitute $\mathbb{E}[n_1] = \frac{n}{2}$ and $\mathbb{E}[n_1^2] = \frac{n^2+n}{4}$:

$$\mathbb{E}[X_3] = \frac{n(n-2)}{6}(n-1+\frac{3}{4}(n+1)-\frac{3}{2}n)a^3 + \frac{n(n-2)(\frac{n}{2}-\frac{n+1}{4})}{2} ab^2$$

$$= \frac{n(n-1)(n-2)}{24} a^3 + \frac{n(n-1)(n-2)}{8} ab^2 = \frac{n(n-1)(n-2)}{24}(a^3+3ab^2)$$

The Binary Symmetric Stochastic Block Model

Model Calibration: Unsupervised Learning (2)

Assuming the graph has m 2-cliques (=edges) and t 3-cliques (=triangles) then by the moment matching method:

$$m=\frac{n(n-1)}{4}(a+b) \quad , \quad t=\frac{n(n-1)(n-2)}{24}(a^3+3ab^2)$$

Note: the $SSBM(n,2,a,b)$ class reduces to the Erdös-Renyi class $\mathcal{G}_{n,p}$ if $a=b=p$.

From where we solve for a and b in terms of n, m and t: Let $c_1=\frac{4m}{n(n-1)}$ and $c_2=\frac{24t}{n(n-1)(n-2)}$. Thus $b=c_1-a$ and

$$4a^3-6c_1a^2+3c_1^2a-c_2=0 \Rightarrow (2a-c_1)^3+c_1^3-2c_2=0$$

Thus:

$$a_{MM}=\frac{1}{2}\left(c_1+\sqrt[3]{2c_2-c_1^3}\right) \quad , \quad b_{MM}=\frac{1}{2}\left(c_1-\sqrt[3]{2c_2-c_1^3}\right)$$

The Binary Symmetric Stochastic Block Model

Model Calibration: Unsupervised Learning. Modified Estimator

The closed form expression deduced earlier using the moment matching method may produce un-feasible solutions. Specifically, the estimates a_{MM}, b_{MM} may not remain in the range $[0,1]$. Now we derive a modifed estimator that satisfies the feasibility constraints $a, b \in [0,1]$.
Our designing principle was to satisfy *exactly*:

$$m = \mathbb{E}[X_2] \quad , \quad t = \mathbb{E}[X_3]$$

Instead the modified estimator will satisfy the first constraint exactly, but will strive to satisfy the second constraint as much as possible, Specifically, it solves the following optimization problem:

$$\begin{array}{cc} \textit{minimize} & |\mathbb{E}[X_3] - t| \\ \text{subject to :} & \\ m = \mathbb{E}[X_2] & \\ 0 \leq a, b \leq 1 & \end{array}$$

The Binary Symmetric Stochastic Block Model

Model Calibration: Unsupervised Learning. Modified Estimator (2)

Substituting $a+b=2p=\frac{4m}{n(n-1)}$ into the objective function, after a bit of algebra we obtain:

$$\frac{6}{n(n-1)(n-2)}|t-\mathbb{E}[X_3]|=|(a-p)^3-\delta|$$

where $p=\frac{2m}{n(n-1)}$, $\delta=\frac{6t}{n(n-1)(n-2)}-p^3$. Let $P(x)=(x-p)^3-\delta$. Note $P'(x)=3(x-p)^2\geq 0$. Hence $x\mapsto P(x)$ is monotone increasing (in fact, strictly increasing).

On the other hand, $b=2p-a$ and the constraint $b\in[0,1]$ imply $0\leq 2p-a\leq 1$. Since $a\in[0,1]$ we obtain:

$$max(0,2p-1)\leq a\leq min(1,2p)$$

With $A_1=max(0,2p-1)$ and $A_2=min(1,2p)$ we obtain:

$$\begin{array}{cc} minimize & |P(a)| \\ A_1\leq a\leq A_2 & \end{array}$$

The Binary Symmetric Stochastic Block Model Calibration

Algorithm 1

The last optimization probem can be solved exactly.
The solutions is as follows:

Algorithm (1)

Input: n, m, t.

1. *Compute:*

$$p = \frac{2m}{n(n-1)}, \delta = \frac{6t}{n(n-1)(n-2)} - p^3$$

$$A_1 = max(0, 2p-1), A_2 = min(1, 2p)$$

$$P(A_1) = (A_1) - p)^3 - \delta, P(A_2) = (A_2 - p)^3 - \delta.$$

The Binary Symmetric Stochastic Block Model Calibration

Algorithm 1 - cont'ed

Algorithm (1 continued)

2. *Test and compute the* Constrained Moment Matching *estimates:*
   - *If* $P(A_1) \leq 0 \leq P(A_2)$ *then*

$$a_{CMM} = p + \sqrt[3]{\delta} \quad , \quad b_{CMM} = p - \sqrt[3]{\delta}$$

   - *If* $P(A_2) < 0$ *then*

$$a_{CMM} = A_2 \quad , \quad b_{CMM} = 2p - A_2$$

   - *If* $P(A_1) > 0$ *then*

$$a_{CMM} = A_1 \quad , \quad b_{CMM} = 2p - A_1$$

Output: a_{CMM} *and* b_{CMM}.

The Binary Symmetric Stochastic Block Model Calibration

Algorithm 2

While the Algorithm 1 produces estimates $a_{CMM}, b_{CMM} \in [0,1]$ it is often the case that one would like to obtain $a, b > 0$. The following algorithm provides such an "engineering fix":

Algorithm (2)

Input: n, m, t.

1. *Compute:*

$$p = \frac{2m}{n(n-1)}, \delta = \frac{6t}{n(n-1)(n-2)} - p^3$$

$$A_1 = max(0, 2p-1), A_2 = min(1, 2p)$$

$$P(A_1) = (A_1) - p)^3 - \delta, P(A_2) = (A_2 - p)^3 - \delta.$$

The Binary Symmetric Stochastic Block Model Calibration

Algorithm 2 - cont'ed

Algorithm (2 continued)

2. *Test and compute:*
   - *If* $P(A_1) \leq 0 \leq P(A_2)$ *then*

$$a_{CMM} = p + \sqrt[3]{\delta} \quad , \quad b_{CMM} = p - \sqrt[3]{\delta}$$

   - *If* $P(A_2) < 0$ *then*

$$a_{CMM} = A_2 \quad , \quad b_{CMM} = 2p - A_2$$

   - *If* $P(A_1) > 0$ *then*

$$a_{CMM} = A_1 \quad , \quad b_{CMM} = 2p - A_1$$

The Binary Symmetric Stochastic Block Model Calibration

Algorithm 2 - cont'ed

Algorithm (2 continued)

3. *Adjust to produce the* Modified Constrained Moment Matching *estimates*
   - *If* $0 < a_{CMM}, b_{CMM}$ *then*
   $$a_{MCMM} = a_{CMM} \quad , \quad b_{MCMM} = b_{CMM}$$
   - *If* $b_{CMM} = 0$ *then*
   $$a_{MCMM} = 0.99 a_{CMM} \quad , \quad b_{MCMM} = 0.01 a_{CMM}$$
   - *If* $a_{CMM} = 0$ *then*
   $$a_{MCMM} = 0.01 b_{CMM} \quad , \quad b_{MCMM} = 0.99 b_{CMM}$$

Output: a_{MCMM} *and* b_{MCMM}.

The Stochastic Block Model

Types of Community Detection Algorithms

Types of *algorithm*:
Let $(Z, G) \sim SBM(n, \mathfrak{p}, Q)$. Then the following recovery requirements are solved if there exists an algorithm that takes G as input and outputs $\hat{Z} = \hat{Z}(G)$ such that:

- **Exact recovery**: $P\{Agr(Z, \hat{Z}) = 1\} = 1 - o(1)$
- **Almost exact recovery**: $P\{Agr(Z, \hat{Z}\} = 1 - o(1)) = 1 - o(1)$
- **Partial recovery**: $P\{Agr(Z, \hat{Z}) \geq \alpha\} = 1 - o(1)$, $\alpha \in (0, 1)$.

Note these definitions apply to an algorithm, where probabilities are computed over all realizations of $SBM(n, \mathfrak{p}, Q)$ model.

The Symmetric Stochastic Block Model $SSBM(n, 2, a, b)$

Expectation of number of 4-cliques (1)

Under $SSBM(n, 2, a, b)$ the conditional expectation of X_4 given the size n_1 of the first community, is given by the following formula:

$$\mathbb{E}[X_4|n_1] = \binom{n_1}{4} a^6 + \binom{n_1}{3} n_2 a^3 b^3 + \binom{n_1}{2}\binom{n_2}{2} a^2 b^4 + \\ + n_1 \binom{n_2}{3} a^3 b^3 + \binom{n_2}{4} a^6$$

where the terms represent the cases when all four vertices are in community 1, three vertices in community 1 and one vertex in community 2, two vertices in each community, one vertex in community 1 and three in community 2, and finally, all four vertices are in community 2.
Next, the expectation of the number of 4-cliques given parameters a, b is obtained by iterating the expectation operator over n_1:

$$\mathbb{E}[X_4; a, b] = \mathbb{E}[\mathbb{E}[X_4|n_1]]$$

The Symmetric Stochastic Block Model $SSBM(n,2,a,b)$

Expectation of number of 4-cliques (2)

Since n_1 follows the binomial distribution $B(n,\frac{1}{2})$,

$$\mathbb{E}[n_1]=\frac{n}{2}\ ,\ \mathbb{E}[n_1^2]=\frac{n^2+n}{4}$$

$$\mathbb{E}[n_1^3]=\frac{n^2(n+3)}{8}\ ,\ \mathbb{E}[n_1^4]=\frac{n(n+1)(n^2+5n-2)}{16}$$

These expressions come from the moment generating function of the binomial distribution $M_X(t)=(1-p+pe^t)^n$ which for $p=\frac{1}{2}$ becomes $M_{n_1}(t)=\frac{1}{2^n}(1+e^t)^n$. Then the k^{th} moment is given by

$$\mathbb{E}[n_1^k]=\frac{d^k}{dt^k}M_{n_1}(t)|_{t=0}$$

See: http://mathworld.wolfram.com/BinomialDistribution.html for details. The expectation over n_1 is obtained by substituting $n_2=n-n_1$, expanding the expression of $\mathbb{E}[X_4|n_1]$ and then using the moments of n_1, n_1^2, n_1^3, n_1^4.

The Symmetric Stochastic Block Model $SSBM(n, 2, a, b)$

Expectation of number of 4-cliques (3)

Expanding, making the substitution $n_2 = n - n_1$ and combining the tems we get:

$$\mathbb{E}[X_4|n_1] = \frac{a^6}{24}\left(2n_1^4 - 4nn_1^3 + (6n^2 - 18n + 22)n_1^2 + (-4n^3 + 18n^2 - 22n)n_1 \right.$$

$$\left. + n^4 - 6n^3 + 11n^2 - 6n\right) +$$

$$+ \frac{a^3b^3}{6}\left(-2n_1^4 + 4nn_1^3 + (-3n^2 + 3n - 4)n_1^2 + (n^3 - 3n^2 + 4n)n_1\right)$$

$$+ \frac{a^2b^4}{4}\left(n_1^4 - 2nn_1^3 + (n^2 + n - 1)n_1^2 + (-n^2 + n)n_1\right)$$

The Symmetric Stochastic Block Model $SSBM(n, 2, a, b)$

Expectation of number of 4-cliques (4)

$$\mathbb{E}[X_4] = \frac{a^6}{24}\Big(2\mathbb{E}[n_1^4] - 4n\mathbb{E}[n_1^3] + (6n^2 - 18n + 22)\mathbb{E}[n_1^2]$$

$$+(-4n^3 + 18n^2 - 22n)\mathbb{E}[n_1] + n^4 - 6n^3 + 11n^2 - 6n\Big) +$$

$$+\frac{a^3b^3}{6}\Big(-2\mathbb{E}[n_1^4] + 4n\mathbb{E}[n_1^3] + (-3n^2 + 3n - 4)\mathbb{E}[n_1^2] + (n^3 - 3n^2 + 4n)\mathbb{E}[n_1]\Big)$$

$$+\frac{a^2b^4}{4}\Big(\mathbb{E}[n_1^4] - 2n\mathbb{E}[n_1^3] + (n^2 + n - 1)\mathbb{E}[n_1^2] + (-n^2 + n)\mathbb{E}[n_1]\Big)$$

where the expectations $\mathbb{E}[n_1]$, $\mathbb{E}[n_1^2]$, $\mathbb{E}[n_1^3]$ and $\mathbb{E}[n_1^4]$ have been computed before.

Numerical Computation of Number of Cliques

An Iterative Algorithm

We discuss two algorithms to compute X_q: iterative, and adjacency matrix based algorithm.

Framework: we are given a sequence $(G_t)_{t\geq 0}$ of graphs on n vertices, where G_{t+1} is obtained from G_t by adding one additional edge: $G_t = (\mathcal{V}, \mathcal{E}_t)$, $\emptyset = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots$ and $|\mathcal{E}_t| = t$.

Iterative Algorithm: Assume we know $X_q(G_t)$, the number of q-cliques of graph G_t. Then $X_q(G_{t+1}) = X_q(G_t) + D_q(e; G_t)$ where $D_q(e; G_t)$ denotes the number of q-cliques in G_{t+1} formed by the additional edge $e \in \mathcal{E}_{t+1} \setminus \mathcal{E}_t$.

Computation of Number of Cliques

An Analytic Formula

Laplace Matrix $\Delta = D - A$ contains all connectivity information.
Idea: Note the (i,j) element of A^2 is

$$(A^2)_{i,j} = \sum_{k=1}^{n} A_{i,k} A_{k,j} = |\{k : i \sim k \sim j\}|.$$

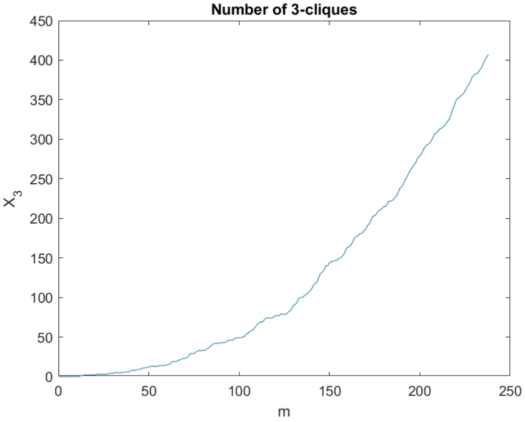
This means $(A^2)_{i,j}$ is the number of paths of length 2 that connect i to j.
Hence $m = \frac{1}{2} trace(A^2)$.
Remark: The diagonal elements of $A(A^2 - D)$ represent twice the number of 3-cycles (= 3-cliques) that contain that particular vertex.
Conclusion:

$$X_3 = \frac{1}{6} trace\{A(A^2 - D)\} = \frac{1}{6} trace(A^3).$$

Numerical results

Graph of X_3 for the BKOFF dataset

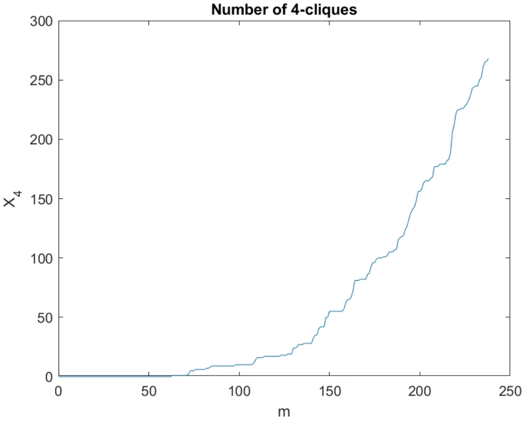
Recall the dataset Bernard & Killworth Office. Weighted graph: Ordered $m = 238$ edges for $n = 40$ nodes. The plot of X_3 the number of 3-cliques:



Numerical results

Plot of X_4 for the BKOFF dataset

Weighted graph: Ordered $m = 238$ edges for $n = 40$ nodes. The plot of X_4 the number of 4-cliques:



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