Portfolios that Contain Risky Assets
9.2. Mean-Variance Estimators for Kelly Objectives

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Math 420: *Mathematical Modeling*
April 15, 2022 version

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Part II: Probabilistic Models

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9. Kelly Objectives for Portfolio Models

Mean-Variance Estimators for Kelly Objectives

Introduction (Means and Volatilities)

Consider portfolios built from N risky assets and possibly some risk-free assets. Given a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ of the risky assets and positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, define the return sample mean $\mathbf{m}$ and sample variance $\mathbf{V}$ by

$$\mathbf{m} = \sum_{d=1}^{D} w_d \, \mathbf{r}(d) \,, \qquad \mathbf{V} = \sum_{d=1}^{D} w_d \left(\mathbf{r}(d) - \mathbf{m}\right) \left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathrm{T}} . \tag{1.1}$$

A Markowitz portfolio with a risk-free return r_{rf} and a risky asset allocation $\mathbf{f}$ has the return mean and volatility estimators

$$\hat{\mu} = r_{\mathrm{rf}} + \mathbf{m}^{\mathrm{T}} \mathbf{f} \,, \qquad \hat{\sigma} = \sqrt{\mathbf{f}^{\mathrm{T}} \mathbf{V} \, \mathbf{f}} \,. \tag{1.2}$$

Remark. The formulas for $\mathbf{m}$ and $\hat{\mu}$ are unbiased IID estimators, while those for $\mathbf{V}$ and $\hat{\sigma}$ are biased IID estimators. These biased estimators are what arise naturally in what follows.

Introduction

A Markowitz portfolio with a risk-free return r_{rf} and a risky asset allocation $\mathbf{f}$ is said to be solvent if

$$1 + r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} > 0 \quad \forall d\,. \tag{1.3}$$

Recall that r_{rf} is given in terms of the allocations of any risk-free assets by

$$r_{\mathrm{rf}} = \begin{cases} 0 & \text{when } \mathbf{f} \in \mathcal{M}\,, \\ \mu_{\mathrm{rf}} f^{\mathrm{rf}} & \text{when } (\mathbf{f}, f^{\mathrm{rf}}) \in \mathcal{M}_1\,, \\ \mu_{\mathrm{si}} f^{\mathrm{si}} + \mu_{\mathrm{cl}} f^{\mathrm{cl}} & \text{when } (\mathbf{f}, f^{\mathrm{si}}, f^{\mathrm{cl}}) \in \mathcal{M}_2\,. \end{cases} \tag{1.4}$$

The Kelly objective for every solvent Markowitz portfolio is

$$\hat{\gamma} = \sum_{d=1}^{D} w_d \log\Big(1 + r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\Big)\,. \tag{1.5}$$

Introduction

The Kelly strategy is to maximize $\hat{\gamma}$ over a convex subset Π of all solvent Markowitz allocations. This maximizer can be found numerically by convex optimization methods that are typically covered in graduate courses.

Rather than seek the maximizer of $\hat{\gamma}$ over Π, our strategy will be to replace the estimator $\hat{\gamma}$ with a new estimator for which finding the maximizer is easier. The hope is that the maximizer of $\hat{\gamma}$ and the maximizer of the new estimator will be close.

This strategy rests upon the fact that $\hat{\gamma}$ is itself an approximation. The uncertainties associated with it will translate into uncertainities about its maximizer. The hope is that the difference between the maximizer of $\hat{\gamma}$ and that of the new estimator will be within these uncertainties.

Introduction

We will derive some *mean-variance estimators* for γ in the form

$$\hat{\gamma} = G(\hat{\sigma}, \hat{\mu}) \ , \tag{1.6a}$$

where $\hat{\sigma}$ and $\hat{\mu}$ are given by (1.2) and $G(\sigma, \mu)$ is a function that is defined over a convex subset Σ of the $\sigma\mu$-plane over which

- $G(\sigma, \mu)$ is a strictly decreasing function of σ,
- $G(\sigma, \mu)$ is a strictly increasing function of μ, (1.6b)
- $G(\sigma, \mu)$ is a concave function of (σ, μ).

The monotonicity properties insure that $\hat{\gamma}$ is larger for more efficient portfolios, which implies that if its maximizer over Π exists and has $(\hat{\sigma}, \hat{\mu}) \in \Sigma$ then it will lie on the efficient frontier of Π. Some of these estimators will satisfy the Jensen inequality bound

$$\hat{\gamma} \leq \log(1 + \hat{\mu}) \ . \tag{1.7}$$

Portfolios with no Risk-Free Assets (Introduction)

For simplicity we will start in the setting of solvent Markowitz portfolios without risk-free assets. In that case the portfolio with allocation $\mathbf{f} \in \mathcal{M}$ has the return mean and volatility estimators $\hat{\mu}(\mathbf{f})$ and $\hat{\sigma}(\mathbf{f})$ given by

$$\hat{\mu}(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \hat{\sigma}(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}\,. \tag{2.8}$$

The set of solvent Markowitz allocations for portfolios without risk-free assets is

$$\Omega = \left\{ \mathbf{f} \in \mathcal{M} \,:\, 1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} > 0 \;\; \forall d \right\}. \tag{2.9}$$

For every $\mathbf{f} \in \Omega$ the growth rate mean sample estimator is

$$\hat{\gamma}(\mathbf{f}) = \sum_{d=1}^{D} w_d \log\left(1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\right)\,. \tag{2.10}$$

Our goal is to derive estimators for $\hat{\gamma}(\mathbf{f})$ that depend only on $\hat{\mu}(\mathbf{f})$ and $\hat{\xi}(\mathbf{f})$.

Portfolios with no Risk-Free Assets (Quadratic Estimator)

A strategy introduced by Markowitz in his 1959 book is to estimate $\hat{\gamma}(\mathbf{f})$ by using the *second-order Taylor approximation* of $\log(1+r)$ for small r. This approximation is

$$\log(1+r) \approx r - \tfrac{1}{2}r^2 \,. \tag{2.11}$$

When it is used in (2.10) we obtain the *quadratic estimator*

$$\begin{aligned}
\hat{\gamma}_{\mathrm{q}}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \left(\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\right)^2 \right) \\
&= \left(\sum_{d=1}^{D} w_d \, \mathbf{r}(d) \right)^{\mathrm{T}} \mathbf{f} - \tfrac{1}{2}\, \mathbf{f}^{\mathrm{T}} \left(\sum_{d=1}^{D} w_d \, \mathbf{r}(d)\, \mathbf{r}(d)^{\mathrm{T}} \right) \mathbf{f} \\
&= \mathbf{m}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\mathbf{f}^{\mathrm{T}} \left(\mathbf{m}\mathbf{m}^{\mathrm{T}} + \mathbf{V} \right) \mathbf{f} \\
&= \mathbf{m}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mathbf{m}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f} \,.
\end{aligned} \tag{2.12}$$

Portfolios with no Risk-Free Assets (Taylor Table)

Table 1. The first three Taylor polynomial approximations to $\log(1+r)$.

r	$r - \frac{1}{2}r^2$	$r - \frac{1}{2}r^2 + \frac{1}{3}r^3$	$\log(1+r)$
$-.5$	$-.62500$	$-.66667$	$-.69315$
$-.4$	$-.48000$	$-.50133$	$-.51083$
$-.3$	$-.34500$	$-.35400$	$-.35667$
$-.2$	$-.22000$	$-.22267$	$-.22314$
$-.1$	$-.10500$	$-.10533$	$-.10536$
$.0$	$.00000$	$.00000$	$.00000$
$.1$	$.09500$	$.09533$	$.09531$
$.2$	$.18000$	$.18267$	$.18232$
$.3$	$.25500$	$.26400$	$.26236$
$.4$	$.32000$	$.34133$	$.33647$
$.5$	$.37500$	$.41667$	$.40547$

Portfolios with no Risk-Free Assets (Second-Order Taylor)

Table 1 shows that the second-order Taylor approximation to $\log(1+r)$ is within 1.5% when $r \in (-\frac{1}{5}, \frac{1}{4})$, within 7.5% when $r \in (-\frac{1}{3}, \frac{1}{2})$, and within 10% when $r \in (-\frac{1}{2}, 1)$. It is worse outside of these intervals.

Remark. This observation suggests that the quadratic estimator $\hat{\gamma}_{\mathrm{q}}(\mathbf{f})$ given by (2.12) might only be trusted when the class of portfolio allocations being considered lies within

$$\left\{\mathbf{f} \in \mathcal{M} \; : \; -\tfrac{1}{3} \leq \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \leq \tfrac{1}{2} \quad \forall d\right\}.$$

Such a restriction is usually satisfied by $\mathbf{f} \in \Lambda$, but is often not satisfied by highly leveraged portfolios. Careful investors who are highly leveraged do not use the quadratic estimator.

Portfolios with no Risk-Free Assets (Taylor Bounds)

Remark. The first-order and third-order Taylor approximations are upper bounds because

$$\log(1+r) \leq r \qquad \text{for every } r > -1\,,$$
$$\log(1+r) \leq r - \tfrac{1}{2}r^2 + \tfrac{1}{3}r^3 \qquad \text{for every } r > -1\,.$$

The second-order Taylor approximation is an upper bound for $r \in (-1, 0)$ and a lower bound for $r > 0$ because

$$\log(1+r) < r - \tfrac{1}{2}r^2 \qquad \text{for every } r \in (-1, 0)\,,$$
$$\log(1+r) > r - \tfrac{1}{2}r^2 \qquad \text{for every } r > 0\,.$$

These facts are reflected by the values given in **Table 1**.

Portfolios with no Risk-Free Assets (Quadratic Estimator)

The *quadratic estimator* (2.12) is

$$\hat{\gamma}_{\mathrm{q}}(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mathbf{m}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}\,. \tag{2.13a}$$

This has the mean-variance form (1.6a) with

$$G_{\mathrm{q}}(\sigma,\mu) = \mu - \tfrac{1}{2}\,\mu^2 - \tfrac{1}{2}\,\sigma^2\,, \tag{2.13b}$$

which has all the properties (1.6b) over the set

$$\Sigma_{\mathrm{q}} = \Big\{(\sigma,\mu) \,:\, \sigma \geq 0\,,\ \mu \leq 1\Big\}\,. \tag{2.13c}$$

It does not satisfy the Jensen inequality bound (1.7).

Portfolios with no Risk-Free Assets (Parabolic Estimator)

Because it is often the case that

$$\left(\mathbf{m}^{\mathrm{T}}\mathbf{f}\right)^2 \quad \text{is much smaller than} \quad \mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}\,,$$

it is tempting to drop the $(\mathbf{m}^{\mathrm{T}}\mathbf{f})^2$ term in (2.13). This leads to the *parabolic estimator*

$$\hat{\gamma}_{\mathrm{p}}(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}\,. \tag{2.14a}$$

This has the mean-variance form (1.6a) with

$$G_{\mathrm{p}}(\sigma,\mu) = \mu - \tfrac{1}{2}\,\sigma^2\,, \tag{2.14b}$$

which has all the properties (1.6b) over the set

$$\Sigma_{\mathrm{p}} = \Big\{(\sigma,\mu)\,:\,\sigma \geq 0\Big\}\,. \tag{2.14c}$$

Remark. While this estimator is commonly used, there are many times when it is not good. It is particularly bad in a bubble. We will see that using it can lead to overbetting at times when overbetting is very risky.

Portfolios with no Risk-Free Assets (Taylor Estimator)

Another strategy is to use the *second-order Taylor approximation of $\log(1 + r)$ centered at $\hat{\mu}$* to estimate $\hat{\gamma}(\mathbf{f})$. Because $\hat{\mu} = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, this approximation is

$$\log(1+r) \approx \log\left(1+\mathbf{m}^{\mathrm{T}}\mathbf{f}\right) + \frac{(\mathbf{r}(d)-\mathbf{m})^{\mathrm{T}}\mathbf{f}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}} - \frac{1}{2}\,\frac{\left((\mathbf{r}(d)-\mathbf{m})^{\mathrm{T}}\mathbf{f}\right)^2}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2}\,.$$

When this approximation is used in (2.10) we obtain the *Taylor estimator*

$$\hat{\gamma}_{\mathrm{t}}(\mathbf{f}) = \log\left(1+\mathbf{m}^{\mathrm{T}}\mathbf{f}\right) - \frac{1}{2}\,\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2}\,, \tag{2.15a}$$

which is defined over the set

$$\Omega_{\mathrm{t}} = \left\{\mathbf{f} \in \mathbb{R}^N \;:\; 1+\mathbf{m}^{\mathrm{T}}\mathbf{f} > 0\right\}. \tag{2.15b}$$

This set contains Ω, the set of allocations for solvent Markowitz portfolios without risk-free assets.

Portfolios with no Risk-Free Assets (Taylor Estimator)

The *Taylor estimator* (2.15a) is

$$\hat{\gamma}_t(\mathbf{f}) = \log\left(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f})^2}\,. \tag{2.16a}$$

This has the mean-variance form (1.6a) with

$$G_t(\sigma, \mu) = \log(1 + \mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1 + \mu)^2}\,, \tag{2.16b}$$

which has all the properties (1.6b) over the set

$$\Sigma_t = \left\{(\sigma, \mu)\;:\; 1 + \mu > 0\,,\; 1 + \mu \geq \sigma \geq 0\right\}. \tag{2.16c}$$

It satisfies the Jensen inequality bound (1.7).

Portfolios with no Risk-Free Assets (Taylor Estimator)

The Taylor estimator $\hat{\gamma}_t(\mathbf{f})$ given by (2.15a) is not concave over the set Ω_t given by (2.15a) on which it is defined. Moreover, it does not generally have a maximum over Ω_t. This makes it a poor replacement for $\hat{\gamma}(\mathbf{f})$ as an objective function over the entire set Ω, which is contained within Ω_t.

The Hessian of $\hat{\gamma}_t(\mathbf{f})$ over Ω_t is

$$\nabla_{\mathbf{f}}^2 \hat{\gamma}_t(\mathbf{f}) = -\frac{\mathbf{m}\,\mathbf{m}^{\mathrm{T}} + \mathbf{V}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2} + 2\frac{\mathbf{V}\,\mathbf{f}\,\mathbf{m}^{\mathrm{T}} + \mathbf{m}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^3} - 3\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}\,\mathbf{m}\,\mathbf{m}^{\mathrm{T}}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^4}\,.$$

Portfolios with no Risk-Free Assets (Taylor Hessian)

This can be expressed as

$$
\begin{aligned}
\nabla_{\mathbf{f}}^2 \hat{\gamma}_t(\mathbf{f}) &= -\left(1 - \frac{\mathbf{f}^{\mathrm{T}} \mathbf{V}\, \mathbf{f}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^2}\right) \frac{\mathbf{m}\, \mathbf{m}^{\mathrm{T}}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^2} \\
&\quad - \frac{\mathbf{V}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^2} + 2\, \frac{\mathbf{V}\, \mathbf{f}\, \mathbf{m}^{\mathrm{T}} + \mathbf{m}\, \mathbf{f}^{\mathrm{T}} \mathbf{V}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^3} - 4\, \frac{\mathbf{f}^{\mathrm{T}} \mathbf{V}\, \mathbf{f}\, \mathbf{m}\, \mathbf{m}^{\mathrm{T}}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^4} \,. \\
&= -\left(1 - \frac{\mathbf{f}^{\mathrm{T}} \mathbf{V}\, \mathbf{f}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^2}\right) \frac{\mathbf{m}\, \mathbf{m}^{\mathrm{T}}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^2} \\
&\quad - \left(\mathbf{I} - \frac{2\, \mathbf{f}\, \mathbf{m}^{\mathrm{T}}}{1 + \mathbf{m}^{\mathrm{T}} \mathbf{f}}\right)^{\mathrm{T}} \frac{\mathbf{V}}{(1 + \mathbf{m}^{\mathrm{T}} \mathbf{f})^2} \left(\mathbf{I} - \frac{2\, \mathbf{f}\, \mathbf{m}^{\mathrm{T}}}{1 + \mathbf{m}^{\mathrm{T}} \mathbf{f}}\right) .
\end{aligned}
\tag{2.17}
$$

This is clearly nonpositive definite over the set

$$
\Pi_t = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \sqrt{\mathbf{f}^{\mathrm{T}} \mathbf{V}\, \mathbf{f}} \leq 1 + \mathbf{m}^{\mathrm{T}} \mathbf{f} \right\} . \tag{2.18}
$$

Portfolios with no Risk-Free Assets (Taylor Hessian)

Fact 1. If $\mathbf{V}$ is positive definite then the Hessian $\nabla_{\mathbf{f}}^2 \hat{\gamma}_{\mathrm{t}}(\mathbf{f})$ is negative definite for some $\mathbf{f} \in \Pi_{\mathrm{t}}$ if and only if either

$$\mathbf{m}^{\mathrm{T}}\mathbf{f} \neq 1 \qquad \text{or} \qquad \mathbf{f} \text{ is in the interior of } \Pi_{\mathrm{t}} \,.$$

Proof. Let $\mathbf{f} \in \Pi_{\mathrm{t}}$. From (2.17) and (2.18) we see for any $\mathbf{y} \in \mathbb{R}^N$ that

$$\mathbf{y}^{\mathrm{T}} \nabla_{\mathbf{f}}^2 \hat{\gamma}_{\mathrm{t}}(\mathbf{f})\, \mathbf{y} = 0$$

if and only if

$$\left(1 - \frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2}\right)\left(\mathbf{m}^{\mathrm{T}}\mathbf{y}\right)^2 = 0\,, \tag{2.19a}$$

and

$$\mathbf{y}^{\mathrm{T}}\left(\mathbf{I} - \frac{2\,\mathbf{f}\,\mathbf{m}^{\mathrm{T}}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}}\right)^{\mathrm{T}} \mathbf{V} \left(\mathbf{I} - \frac{2\,\mathbf{f}\,\mathbf{m}^{\mathrm{T}}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}}\right)\mathbf{y} = 0\,. \tag{2.19b}$$

Portfolios with no Risk-Free Assets (Taylor Hessian)

If $\mathbf{f}$ satisfies $\mathbf{m}^{\mathrm{T}}\mathbf{f} \neq 1$ then

$$\left(\mathbf{I} - \frac{2\,\mathbf{f}\,\mathbf{m}^{\mathrm{T}}}{1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}}\right)^{-1} = \mathbf{I} + \frac{2\,\mathbf{f}\,\mathbf{m}^{\mathrm{T}}}{1 - \mathbf{m}^{\mathrm{T}}\mathbf{f}}\,,$$

whereby (2.19b) alone implies that $\mathbf{y} = \mathbf{0}$.

If $\mathbf{f}$ is in the interior of Π_{t} then we see from (2.18) that

$$1 - \frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f})^2} > 0\,,$$

whereby (2.19a) implies that $\mathbf{m}^{\mathrm{T}}\mathbf{y} = 0$, which then implies that (2.19b) reduces to $\mathbf{y}^{\mathrm{T}}\mathbf{V}\,\mathbf{y} = 0$, which implies that $\mathbf{y} = \mathbf{0}$.

Therefore the Hessian is negative definite because

$$\mathbf{y}^{\mathrm{T}}\nabla_{\mathbf{f}}^2\hat{\gamma}_{\mathrm{t}}(\mathbf{f})\,\mathbf{y} = 0 \quad \Longrightarrow \quad \mathbf{y} = \mathbf{0}\,.$$

Portfolios with no Risk-Free Assets (Taylor Hessian)

Conversely, suppose that

$$\mathbf{m}^{\mathrm{T}}\mathbf{f} = 1 \qquad \text{and} \qquad \mathbf{f} \text{ is not in the interior of } \Pi_{\mathrm{t}} \,.$$

Because $\mathbf{f}$ is not in the interior of Π_{t}, we see from (2.18) that

$$1 - \frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f})^2} = 0 \,,$$

whereby (2.19a) is satisfied for every $\mathbf{y} \in \mathbb{R}^N$.

Because $\mathbf{m}^{\mathrm{T}}\mathbf{f} = 1$, we know $\mathbf{f} \neq \mathbf{0}$. Moreover, (2.19b) is satisfied by $\mathbf{y} = \mathbf{f}$. Because both (2.19a) and (2.19b) are satisfied, we have shown that

$$\mathbf{f}^{\mathrm{T}}\nabla_{\mathbf{f}}^2\hat{\gamma}_{\mathrm{t}}(\mathbf{f})\,\mathbf{f} = 0 \quad \text{and} \quad \mathbf{f} \neq \mathbf{0} \,,$$

whereby the Hessian is not negative definite.

Both directions of **Fact 1** have now been proved.

Portfolios with no Risk-Free Assets (Taylor Domain)

Remark. The set Π_t defined by (2.18) is convex. Indeed, if $\mathbf{f}_0$, $\mathbf{f}_1 \in \Pi_t$ and we set $\mathbf{f}_t = (1-t)\mathbf{f}_0 + t\mathbf{f}_1$ then for every $t \in (0,1)$ then we have

$$
\begin{aligned}
\mathbf{f}_t^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_t &= (1-t)^2\mathbf{f}_0^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_0 + 2t(1-t)\mathbf{f}_0^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_1 + t^2\mathbf{f}_1^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_1 \\
&\leq (1-t)^2\mathbf{f}_0^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_0 + 2t(1-t)\sqrt{\mathbf{f}_0^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_0}\sqrt{\mathbf{f}_1^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_1} + t^2\mathbf{f}_1^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_1 \\
&= \left((1-t)\sqrt{\mathbf{f}_0^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_0} + t\,\sqrt{\mathbf{f}_1^{\mathrm{T}}\mathbf{V}\,\mathbf{f}_1}\right)^2 \\
&\leq \Big((1-t)(1+\mathbf{m}^{\mathrm{T}}\mathbf{f}_0) + t\,(1+\mathbf{m}^{\mathrm{T}}\mathbf{f}_1)\Big)^2 = (1+\mathbf{m}^{\mathrm{T}}\mathbf{f}_t)^2\,.
\end{aligned}
$$

The constraint for the set Π_t can be expressed as a linear constraint $0 < 1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}$ plus a quadratic constraint $\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f} \leq (1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2$. This quadratic constraint can become nondefinite and thereby harder to use.

Portfolios with no Risk-Free Assets (Reasonable Estimator)

We now introduce an estimator with better properties that uses the first term from the Taylor estimator (2.15a) and the volatility term from the quadratic estimator (2.12). This leads to the *reasonable estimator*

$$\hat{\gamma}_{\mathrm{r}}(\mathbf{f}) = \log\left(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}\,, \tag{2.20a}$$

which is also defined over the set $\Omega_{\mathrm{r}} = \Omega_{\mathrm{t}}$ given by (2.15b). This has the mean-variance form (1.6a) with

$$G_{\mathrm{r}}(\sigma, \mu) = \log(1 + \mu) - \tfrac{1}{2}\,\sigma^2\,, \tag{2.20b}$$

which has all the properties (1.6b) over the set

$$\Sigma_{\mathrm{r}} = \Big\{(\sigma, \mu) \,:\, \sigma \geq 0\,,\; 1 + \mu > 0\Big\}\,. \tag{2.20c}$$

It satisfies the Jensen inequality bound (1.7).

Portfolios with no Risk-Free Assets (Sensible Estimator)

Another growth rate mean estimator with good properties can be obtained by a different modification of (2.15) — namely, the *sensible estimator*

$$\hat{\gamma}_{\mathrm{s}}(\mathbf{f}) = \log\left(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}}\,, \tag{2.21a}$$

which is also defined over the set $\Omega_{\mathrm{s}} = \Omega_{\mathrm{t}}$ given by (2.15b). This has the mean-variance form (1.6a) with

$$G_{\mathrm{s}}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+\mu}\,, \tag{2.21b}$$

which has all the properties (1.6b) over the set

$$\Sigma_{\mathrm{s}} = \left\{(\sigma,\mu)\,:\,\sigma \geq 0\,,\,1+\mu > 0\right\}. \tag{2.21c}$$

It satisfies the Jensen inequality bound (1.7).

Portfolios with no Risk-Free Assets (Sensible Estimator)

Fact 2. $\hat{\gamma}_{\mathrm{s}}(\mathbf{f})$ is strictly concave over the set Ω_{s}.

Proof of Fact 2. We will show that $\hat{\gamma}_{\mathrm{s}}(\mathbf{f})$ is the sum of two functions, one of which is concave and the other of which is strictly concave over Ω_{s}.

The function $\log\left(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\right)$ is infinitely differentiable over Ω_{s} with

$$\nabla_{\mathbf{f}} \log\left(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\right) = \frac{\mathbf{m}}{1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}},$$

$$\nabla_{\mathbf{f}}^2 \log\left(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\right) = -\frac{\mathbf{m}\,\mathbf{m}^{\mathrm{T}}}{(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f})^2}.$$

Because its Hessian is nonpositive definite, the function $\log(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f})$ is concave over Ω_{s}.

Portfolios with no Risk-Free Assets (Sensible Estimator)

The harder part of the proof of **Fact 2** is to show that

$$\frac{1}{2}\,\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}} \quad \text{is strictly convex over } \Omega_{\mathrm{s}}\,. \tag{2.22}$$

This function is infinitely differentiable over Ω_{s} with

$$\nabla_{\mathbf{f}}\left(\frac{1}{2}\,\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}}\right) = \frac{\mathbf{V}\,\mathbf{f}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}} - \frac{1}{2}\,\frac{(\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f})\,\mathbf{m}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2}$$

$$\nabla_{\mathbf{f}}^2\left(\frac{1}{2}\,\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}}\right) = \frac{\mathbf{V}}{1+\mathbf{m}^{\mathrm{T}}\mathbf{f}} - \frac{\mathbf{V}\,\mathbf{f}\,\mathbf{m}^{\mathrm{T}} + \mathbf{m}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^2} + \frac{(\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f})\,\mathbf{m}\,\mathbf{m}^{\mathrm{T}}}{(1+\mathbf{m}^{\mathrm{T}}\mathbf{f})^3}\,.$$

We will show this Hessian is negative definite by using two general facts that we will state and prove before finishing the proof of **Fact 2**.

Portfolios with no Risk-Free Assets (Sensible Estimator)

Fact 3. Let $\mathbf{b}$, $\mathbf{x} \in \mathbb{R}^N$ such that $1 + \mathbf{b}^{\mathrm{T}}\mathbf{x} > 0$. Then $\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}}$ is invertible with

$$\left(\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}}\right)^{-1} = \mathbf{I} - \frac{\mathbf{x}\,\mathbf{b}^{\mathrm{T}}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}}\,. \tag{2.23}$$

Proof of Fact 3. Just check that

$$\begin{aligned}
\left(\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}}\right)\left(\mathbf{I} - \frac{\mathbf{x}\,\mathbf{b}^{\mathrm{T}}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}}\right) &= \left(\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}}\right) - \frac{\left(\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}}\right)\mathbf{x}\,\mathbf{b}^{\mathrm{T}}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}} \\
&= \mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}} - \frac{\mathbf{x}\,\mathbf{b}^{\mathrm{T}} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}}\mathbf{x}\,\mathbf{b}^{\mathrm{T}}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}} \\
&= \mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}} - \frac{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}}\,\mathbf{x}\,\mathbf{b}^{\mathrm{T}} = \mathbf{I}\,.
\end{aligned}$$

The assertions of **Fact 3** then follow. □

Portfolios with no Risk-Free Assets (Sensible Estimator)

Fact 4. Let $\mathbf{A} \in \mathbb{R}^{N\times N}$ be symmetric and positive definite. Let $\mathbf{b} \in \mathbb{R}^N$. Let $\mathbb{X}$ be the half-space given by

$$\mathbb{X} = \left\{ \mathbf{x} \in \mathbb{R}^N \;:\; 1 + \mathbf{b}^{\mathrm{T}}\mathbf{x} > 0 \right\}.$$

Then

$$\phi(\mathbf{x}) = \tfrac{1}{2}\,\frac{\mathbf{x}^{\mathrm{T}}\mathbf{A}\,\mathbf{x}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}} \quad \text{is strictly convex over } \mathbb{X}\,.$$

Proof of Fact 4. The function $\phi(\mathbf{x})$ is infinitely differentiable over $\mathbb{X}$ with

$$\nabla_{\mathbf{x}}\,\phi(\mathbf{x}) = \frac{\mathbf{A}\,\mathbf{x}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}} - \tfrac{1}{2}\,\frac{(\mathbf{x}^{\mathrm{T}}\mathbf{A}\,\mathbf{x})\,\mathbf{b}}{(1 + \mathbf{b}^{\mathrm{T}}\mathbf{x})^2}\,,$$

$$\nabla_{\mathbf{x}}^2\phi(\mathbf{x}) = \frac{\mathbf{A}}{1 + \mathbf{b}^{\mathrm{T}}\mathbf{x}} - \frac{\mathbf{A}\,\mathbf{x}\,\mathbf{b}^{\mathrm{T}} + \mathbf{b}\,\mathbf{x}^{\mathrm{T}}\mathbf{A}}{(1 + \mathbf{b}^{\mathrm{T}}\mathbf{x})^2} + \frac{(\mathbf{x}^{\mathrm{T}}\mathbf{A}\,\mathbf{x})\,\mathbf{b}\,\mathbf{b}^{\mathrm{T}}}{(1 + \mathbf{b}^{\mathrm{T}}\mathbf{x})^3}\,.$$

Portfolios with no Risk-Free Assets (Sensible Estimator)

Then, by using (2.23) of **Fact 3**, the Hessian can be expressed as

$$\nabla_{\mathbf{x}}^2 \phi(\mathbf{x}) = \left(\mathbf{I} - \frac{\mathbf{b}\,\mathbf{x}^{\mathrm{T}}}{1+\mathbf{b}^{\mathrm{T}}\mathbf{x}} \right) \frac{\mathbf{A}}{1+\mathbf{b}^{\mathrm{T}}\mathbf{x}} \left(\mathbf{I} - \frac{\mathbf{x}\,\mathbf{b}^{\mathrm{T}}}{1+\mathbf{b}^{\mathrm{T}}\mathbf{x}} \right)$$
$$= \left(\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}} \right)^{-\mathrm{T}} \frac{\mathbf{A}}{1+\mathbf{b}^{\mathrm{T}}\mathbf{x}} \left(\mathbf{I} + \mathbf{x}\,\mathbf{b}^{\mathrm{T}} \right)^{-1} .$$

Because $\mathbf{A}$ is positive definite and $1 + \mathbf{b}^{\mathrm{T}}\mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{X}$, this shows that $\nabla_{\mathbf{x}}^2 \phi(\mathbf{x})$ is positive definite for every $\mathbf{x} \in \mathbb{X}$. Therefore $\phi(\mathbf{x})$ is strictly convex over $\mathbb{X}$, thereby proving **Fact 4**. □

Proof of Fact 2 Finish. By setting $\mathbf{A} = \mathbf{V}$ and $\mathbf{b} = \mathbf{m}$ in **Fact 4** and using the fact that the negative of a strictly convex function is strictly concave, we establish (2.23), thereby finishing the proof of **Fact 2**. □

Portfolios with no Risk-Free Assets (Sensible Estimator)

Remark. By setting $\mathbf{A} = \alpha \mathbf{V}$ and $\mathbf{b} = \beta\, \mathbf{m}$ in **Fact 4** for any positve α and β and using the fact that the negative of a strictly convex function is strictly concave, the sensible estimator (2.21a) can be extended to a family in the form

$$\hat{\gamma}_{\alpha\beta}(\mathbf{f}) = \log\Big(1 + \mathbf{m}^{\mathrm{T}}\mathbf{f}\Big) - \tfrac{1}{2}\,\frac{\alpha\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{1 + \beta\,\mathbf{m}^{\mathrm{T}}\mathbf{f}}\,.$$

This has the mean-variance form (1.6a) with

$$G_{\alpha\beta}(\sigma, \mu) = \log(1 + \mu) - \tfrac{1}{2}\,\frac{\alpha\,\sigma^2}{1 + \beta\,\mu}\,,$$

which has all the properties (1.6b) over the set

$$\Sigma_{\alpha\beta} = \Big\{(\sigma, \mu)\,:\,\sigma \geq 0\,,\ 1 + \mu > 0\,,\ 1 + \beta\,\mu > 0\Big\}\,.$$

It satisfies the Jensen inequality bound (1.7).

Portfolios with Risk-Free Assets

We now extend the estimators derived in the last section to solvent Markowitz portfolios with risk-free assets. Specifically, for a portfolio with risk-free return r_{rf} and risky asset allocation $\mathbf{f}$ we will use the sample estimator $\hat{\gamma}$ to derive new estimators of γ in terms of the return mean and volatility estimators given by

$$\hat{\mu} = r_{\mathrm{rf}} + \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \hat{\sigma} = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}\,, \tag{3.24}$$

where $\mathbf{m}$ and $\mathbf{V}$ are given by

$$\mathbf{m} = \sum_{d=1}^{D} w_d \mathbf{r}(d)\,, \qquad \mathbf{V} = \sum_{d=1}^{D} w_d \left(\mathbf{r}(d) - \mathbf{m}\right)\left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathrm{T}}. \tag{3.25}$$

These new return mean-variance estimators of γ will allow us to work within the framework of Markowitz portfolio theory.

Portfolios with Risk-Free Assets

The portfolio with risk-free return r_{rf} and risky asset allocation $\mathbf{f}$ has the return history $\{r(d)\}_{d=1}^{D}$ with

$$r(d) = \hat{\mu}(\mathbf{f}) + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f} \,,$$

where $\tilde{\mathbf{r}}(d) = \mathbf{r}(d) - \mathbf{m}$. In words, $\tilde{\mathbf{r}}(d)$ is the deviation of $\mathbf{r}(d)$ from its sample mean $\mathbf{m}$. Then we can write

$$\begin{aligned} \log(1 + r(d)) = \log(1 + \hat{\mu}) + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}} \\ - \left(\frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}} - \log\left(1 + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}}\right)\right) \,. \end{aligned} \tag{3.26}$$

Notice that the last term on the first line has sample mean zero while the concavity of the function $r \mapsto \log(1 + r)$ implies that $r - \log(1 + r) \geq 0$, which implies that the term on the second line is nonpositive.

Portfolios with Risk-Free Assets

Therefore by taking the sample mean of (3.26) we obtain

$$\begin{aligned}\hat{\gamma} &= \sum_{d=1}^{D} w_d \log(1 + r(d)) \\ &= \log(1+\hat{\mu}) - \sum_{d=1}^{D} w_d \left(\frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1+\hat{\mu}} - \log\left(1 + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1+\hat{\mu}}\right)\right) .\end{aligned} \tag{3.27}$$

The last sum will be positive whenever $\mathbf{f} \neq \mathbf{0}$ and $\mathbf{V}$ is positive definite.

Remark. By dropping the last term in the foregoing calculation we get an alternative proof of the Jensen inequality bound, which was originally proved using the Jensen inequality. Indeed, (3.27) can be viewed as an improvement upon that bound.

Portfolios with Risk-Free Assets

We can estimate $\hat{\gamma}$ using the second-order Taylor approximation of $\log(1+r)$ for small r. This approximation is

$$\log(1+r) \approx r - \tfrac{1}{2}r^2 \,. \tag{3.28}$$

When this approximation is used inside the sum of (3.27) we obtain

$$\hat{\gamma} \approx \log(1+\hat{\mu}) - \tfrac{1}{2}\sum_{d=1}^{D} w_d \left(\frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1+\hat{\mu}}\right)^2 .$$

This leads to the *Taylor estimator*

$$\hat{\gamma}_{\mathrm{t}} = \log(1+\hat{\mu}) - \tfrac{1}{2}\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}}{(1+\hat{\mu})^2} \,. \tag{3.29}$$

While this estimator is defined over $1+\hat{\mu} > 0$, it is strictly concave where in addition

$$\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \leq 1+\hat{\mu} \,. \tag{3.30}$$

Portfolios with Risk-Free Assets

We can derive other estimators from the Taylor estimator.

The analog of the sensible estimator (2.21a) is

$$\hat{\gamma}_{\rm s} = \log(1+\hat{\mu}) - \frac{1}{2}\,\frac{\mathbf{f}^{\rm T}\mathbf{V}\,\mathbf{f}}{1+\hat{\mu}} \quad \text{over} \quad 1+\hat{\mu} > 0\,. \tag{3.31a}$$

The analog of the reasonable estimator (2.20a) is

$$\hat{\gamma}_{\rm r} = \log(1+\hat{\mu}) - \tfrac{1}{2}\,\mathbf{f}^{\rm T}\mathbf{V}\,\mathbf{f} \qquad \text{over} \quad 1+\hat{\mu} > 0\,. \tag{3.31b}$$

The analog of the quadratic estimator (2.13a) is

$$\hat{\gamma}_{\rm q} = \hat{\mu} - \tfrac{1}{2}\,\hat{\mu}^2 - \tfrac{1}{2}\,\mathbf{f}^{\rm T}\mathbf{V}\,\mathbf{f} \qquad \text{over} \quad \hat{\mu} \leq 1\,. \tag{3.31c}$$

The analog of the parabolic estimator (2.14a) is

$$\hat{\gamma}_{\rm p} = \hat{\mu} - \tfrac{1}{2}\,\mathbf{f}^{\rm T}\mathbf{V}\,\mathbf{f}\,. \tag{3.31d}$$

Portfolios with Risk-Free Assets

The derivations of these estimators each assume that $|\hat{\mu}| \ll 1$.

- The sensible estimator (3.31a) derives from the Taylor estimator (3.29) by replacing the $(1+\hat{\mu})^2$ in the denominator under $\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}$ with $1+\hat{\mu}$.
- The reasonable estimator (3.31b) derives from the sensible estimator (3.31a) by dropping the $\hat{\mu}$ term in the denominator under $\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}$.
- The quadratic estimator (3.31c) derives from the reasonable estimator (3.31b) by replacing $\log(1+\hat{\mu})$ with its second-order Taylor polynomial approximation $\hat{\mu} - \frac{1}{2}\,\hat{\mu}^2$. The result is an increasing function of $\hat{\mu}$ when $\hat{\mu} \leq 1$.
- The parabolic estimator (3.31d) derives from the quadratic estimator (3.31c) by making the additional assumption that $\hat{\mu}^2 \ll \mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}$ and dropping the $\hat{\mu}^2$ term.

Portfolios with Risk-Free Assets

The parabolic, quadratic, reasonable, sensible and Taylor estimators each have the mean-variance form (1.6a) with $G(\sigma,\mu)$ and Σ given by

$$G_{\rm p}(\sigma,\mu) = \mu - \tfrac{1}{2}\,\sigma^2 \qquad \text{over } \sigma \geq 0\,; \tag{3.32a}$$

$$G_{\rm q}(\sigma,\mu) = \mu - \tfrac{1}{2}\,\mu^2 - \tfrac{1}{2}\,\sigma^2 \qquad \text{over } \sigma \geq 0,\ \mu \leq 1\,; \tag{3.32b}$$

$$G_{\rm r}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\sigma^2 \qquad \text{over } \sigma \geq 0,\ 1+\mu > 0\,; \tag{3.32c}$$

$$G_{\rm s}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+\mu} \qquad \text{over } \sigma \geq 0,\ 1+\mu > 0\,; \tag{3.32d}$$

$$G_{\rm t}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} \qquad \text{over } \begin{cases} 1+\mu \geq \sigma \geq 0\,, \\ 1+\mu > 0\,. \end{cases} \tag{3.32e}$$

The properties (1.6b) hold over the given domain Σ for each estimator.

Portfolios with Risk-Free Assets

The functions $G(\sigma, \mu)$ and domains Σ given by (3.32) are identical to those that arose for the case with no risk-free assets.

The reasonable, sensible and Taylor estimators given by (3.32c), (3.32d), and (3.32e) respectively each satisfy the Jensen inequality bound (1.7).

The quadratic and parabolic estimators given by (3.32a) and (3.32b) respectively each satisfy analogs of the Jensen inequality bound obtained by replacing the $\log(1 + \hat{\mu})$ in it by an appropriate Taylor approximation.

The parabolic, quadratic, reasonable and sensible estimators given by (3.32a), (3.32b), (3.32c) and (3.32d) respectively are each strictly concave functions with a unique global maximizer over the allocation domain over which it is defined.

The Taylor estimator (3.32e) is strictly concave with a unique maximizer over the allocation domain that satisfies the additional condiition (3.30).