Portfolios that Contain Risky Assets: 8.3. Kelly Criterion for IID Models

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Kelly Criterion for IID Models

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Introduction

Our general approach to portfolio management will be to select an allocation **f** that maximizes some objective function. The *Law of Large Numbers* for IID models suggests that we might want to pick **f** to maximize γ. However, a difficulty with using this strategy is that we do not know γ. Rather, we will develop strategies that maximize one of a family objective functions that are built from $\hat{\gamma}$ and $\hat{\theta}$.

In 1956 John Kelly, a colleague of Claude Shannon at Bell Labs, used the Law of Large Numbers to devise optimal betting strategies for a class of games of chance. A strategy that tries to maximize γ became known as the *Kelly criterion, Kelly strategy*, or *Kelly bet*. In practice they had to employ modifications of the Kelly criterion. Such strategies were subsequently adopted by Claude Shannon, Ed Thorp, and others to win at blackjack, roullette, and other casino games. These exploits are documented in Ed Thorpe's 1962 book *Beat the Dealer*.

Introduction

Because many casinos were controlled by organized crime at that time, using these strategies could adversely affect the user's health. Claude Shannon, Ed Thorp, and others soon realized that it was better for both their health and their wealth to apply the Kelly criterion to winning on Wall Street. Ed Thorpe laid out a strategy to do this in his 1967 book *Beat the Market*.

He went on to run the first quantiative hedge fund, Princeton Newport Partners, which introduced statistical arbitrage strategies to Wall Street. This history is told in Scott Peterson's 2010 book *The Quants* and in Ed Thorp's 2017 book *A Man for All Markets.*

Here we introduce these ideas in the setting of a simple betting game.

Kelly Criterion for a Simple Game (The Game)

Before showing how the Kelly criterion is applied to balancing portfolios with risky assets, we will show how it is applied to a simple betting game.

Consider a game in which each time that we place a bet:

(i) the probability of winning is $p \in (0, 1)$,

(ii) the probability of losing is $q = 1 - p$,

(iii) when we win there is a positive return r on our bet.

We start with a bankroll of cash and the game ends when the bankroll is gone. Suppose that you know p and r. We would like answers to the following questions.

1. When should we play?,
2. When we do play, what fraction of our bankroll should we bet?,

Kelly Criterion for a Simple Game (One Outcome)

The game is clearly an IID process. Because each time we play we are faced with the same questions and will have no addtional helpful information, the answers will be the same each time. Therefore we only consider strategies in which we bet a fixed fraction f of our bankroll.

- If $f = 0$ then we are not betting.
- If $f = 1$ then we are betting our entire bankroll. (This is a foolish long term strategy because we will go broke the first time we lose.)
- If $f \in [0, 1)$ then
  - when we win our bankroll increases by a factor of $1 + fr$,
  - when we lose our bankroll decreases by a factor of $1 - f$.

Kelly Criterion for a Simple Game (Many Outcomes)

Therefore if we bet n times and win m times (hence, lose $n-m$ times) then our bankroll changes by a factor of

$$(1+fr)^m(1-f)^{n-m}\,.$$

The Kelly criterion is to pick $f \in [0,1)$ to maximize this factor for large n.

This is equivalent to maximizing the log of this factor, which is

$$\begin{aligned} m\log(1+fr) &+ (n-m)\log(1-f) \\ &= \left(\frac{m}{n}\log(1+fr) + \left(1-\frac{m}{n}\right)\log(1-f)\right)\, n\,. \end{aligned} \tag{2.1}$$

Kelly Criterion for a Simple Game (Kelly)

For the *binomial distribution* the probability of m wins in n trys is

$$\frac{n!}{m!\,(n-m)!}\,p^m\,(1-p)^{n-m}\,.$$

The mean and variance of this distribution are

$$\mathrm{Ex}(m) = n\,p\,, \qquad \mathrm{Vr}(m) = n\,p\,(1-p)\,.$$

For every $\delta > 0$ the Chebyshev inequality yields

$$\Pr\left\{\left|\frac{m}{n} - p\right| \geq \delta\right\} \leq \frac{p\,(1-p)}{n\,\delta^2}\,. \tag{2.2}$$

This gives the law of large numbers, that

$$\lim_{n\to\infty} \frac{m}{n} = p\,. \tag{2.3}$$

Kelly Criterion for a Simple Game (Kelly)

The law of large numbers (2.3) implies that for large n the quantity (2.1) can be approximated by

$$\begin{aligned} & m \log(1 + fr) + (n - m) \log(1 - f) \\ & \qquad \sim \Big(p \log(1 + fr) + (1 - p) \log(1 - f) \Big)\, n \,. \end{aligned}$$

The Kelly criterion says to pick $f \in [0, 1)$ to maximize the growth rate

$$\gamma(f) = p \log(1 + fr) + (1 - p) \log(1 - f) \,. \qquad (2.4)$$

This is now an exercise from first semester calculus.

Kelly Criterion for a Simple Game (Strict Concavity)

Notice from (2.4) that $\gamma(0) = 0$ and that

$$\lim_{f \nearrow 1} \gamma(f) = -\infty \,.$$

Also notice that for every $f \in [0, 1)$ we have

$$\gamma'(f) = \frac{pr}{1 + fr} - \frac{1 - p}{1 - f} \,,$$
$$\gamma''(f) = -\frac{pr^2}{(1 + fr)^2} - \frac{1 - p}{(1 - f)^2} \,.$$

Because $\gamma''(f) < 0$ over $[0, 1)$, we see that

- $\gamma(f)$ is strictly concave over $[0, 1)$ and
- $\gamma'(f)$ is strictly decreasing over $[0, 1)$.

Kelly Criterion for a Simple Game ($\gamma'(0) \leq 0$)

We will break our analysis into two cases:

- $\gamma'(0) \leq 0$,
- $\gamma'(0) > 0$.

Fact 1. If $\gamma'(0) = pr - (1-p) = p(1+r) - 1 \leq 0$ then

- the unique maximizer of $\gamma(f)$ over $[0,1)$ is $f = 0$ and
- the maximum of $\gamma(f)$ over $[0,1)$ is $\gamma(0) = 0$.

Proof. Because $\gamma'(f)$ is strictly decreasing over over $[0,1)$ while $\gamma'(0) \leq 0$, we see that

- $\gamma'(f) < 0$ over $(0,1)$ and
- $\gamma(f)$ is strictly deceasing over $[0,1)$.

Therefore the unique maximizer of $\gamma(f)$ over $[0,1)$ is $f = 0$ and the maximum of $\gamma(f)$ over $[0,1)$ is $\gamma(0) = 0$. □

Kelly Criterion for a Simple Game ($\gamma'(0) > 0$)

Fact 2. If $\gamma'(0) = pr - (1-p) = p(1+r) - 1 > 0$ then $\gamma(f)$ has a unique maximizer at $f = f_* \in (0,1)$ given by

$$f_* = \frac{p(1+r)-1}{r}\,. \tag{2.5}$$

Proof. Any critical point f_* of $\gamma(f)$ must satisfy

$$\begin{aligned} 0 = \gamma'(f_*) &= \frac{pr}{1+f_*r} - \frac{1-p}{1-f_*} \\ &= \frac{pr(1-f_*) - (1-p)(1+f_*r)}{(1+f_*r)(1-f_*)} = \frac{p(1+r) - 1 - f_*r}{(1+f_*r)(1-f_*)}\,. \end{aligned}$$

Upon solving this equation for f_* we obtain (2.5). The fact that f_* is a unique maximizer follows from the strict concavity of $\gamma(f)$ over $[0,1)$. □

Kelly Criterion for a Simple Game (Kelly Bet)

Remark. When $p(1+r)-1>0$ we see from (2.5) that f_* satisfies

$$0 < f_* = \frac{p(1+r)-1}{r} = p - \frac{1-p}{r} < p < 1\,.$$

Fact 1 and **Fact 2** yield the optimal *Kelly betting strategy*,

$$f_* = \begin{cases} 0 & \text{if } p(1+r)-1 \leq 0\,, \\ \dfrac{p(1+r)-1}{r} & \text{if } p(1+r)-1 > 0\,. \end{cases} \tag{2.6}$$

The maximum growth rate (details not shown) when $p(1+r)-1>0$ is

$$\gamma(f_*) = p\log\big(p(1+r)\big) + (1-p)\log\left((1-p)\,\frac{1+r}{r}\right) > 0\,. \tag{2.7}$$

Remark. In practice this strategy is far from ideal for reasons that we will discuss in the next section.

Kelly Criterion for a Simple Game (Edge over Odds)

Remark. Some bettors call r the *odds* because the return r on a winning wager is usually chosen so that the ratio $r : 1$ reflects a probability of winning. The expected return on each amount wagered is $pr - (1 - p)$. This is the probability of winning, p, times the return of a win, r, plus the probability of losing, $1 - p$, times the return of a loss, -1. Some bettors call this quantity the *edge* when it is postive. Notice that $pr - (1 - p) = p(1 + r) - 1$ is the numerator of f_* given by (2.5), while r is the denominator of f_* given by (2.5). Then strategy (2.6) can be expressed in this language as follows.

1. Do not bet unless we have an edge.
2. If we have an edge then bet $f_* = \dfrac{\text{edge}}{\text{odds}}$ of our bankroll.

This view of the Kelly criterion is popular, but is not very helpful when trying to apply it to more complicated games.

Kelly Criterion in Practice (Introduction)

In most betting games played at casinos the players do not have an edge unless they can use information that is not used by the house when computing the odds. For example, card counting strategies can allow a blackjack player to compute a more accurate probability of winning than the one used by the house when it computed the odds.

Kelly bettors will not make a serious wager until they are very sure that they have an edge, and then they will use the Kelly criterion to size their bet. Because their algorithm yields an approximation of their edge, they are not sure of their true Kelly optimal bet. Because there is a big downside to betting more that the true Kelly optimal bet, their bet is typically a fraction of the Kelly optimal bet.

Kelly Criterion in Practice (Another Game)

We will illustrate these ideas with a modification of the simple game from the last section. Specifically, suppose that the game is the same except for the fact that we are not told p. Rather, we are told that $r = 0.125$ and that the player won 225 times the last 250 times the game was played.

Based on the information that the player won 225 out of 250 times the game was played, we guess that $p = 0.9$. With this value of p we see that

$$p(1+r) - 1 = 0.9(1+0.125) - 1 = \tfrac{9}{10} \cdot \tfrac{9}{8} - 1 = \tfrac{1}{80} = 0.0125\,.$$

Based on this calculation, we have an edge, so we will play and the optimal bet is

$$f_* = \frac{p(1+r) - 1}{r} = \frac{\frac{1}{80}}{\frac{1}{8}} = \frac{1}{10} = 0.1\,.$$

Therefore the Kelly strategy is to bet $\frac{1}{10}$ of our bankroll each time and expect a growth rate of $\gamma(.1) \approx 0.0006442$.

Kelly Criterion in Practice (Ugly Outcome)

However, suppose that the previous players had just gotten lucky and that in fact $p = 0.875$. With this value of p we see that

$$p(1+r) - 1 = 0.875(1 + 0.125) - 1 = \tfrac{7}{8} \cdot \tfrac{9}{8} - 1 = -\tfrac{1}{64}\,.$$

Therefore we do not have an edge and we should not play!

The difference between 0.9 and 0.875 is not large in the sense that it is not an unreasonable error based on only 250 observations. For example, the Chebyshev inequality (2.2) with $n = 250$, $p = \frac{7}{8}$ and $\delta = \frac{1}{40}$ gives an upper bound on the probability of an error at least that large of only

$$\frac{p\,(1-p)}{n\,\delta^2} = \frac{\frac{7}{64}}{250\,\frac{1}{1600}} = \frac{7}{10} = 0.7\,.$$

If we bet $\frac{1}{10}$ of our bankroll each time then our bankroll will be greatly diminished before we have played enough to realize that there is no edge!

Kelly Criterion in Practice (Bad Outcome)

Now suppose that in fact $p = 0.895$. With this value of p we see that

$$p(1+r) - 1 = 0.895(1+0.125) - 1 = 0.006875\,,$$

whereby we have an edge. However, the optimal bet is

$$f_* = \frac{p(1+r)-1}{r} = \frac{0.006875}{0.125} = 0.055 = \tfrac{11}{200}\,.$$

Because

$$\gamma(0.1) \approx 0.0000553 \qquad \text{and} \qquad \gamma(0.055) \approx 0.0001922\,,$$

if we bet $\frac{1}{10}$ of our bankroll each time expecting a growth rate of about 0.0006442 then our bankroll will suffer slower growth rate of 0.0000553 before we have played enough to realize that p is lower than .9. Such an outcome should not be surprising. For example, the Chebyshev inequality (2.2) with $n = 250$, $p = \frac{179}{200}$ and $\delta = \frac{1}{200}$ gives an upper bound on the probability of an error at least that large that is greater than 1!

Kelly Criterion in Practice (Fractional Kelly)

In this game both the edge and the odds are small. Small uncertainties in our estimation of p can lead to large uncertainties in our estimation of f_*. If we overestimate f_* enough then we are certain to do poorly or lose. Betting more than the true f_* is called *overbetting*. If we underestimate f_* then we will certainly win, just at less than the optimal rate.

Because of this asymmetry, it is wise to bet a fraction of the optimal Kelly bet when we are uncertain of our edge. The greater the uncertainty, the smaller the fraction that should be used. Fractions ranging from $\frac{1}{3}$ to $\frac{1}{10}$ or smaller are common, depending on the uncertainty. These are called *fractional Kelly strategies*.