Portfolios that Contain Risky Assets: 8.2. Law of Large Numbers for IID Models

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Part II: Probabilistic Models

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8. Independent, Identically-Distributed Models for Portfolios

Law of Large Numbers for IID Models

Inroduction

We now consider a market with N risky assets. Let $\{s_i(d)\}_{d=0}^{D}$ be the share price history of asset i. The associated return and growth rate histories are $\{r_i(d)\}_{d=1}^{D}$ and $\{x_i(d)\}_{d=1}^{D}$ where

$$r_i(d) = \frac{s_i(d)}{s_i(d-1)} - 1\,, \qquad x_i(d) = \log\left(\frac{s_i(d)}{s_i(d-1)}\right) .$$

Because each $s_i(d)$ is positive, each $r_i(d)$ is in $(-1, \infty)$, and each $x_i(d)$ is in $(-\infty, \infty)$. Let $\mathbf{r}(d)$ and $\mathbf{x}(d)$ be the N-vectors

$$\mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix}, \qquad \mathbf{x}(d) = \begin{pmatrix} x_1(d) \\ \vdots \\ x_N(d) \end{pmatrix} .$$

The market return and growth rate histories can then be expressed compactly as $\{\mathbf{r}(d)\}_{d=1}^{D}$ and $\{\mathbf{x}(d)\}_{d=1}^{D}$ respectively.

Introduction

If we assume that the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ is an IID sample drawn from a probability density $q(\mathbf{R})$ then the associated return mean $\boldsymbol{\mu}$ and return variance $\boldsymbol{\Xi}$ have unbiased estimators given in terms of $\mathbf{m}$ and $\mathbf{V}$ by

$$\hat{\boldsymbol{\mu}} = \mathbf{m}\,, \qquad \hat{\boldsymbol{\Xi}} = \frac{1}{1-\bar{w}}\,\mathbf{V}\,.$$

Moreover, the Markowitz portfolio with risk-free return r_{rf} and risky asset allocation $\mathbf{f}$ has the return history $\{r(d)\}_{d=1}^{D}$ where

$$r(d) = r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,.$$

This return history is an IID sample drawn from the probability density $q_{(r_{\mathrm{rf}},\mathbf{f})}(R)$ with return mean μ and return variance ξ that have the unbiased estimators

$$\hat{\mu} = r_{\mathrm{rf}} + \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \hat{\xi} = \frac{1}{1-\bar{w}}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,. \tag{1.1}$$

Growth Rate Probability Densities (Portfolio Values)

Now suppose that the returns of an IID model for a solvent portfolio are drawn from a return probability density $q(R)$. Given D samples $\{R_d\}_{d=1}^D$ that are drawn from $q(R)$, the associated simulated *portfolio values* $\{\Pi_d\}_{d=1}^D$ satisfy

$$\Pi_d = \Pi_{d-1}\,(1 + R_d)\,, \quad \text{for } d = 1, \cdots, D\,. \tag{2.2}$$

Recall that for a solvent portfolio we have

$$1 + R_d > 0 \qquad \text{for every } d\,. \tag{2.3}$$

If the inital portfolio value Π_0 is known then we can use induction to show that the portfolio value at the close of day d is

$$\Pi_d = \Pi_0 \prod_{d'=1}^{d} (1 + R_{d'})\,. \tag{2.4}$$

Growth Rate Probability Densities (Growth and Its Rates)

We define the *growth* of the portfolio at the close of day d as

$$\log\left(\frac{\Pi_d}{\Pi_0}\right) = \sum_{d'=1}^{d} \log(1 + R_{d'}) \,. \tag{2.5}$$

Here the solvency condition (2.3) insures that $1 + R_d > 0$ for every d.

We now introduce *growth rates* X_d that are related to the returns R_d by

$$X_d = \log(1 + R_d) \,, \qquad e^{X_d} = 1 + R_d \,. \tag{2.6}$$

In other words, the growth rate X_d yields a return R_d on trading day d. Then the growth of the portfolio at the close of day d that is given by formula (2.5) is expressed simply as

$$\log\left(\frac{\Pi_d}{\Pi_0}\right) = \sum_{d'=1}^{d} X_{d'} \,. \tag{2.7}$$

Growth Rate Probability Densities (Densities)

If the samples $\{R_d\}_{d=1}^{D}$ are drawn from a density $q(R)$ over $(-1, \infty)$ then the $\{X_d\}_{d=1}^{D}$ are drawn from a density $p(X)$ over $(-\infty, \infty)$ where

$$p(X)\,\mathrm{d}X = q(R)\,\mathrm{d}R\,, \tag{2.8a}$$

with X and R related by (2.6) as

$$X = \log(1+R)\,, \qquad R = e^X - 1\,. \tag{2.8b}$$

More explicitly, the densities $p(X)$ and $q(R)$ are related by

$$p(X) = q\left(e^X - 1\right) e^X\,, \qquad q(R) = \frac{p(\log(1+R))}{1+R}\,.$$

Growth Rate Probability Densities (Mean and Variance)

Because our models will involve means and variances, we will require that

$$\int_{-\infty}^{\infty} X^2 p(X)\,\mathrm{d}X = \int_{-1}^{\infty} \log(1+R)^2\, q(R)\,\mathrm{d}R < \infty\,,$$
$$\int_{-\infty}^{\infty} \left(e^X - 1\right)^2 p(X)\,\mathrm{d}X = \int_{-1}^{\infty} R^2 q(R)\,\mathrm{d}R < \infty\,.$$

Then the mean γ and variance θ of X are

$$\begin{aligned} \gamma &= \mathrm{Ex}(X) = \int_{-\infty}^{\infty} X\,p(X)\,\mathrm{d}X\,, \\ \theta &= \mathrm{Vr}(X) = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \int_{-\infty}^{\infty} (X-\gamma)^2\,p(X)\,\mathrm{d}X\,. \end{aligned} \tag{2.9}$$

Growth Rate Probability Densities (Growth Mean)

The big advantage of working with $p(X)$ rather than $q(R)$ is the fact (2.7) that the growth at the close of day d is given by

$$\log\left(\frac{\Pi_d}{\Pi_0}\right) = \sum_{d'=1}^{d} X_{d'} \,.$$

Hence, the growth at the close of day d is the sum of the IID variables X_d. This fact makes it easy to compute the mean and variance of this growth in terms of those of X.

The mean of the growth $\log(\Pi_d/\Pi_0)$ is found to be

$$\mathrm{Ex}\left(\log\left(\frac{\Pi_d}{\Pi_0}\right)\right) = \sum_{d'=1}^{d} \mathrm{Ex}(X_{d'}) = \gamma\, d \,, \qquad (2.10)$$

Growth Rate Probability Densities (Growth Variance)

The variance of the growth $\log(\Pi_d/\Pi_0)$ is found to be

$$
\begin{aligned}
\mathrm{Vr}\left(\log\left(\frac{\Pi_d}{\Pi_0}\right)\right) &= \mathrm{Ex}\left(\left(\sum_{d'=1}^{d} X_{d'} - \gamma\, d\right)^2\right) \\
&= \mathrm{Ex}\left(\left(\sum_{d'=1}^{d} (X_{d'} - \gamma)\right)^2\right) \\
&= \mathrm{Ex}\left(\sum_{d'=1}^{d} \sum_{d''=1}^{d} (X_{d'} - \gamma)\,(X_{d''} - \gamma)\right) \\
&= \sum_{d'=1}^{d} \mathrm{Ex}\left((X_{d'} - \gamma)^2\right) \\
&= \theta\, d\,.
\end{aligned}
\tag{2.11}
$$

Growth Rate Probability Densities (Summary)

Remark. The off-diagonal terms in the foregoing double sum vanish because

$$\mathrm{Ex}\Big((X_{d'} - \gamma)\,(X_{d''} - \gamma)\Big) = 0 \qquad \text{when } d'' \neq d'\,.$$

In summary, formulas (2.10) and (2.11) show that for an IID model the portfolio growth at the close of day d has mean and variance given by

$$\mathrm{Ex}\left(\log\left(\frac{\Pi_d}{\Pi_0}\right)\right) = \gamma\, d\,, \qquad \mathrm{Vr}\left(\log\left(\frac{\Pi_d}{\Pi_0}\right)\right) = \theta\, d\,. \tag{2.12}$$

Law of Large Numbers

Let $\{X_d\}_{d=1}^{\infty}$ be any sequence of IID random variables drawn from a probability density $p(X)$ with mean γ and variance $\theta > 0$. Let $\{Y(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y_d = \frac{1}{d} \sum_{d'=1}^{d} X_{d'} \qquad \text{for every } d = 1,\, 2,\, \cdots \quad .$$

It is easy to check from (2.12) that

$$\mathrm{Ex}(Y_d) = \gamma\,, \qquad \mathrm{Vr}(Y_d) = \frac{\theta}{d}\,. \tag{3.13}$$

Given any $\delta > 0$ the *Law of Large Numbers* states that

$$\lim_{d\to\infty} \mathrm{Pr}\Big\{ |Y_d - \gamma| \geq \delta\,\sqrt{\theta} \Big\} = 0\,. \tag{3.14}$$

Law of Large Numbers (Chebyshev)

Because the mean and variance of Y_d are given in terms of γ and θ by (3.13), the convergence rate of the limit (3.14) can be estimated by the *Chebyshev inequality*, which yields the δ-dependent upper bound

$$\Pr\left\{|Y_d - \gamma| \geq \delta\sqrt{\theta}\right\} \leq \frac{\mathrm{Vr}(Y_d)}{\delta^2\,\theta} = \frac{1}{\delta^2}\,\frac{1}{d}\,. \tag{3.15}$$

Remark. Recall that the Chebyshev inequality (3.15) is easy to derive. Suppose that $p_d(Y)$ is the unknown probability density for Y_d. Because the mean and variance of Y_d are given by (3.13), we have

$$\begin{aligned}\Pr\left\{|Y_d - \gamma| \geq \delta\,\sqrt{\theta}\right\} &= \int_{\{|Y-\gamma|\geq\delta\,\sqrt{\theta}\}} p_d(Y)\,\mathrm{d}Y \\ &\leq \int \frac{|Y-\gamma|^2}{\delta^2\,\theta}\,p_d(Y)\,\mathrm{d}Y = \frac{\mathrm{Vr}(Y_d)}{\delta^2\,\theta} = \frac{1}{\delta^2}\,\frac{1}{d}\,.\end{aligned}$$

Law of Large Numbers

Remark. The probability density $p_d(Y)$ in the previous slide can be expressed in terms of the unknown probability density $p(X)$ as

$$p_d(Y) = \int \cdots \int \delta\left(Y - \frac{1}{d} \sum_{d'=1}^{d} X_{d'} \right) p(X_1) \cdots p(X_d) \, \mathrm{d}X_1 \cdots \mathrm{d}X_d \, ,$$

where $\delta(\,\cdot\,)$ is the Dirac delta distribution introduced earlier.

Remark. The IID model suggests that the growth rate mean γ is a good proxy for the reward of a portfolio and that $\sqrt{\theta}$ is a good proxy for its risk. However, these are not the proxies chosen by MPT.

Law of Large Numbers

Given any sample of IID variables $\{X_d\}_{d=1}^{D}$ drawn from $p(X)$ and any positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, the mean γ and variance θ have the unbiased estimators

$$\hat{\gamma} = \sum_{d=1}^{D} w_d \, X_d \,,$$

$$\hat{\theta} = \frac{1}{1 - \bar{w}} \sum_{d=1}^{D} w_d \, (X_d - \hat{\gamma})^2 \,.$$

Therefore estimators of the proxies γ and $\sqrt{\theta}$ are $\hat{\gamma}$ and $\sqrt{\hat{\theta}}$.

Remark. None of these estimators can be expressed exactly in terms of the return mean and variance estimators given by (1.1).

Normal Growth Rate Model (Introduction)

We can illustrate what is going on with the simple IID model where $p(X)$ is the *normal* or *Gaussian* density with mean γ and variance θ, which is given by

$$p(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) . \tag{4.16}$$

For this IID model many expected values can be computed explicitly.

Fact 1. The return mean μ and variance ξ for the normal growth rate model (4.16) are given by

$$\mu = e^{\gamma + \frac{1}{2}\theta} - 1 , \qquad \xi = e^{2\gamma+2\theta} - e^{2\gamma+\theta} . \tag{4.17}$$

The proof of **Fact 1** uses the following fact.

Normal Growth Rate Model ($\mathrm{Ex}(e^{nX})$)

Fact 2. For every $n \in \{0, 1, \cdots\}$ we have

$$\mathrm{Ex}\left(e^{nX}\right) = \exp\left(n\gamma + \tfrac{1}{2}n^2\theta\right) . \tag{4.18}$$

Proof. Let $n \in \{0, 1, \cdots\}$. Completing the square in the exponent shows

$$\begin{aligned} \mathrm{Ex}\left(e^{nX}\right) &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma)^2}{2\theta} + nX\right) \mathrm{d}X \\ &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-(\gamma+n\theta))^2}{2\theta} + n\gamma + \tfrac{1}{2}n^2\theta\right) \mathrm{d}X \\ &= \exp\left(n\gamma + \tfrac{1}{2}n^2\theta\right) . \end{aligned}$$

This proves **Fact 2**. □

Normal Growth Rate Model (**Fact 1** Proof)

Proof of Fact 1. By (2.8b) we have $R = e^X - 1$. Therefore

$$\mu = \mathrm{Ex}(R) = \mathrm{Ex}\Big(e^X\Big) - 1\,,$$
$$\xi = \mathrm{Vr}(R) = \mathrm{Ex}\Big((R-\mu)^2\Big)$$
$$= \mathrm{Ex}\Big(\Big(e^X - 1 - \mu\Big)^2\Big) = \mathrm{Ex}\Big(e^{2X}\Big) - \mathrm{Ex}\Big(e^X\Big)^2\,.$$

Formulas (4.17) of **Fact 1** then follow because formula (4.18) of **Fact 2** with $n = 1$ and $n = 2$ gives

$$\mathrm{Ex}\Big(e^X\Big) = e^{\gamma + \frac{1}{2}\theta}\,, \qquad \mathrm{Ex}\Big(e^{2X}\Big) = e^{2\gamma + 2\theta}\,.$$

This proves **Fact 1**. □

Normal Growth Rate Model

Remark. Formulas (4.17) can be inverted to obtain

$$\gamma = \log(1+\mu) - \tfrac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right),$$
$$\theta = \log\left(1 + \frac{\xi}{(1+\mu)^2}\right).$$

Because $\gamma + \frac{1}{2}\theta = \log(1+\mu)$, the concavity of log implies that

$$\gamma + \tfrac{1}{2}\theta \leq \mu\,, \qquad \theta \leq \frac{\xi}{(1+\mu)^2}\,. \tag{4.19}$$

Moreover, we see that when $|\mu|$ and ξ are small we have

$$\gamma + \tfrac{1}{2}\theta \approx \mu\,, \qquad \theta \approx \xi\,.$$

Normal Growth Rate Model (Averages of)

Let $\{X_d\}_{d=1}^{\infty}$ be any sequence of IID random variables drawn from $p(X)$ and let $\{Y_d\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y_d = \frac{1}{d}\sum_{d'=1}^{d} X_{d'} \qquad \text{for every } d = 1,\, 2,\, \cdots\,.$$

We can easily check that

$$\mathrm{Ex}(Y_d) = \gamma\,, \qquad \mathrm{Vr}(Y_d) = \frac{\theta}{d}\,.$$

We can also check that

$$\mathrm{Ex}(Y_d|\,Y_{d-1}) = \frac{d-1}{d}\,Y_{d-1} + \frac{1}{d}\,\gamma\,.$$

So the variables Y_d are neither independent nor identically distributed.

Normal Growth Rate Model

It can be shown (the details are not given here) that Y_d is drawn from the normal density with mean γ and variance θ/d, which is given by

$$p_d(Y) = \sqrt{\frac{d}{2\pi\theta}}\,\exp\left(-\frac{(Y-\gamma)^2 d}{2\theta}\right) . \tag{4.20}$$

Because $\Pi_d = \Pi_0 e^{Y_d\, d}$, by completing the square in the exponent we find

$$\begin{aligned}
\mathrm{Ex}(\Pi_d) &= \Pi_0\sqrt{\frac{d}{2\pi\theta}}\int \exp\left(-\frac{(Y-\gamma)^2 d}{2\theta} + Y\,d\right)\mathrm{d}Y \\
&= \Pi_0\sqrt{\frac{d}{2\pi\theta}}\int \exp\left(-\frac{(Y-\gamma-\theta)^2 d}{2\theta} + (\gamma+\tfrac{1}{2}\theta)\,d\right)\mathrm{d}Y \\
&= \Pi_0 \exp\Big((\gamma+\tfrac{1}{2}\theta)\,d\Big) .
\end{aligned}$$

When $d=1$ this recovers formula (4.18) for $n=1$.

Normal Growth Rate Model ()

Because formula (4.20) shows that $p_d(Y)$ becomes sharply peaked around $Y = \gamma$ as d increases, most investors will see a growth rate closer to γ, which is below the rate $\gamma + \frac{1}{2}\theta$ at which the mean portfolio return at the close of day d grows. In particular, most investors will see a return that is below the return mean μ — far below in volatile markets.

- This is because e^X amplifies the tail of the normal density.
- More realistic IID models have a density $p(X)$ that decays more slowly as $X \to \infty$ than a normal density, so this difference will be larger.

Said another way, most investors will not see the same return as Warren Buffett, but his return will boost the return mean.

The normal growth rate model confirms that γ is a better proxy for how well a risky asset might perform than μ because $p_d(Y)$ becomes more peaked around $Y = \gamma$ as d increases. The Law of Large Numbers (3.14) extends this result to IID models that are more realistic.