Portfolios that Contain Risky Assets
7.1. Assessing Identically-Distributed for IID Models

C. David Levermore

University of Maryland, College Park, MD

Math 420: *Mathematical Modeling*
May 9, 2022 version

Portfolios that Contain Risky Assets
Part II: Probabilistic Models

Portfolios that Contain Risky Assets
Part II: Probabilistic Models

7. Assessing Independent, Identically-Distributed Models

Assessing Identically-Distributed for IID Models

Introduction

Independent, Identically Distributed (IID) models of returns make two simplifying assumptions.

1. **Independent.** That what happens on day d is independent of what has happened in the past.
2. **Identically Distributed.** What happens each day is statistically identical to what happens every other day.

In IID models the random numbers $\{R_d\}_{d=1}^{D}$ that mimic a return history are each drawn from $(-1, \infty)$ in accord with the *same* probability density.

The question arises as to how well a given return history $\{r(d)\}_{d=1}^{D}$ is mimiced by such a model. We will present ways by which the validity of each assumption can be assessed.

Introduction

Here we will examine how to assess the validity of the **identically distributed** assumption.

In an IID model the random numbers $\{R_d\}_{d=1}^{D}$ are drawn from $(-1, \infty)$ in accord with the *same* probability density $q(R)$. This means that for every d and every $R \in (-1, \infty)$ we have

$$\Pr\{R_d \leq R\} = \int_{-1}^{R} q(R')\,\mathrm{d}R' .$$

If a return history $\{r(d)\}_{d=1}^{D}$ is consistent with an IID model then every subsample of the return history should behave as if it was drawn from the same probability density. This is the idea of being *identically distributed*.

Introduction

Therefore the question that we must address is how to tell if it is likely that any two samples, $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$, are drawn from the same probability density. We will take three approaches:

- graphical,
- comparing means and variances,
- comparing distributions.

Later, we will examine how to assess the validity of the **independent** assumption. This comes down to understanding how correlated each $r(d)$ is with earlier values, say with $r(d-1)$. We will take three approaches:

- graphical,
- comparing with an autoregressive model,
- comparing autocovariance matrices.

Comparing Means and Variances (Introduction)

Introduction. Given any two samples $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$, their sample means and variances are

$$m_1 = \frac{1}{D_1}\sum_{d=1}^{D_1} r_1(d)\,, \qquad m_2 = \frac{1}{D_2}\sum_{d=1}^{D_2} r_2(d)\,,$$

$$v_1 = \frac{1}{D_1}\sum_{d=1}^{D_1} \big(r_1(d) - m_1\big)^2\,, \qquad v_2 = \frac{1}{D_2}\sum_{d=1}^{D_2} \big(r_2(d) - m_2\big)^2\,.$$

We will assume that $v_1 > 0$ and $v_2 > 0$, which is always the case in practice. If $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$, are drawn from the same probability density then we would expect that m_1 is close to m_2 and v_1 is close to v_2 when D_1 and D_2 are sufficiently large. Our goal is to develop measures of how close m_1 is to m_2 and v_1 is to v_2.

Comparing Means and Variances (Variances)

We begin by assessing the closeness of v_1 and v_2 because it is easier. Because we have assumed that v_1 and v_2 are positive, we can define the relative difference of v_1 and v_2 by the ratio

$$\frac{v_1 - v_2}{v_1 + v_2} \,.$$

This ratio takes values in the interval $(-1, 1)$. When its absolute value is small then v_1 and v_2 are relatively close.

When this ratio is squared we get

$$\frac{(v_1 - v_2)^2}{(v_1 + v_2)^2} = 1 - \frac{4v_1 v_2}{(v_1 + v_2)^2} \,. \tag{2.1}$$

This quantity takes values in the interval $[0, 1)$. Its value is closer to 0 when v_1 and v_2 are relatively closer.

Comparing Means and Variances (Means)

We now assess the closeness of m_1 and m_2. Using relative difference does not work because m_1 and m_2 might have opposite signs and $m_1 + m_2$ might be zero or nearly zero. Rather, because the variances associated with m_1 and m_2 are estimated by $\frac{1}{D_1}v_1$ and $\frac{1}{D_2}v_2$, we use the ratio

$$\frac{(m_1 - m_2)^2}{\frac{1}{D_1}v_1 + \frac{1}{D_2}v_2} .$$

This ratio takes values in the interval $[0, \infty)$. It is close to 0 when $|m_1 - m_2|$ is small compared to either standard deviation.

When this ratio is divided by 1 plus this ratio we get

$$\frac{(m_1 - m_2)^2}{\frac{1}{D_1}v_1 + \frac{1}{D_2}v_2 + (m_1 - m_2)^2} . \tag{2.2}$$

This quantity takes values in the interval $[0, 1)$. Its value is closer to 0 when m_1 and m_2 are relatively closer.

Comparing Distributions (Introduction)

Introduction. Before addressing how likely it is that two samples are drawn from the same probability density, we start with a simpler question. How to compare two probability densities over $(-1,\infty)$, say $q_1(R)$ and $q_2(R)$ where $q_1(R) \geq 0$, $q_2(R) \geq 0$, and

$$\int_{-1}^{\infty} q_1(R)\,\mathrm{d}R = \int_{-1}^{\infty} q_2(R)\,\mathrm{d}R = 1\,.$$

One idea is to compare their distributions $Q_1(R)$ and $Q_2(R)$, which are

$$Q_1(R) = \int_{-1}^{R} q_1(R')\,\mathrm{d}R'\,, \qquad Q_2(R) = \int_{-1}^{R} q_2(R')\,\mathrm{d}R'\,. \tag{3.3a}$$

These are nondecreasing functions of R over $(-1,\infty)$ that satisfy

$$\begin{aligned} \lim_{R\to-1} Q_1(R) &= \lim_{R\to-1} Q_2(R) = 0\,,\\ \lim_{R\to\infty} Q_1(R) &= \lim_{R\to\infty} Q_2(R) = 1\,. \end{aligned} \tag{3.3b}$$

Comparing Distributions (K-S and Kuiper for Densities)

The *Kolmogorov-Smirnov* measure of the closeness of Q_1 and Q_2 is the sup norm of their difference:

$$\|Q_2 - Q_1\|_{\mathrm{KS}} = \sup\Big\{ |Q_2(R) - Q_1(R)| \,:\, R \in (-1,\infty) \Big\}. \tag{3.4a}$$

The *Kuiper* measure of the closeness of Q_1 and Q_2 is

$$\begin{aligned} \|Q_2 - Q_1\|_{\mathrm{Ku}} = \sup\Big\{ Q_2(R) - Q_1(R) \,:\, R \in (-1,\infty) \Big\} \\ + \sup\Big\{ Q_1(R) - Q_2(R) \,:\, R \in (-1,\infty) \Big\}. \end{aligned} \tag{3.4b}$$

Fact 1. These measures satisfy the relations

$$\tfrac{1}{2}\|Q_2 - Q_1\|_{\mathrm{Ku}} \le \|Q_2 - Q_1\|_{\mathrm{KS}} \le \|Q_2 - Q_1\|_{\mathrm{Ku}} \le 1. \tag{3.5}$$

Comparing Distributions (K-S and Kuiper Relations)

Proof. The first inequality in (3.5) should be clear from definitions (3.4). Because $Q_1(R)$ and $Q_2(R)$ satisfy (3.3b) we have

$$\lim_{R\to-1}\Big(Q_2(R)-Q_1(R)\Big)=0\,,\qquad \lim_{R\to\infty}\Big(Q_2(R)-Q_1(R)\Big)=0\,.$$

It follows that the components of the Kuiper measure satisfy

$$\sup\Big\{Q_2(R)-Q_1(R)\;:\;R\in(-1,\infty)\Big\}\geq 0\,,$$
$$\sup\Big\{Q_1(R)-Q_2(R)\;:\;R\in(-1,\infty)\Big\}\geq 0\,.$$

But it can be shown that at least one of these components must equal the Kolmogorov-Smirnov measure. Therefore

$$\|Q_2-Q_1\|_{\mathrm{KS}}\leq\|Q_2-Q_1\|_{\mathrm{Ku}}\,.$$

In order to prove (3.5) it still needs to be shown that $\|Q_2-Q_1\|_{\mathrm{Ku}}\leq 1$. This is left as an exercise.

Comparing Distributions (Cramer-von Mises and L^p)

Remark. There are other ways to measure of the closeness of Q_1 and Q_2. For example, the *Cramer-von Mises* measure is the L^2-norm of their difference:

$$\|Q_2 - Q_1\|_{\mathrm{CvM}} = \left(\int_{-1}^{\infty} \Big(Q_2(R) - Q_1(R) \Big)^2 \mathrm{d}R \right)^{\frac{1}{2}} .$$

This can clearly be generalized to any L^p-norm with respect to any positive measure over $(-1, \infty)$. For example, for every $p \in [1, \infty)$ we have

$$\|Q_2 - Q_1\|_{L^p} = \left(\int_{-1}^{\infty} \Big(Q_2(R) - Q_1(R) \Big)^p \mathrm{d}R \right)^{\frac{1}{p}} .$$

However, we will stick to the Kolmogorov-Smirnov and Kuiper measures.

Comparing Distributions (Emperical Distributions)

Now we return to our original question. Given two samples, $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$, we construct their so-called *emperical distributions*

$$\widehat{Q}_1(R) = \frac{\#\{d \,:\, r_1(d) \leq R\}}{D_1}\,, \qquad \widehat{Q}_2(R) = \frac{\#\{d \,:\, r_2(d) \leq R\}}{D_2}\,. \quad (3.6a)$$

Here $\#S$ denotes the number of elements in a set S. These are analogs of the distributions $Q_1(R)$ and $Q_2(R)$ defined in (3.3a). They are piecewise constant, nondecreasing *staircase functions* of R over $(-1, \infty)$ with steps at $R \in \{r_1(d)\}_{d=1}^{D_1}$ and $R \in \{r_2(d)\}_{d=1}^{D_2}$ respectively. They satisfy

$$\begin{aligned} \lim_{R\to -1} \widehat{Q}_1(R) &= \lim_{R\to -1} \widehat{Q}_2(R) = 0\,, \\ \lim_{R\to \infty} \widehat{Q}_1(R) &= \lim_{R\to \infty} \widehat{Q}_2(R) = 1\,. \end{aligned} \quad (3.6b)$$

Comparing Distributions (Emperical Distributions)

Remark. Emperical distributions in the form (3.6a) approximate any distribution $Q(R)$ in the following sense. Let $\{R_d\}_{d=1}^{\infty}$ be drawn from the distribution $Q(R)$. This means that for every $d \in \mathbb{Z}_+$ we have

$$Q(R) = \mathrm{Pr}\{R_d \leq R\}\,.$$

For every $D \in \mathbb{Z}_+$ define the emperical distribution

$$\widehat{Q}_D(R) = \frac{\#\{d \leq D \,:\, R_d \leq R\}}{D}\,.$$

It will not be shown here, but Kolmogorov and Smirnov showed that

$$\lim_{D \to \infty} \left\| \widehat{Q}_D(R) - Q(R) \right\|_{\mathrm{KS}} = 0\,.$$

Comparing Distributions (K-S and Kuiper for Samples)

Because the Kolmogorov-Smirnov and Kuiper measures quantify the size of the difference $\widehat{Q}_2 - \widehat{Q}_1$, they give us ways to quantify the likelihood that samples are drawn from similar distributions. Because $\widehat{Q}_1$ and $\widehat{Q}_2$ are staircase functions, there are algorithms that evaluate

$$\begin{aligned}
\|\widehat{Q}_2 - \widehat{Q}_1\|_{\mathrm{KS}} &= \max\Big\{ \big|\widehat{Q}_2(R) - \widehat{Q}_1(R)\big| \;:\; R \in (-1,\infty) \Big\} \,. \\
\|\widehat{Q}_2 - \widehat{Q}_1\|_{\mathrm{Ku}} &= \max\Big\{ \widehat{Q}_2(R) - \widehat{Q}_1(R) \;:\; R \in (-1,\infty) \Big\} \\
&\quad + \max\Big\{ \widehat{Q}_1(R) - \widehat{Q}_2(R) \;:\; R \in (-1,\infty) \Big\} \,.
\end{aligned} \tag{3.7}$$

Fortunately statisticians have provided software that efficiently computes these values given any two samples $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$. These are called respectively the *two-sample KS test* and the *two-sample Kuiper test*.

Assessing Identical Distribution (Introduction)

Assessing Identical Distribution. We will now present three ways to assess how much a given return history $\{r(d)\}_{d=1}^{D}$ is consistent with the *identical distribution assumption*. More specifically, we will present:

- a graphical assessment,
- a mean and a variance assessment,
- two distribution assessments.

The first is purely visual, but can be used to build understanding of the data. The other two are analytical. They will yield metrics ω^{m}, ω^{v}, ω^{KS}, and ω^{Ku} of how consistent the given data is with the identical distribution assumption. As before, these metrics will take values in the interval $[0, 1]$ with higher values indicating greater consistency with the identical distribution assumption.

Assessing Identical Distribution (Graphical)

Graphical Assessment. In an IID model the random numbers $\{R_d\}_{d=1}^{D}$ are each drawn from $(-1,\infty)$ in accord with the *same* distribution $Q(R)$. This means that if we scatter plot the points $\{(d,R_d)\}_{d=1}^{D}$ in the dr-plane they will usually be distributed in a way that looks uniform in d.

Therefore if the return history $\{r(d)\}_{d=1}^{D}$ is mimiced by such a model then the points $\{(d,r(d))\}_{d=1}^{D}$ scatter plotted in the dr-plane should appear to be distributed in a way that is uniform in d.

Remark. Of course, determining whether such a scatter plot is distributed in a way that is uniform in d simply by looking at it is subjective. However, sometimes this graphical approach can make it quite clear that the identical distribution assumption is flawed! Henceforth, we will present quantitative approaches.

Assessing Identical Distribution (m and v Metrics)

Mean and Variance Assessments. Given return histories over a year $\{r(d)\}_{d=1}^{D}$, we can split the year into quarters and compare the mean and variance of each quarter with that of another quarter or with that of the other three quarters combined. The maximum of all such comparisons made is the score for the year. For example, using (2.2) and (2.1), for each year we define

$$\omega^{\mathrm{m}} = \max\left\{ \frac{(m_1 - m_2)^2}{\frac{1}{D_1}v_1 + \frac{1}{D_2}v_2 + (m_1 - m_2)^2} \ : \ \text{all comparisons made} \right\} ,$$

$$\omega^{\mathrm{v}} = \max\left\{ \frac{(v_1 - v_2)^2}{(v_1 + v_2)^2} \ : \ \text{all comparisons made} \right\} . \tag{4.8}$$

If we compare quarters with each other then six comparisons are made. If we compare each quarter with the other three quarters combined then four comparisons are made. Notice that the means are closer when ω^{m} is nearer 0, and that the variances are closer when ω^{v} is nearer 0.

Assessing Identical Distribution (KS and Ku Metrics)

Distribution Assessments. Similarly, given return histories over a year $\{r(d)\}_{d=1}^{D}$, we can split the year into quarters and compare the emperical distribution of each quarter with that of another quarter or with that of the other three quarters combined. The maximum of all such comparisons made is the score for the year. For example, for each year we define

$$\begin{aligned} \omega^{\mathrm{KS}} &= \max\left\{ \left\| \widehat{Q}_2 - \widehat{Q}_1 \right\|_{\mathrm{KS}} : \text{all comparisons made} \right\}, \\ \omega^{\mathrm{Ku}} &= \max\left\{ \left\| \widehat{Q}_2 - \widehat{Q}_1 \right\|_{\mathrm{Ku}} : \text{all comparisons made} \right\}. \end{aligned} \tag{4.9}$$

If we choose to compare quarters with each other then six comparisons are made. If we choose to compare each quarter with the other three quarters combined then four comparisons are made. Notice that $\omega^{\mathrm{KS}} \leq \omega^{\mathrm{Ku}} \leq 1$, and that the distributions are closer when ω^{Ku} is nearer 0.