Portfolios that Contain Risky Assets
6.3. Central Moment Estimators for IID Models

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Portfolios that Contain Risky Assets
Part II: Probabilistic Models

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6. Independent, Identically-Distributed Models for Assets

Central Moment Estimators for IID Models

Central Moments (Introduction)

Central Moments. We recall from **Fact 6** that if $\mathrm{Ex}(\Psi^4) < \infty$ then the unbiased variance estimator $\widehat{\mathrm{Vr}}(\Psi)$ has variance given by

$$\begin{aligned}\mathrm{Vr}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big) = {} & \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\mathrm{Vr}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2\Big) \\ & + 2\,\frac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\mathrm{Vr}(\Psi)^2\,,\end{aligned} \tag{1.1a}$$

where $\bar{w}$, $\overline{w^2}$, and $\overline{w^3}$ are given by

$$\bar{w} = \sum_{d=1}^{D} w_d^2\,, \qquad \overline{w^2} = \sum_{d=1}^{D} w_d^3\,, \qquad \overline{w^3} = \sum_{d=1}^{D} w_d^4\,. \tag{1.1b}$$

This suggests that it would be useful to have estimators for

$$\mathrm{Vr}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2\Big)\,, \qquad \mathrm{Vr}(\Psi)^2\,. \tag{1.2}$$

Central Moments (Definition)

For any function $\psi : (-1, \infty) \to \mathbb{R}$, the variance of $\Psi = \psi(R)$ with respect to the probability density $q(R)$ is given by

$$\mathrm{Vr}(\Psi) = \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2\Big) = \int_{-1}^{\infty} \big(\psi(R) - \mathrm{Ex}(\Psi)\big)^2 q(R)\,\mathrm{d}R\,.$$

Given any $k \in \mathbb{N}$ the *k^{th} central moment* of $\Psi = \psi(R)$ with respect to the probability density $q(R)$ is defined by

$$\mathrm{Cn}_k(\Psi) = \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^k\Big) = \int_{-1}^{\infty} \big(\psi(R) - \mathrm{Ex}(\Psi)\big)^k q(R)\,\mathrm{d}R\,. \quad (1.3)$$

The first three are $\mathrm{Cn}_0(\Psi) = 1$, $\mathrm{Cn}_1(\Psi) = 0$, and $\mathrm{Cn}_2(\Psi) = \mathrm{Vr}(\Psi)$.
For each $k > 2$ the central moment $\mathrm{Cn}_k(\Psi)$ gives an alternative measure of the variation of Ψ about its expected value $\mathrm{Ex}(\Psi)$.

Central Moments (Cubic and Quartic)

The cubic and quartic central moments are the next most important after the variance. We denote them by

$$\begin{aligned} \mathrm{Cb}(\Psi) &= \mathrm{Cn}_3(\Psi) = \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^3 \Big) \,, \\ \mathrm{Qr}(\Psi) &= \mathrm{Cn}_4(\Psi) = \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^4 \Big) \,. \end{aligned} \tag{1.4}$$

Then the first quantity on the right-hand side of (1.1a) can be expressed as

$$\begin{aligned} \mathrm{Vr}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2 \Big) &= \mathrm{Ex}\Bigg(\bigg(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2 - \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2 \Big) \bigg)^2 \Bigg) \\ &= \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^4 \Big) - \mathrm{Ex}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2 \Big)^2 \\ &= \mathrm{Qr}(\Psi) - \mathrm{Vr}(\Psi)^2 \,. \end{aligned}$$

Central Moments (Dispersion Decomposition)

This shows that the quartic central moment has the decomposition

$$\mathrm{Qr}(\Psi) = \mathrm{Ds}(\Psi) + \mathrm{Vr}(\Psi)^2 \,, \tag{1.5a}$$

where

$$\mathrm{Ds}(\Psi) = \mathrm{Vr}\Big(\big(\Psi - \mathrm{Ex}(\Psi)\big)^2\Big) \,. \tag{1.5b}$$

This quantity does not have a widely used name. We will call it the *dispersion* of Ψ and call (1.5a) the *dispersion deomposition* of $\mathrm{Qr}(\Psi)$. Equation (1.1a) can then be expressed as

$$\mathrm{Vr}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big) = \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\mathrm{Ds}(\Psi) + 2\,\frac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\mathrm{Vr}(\Psi)^2 \,, \tag{1.6}$$

where $\bar{w}$, $\overline{w^2}$, and $\overline{w^3}$ are given by (1.1b).

Central Moments (Sample Central Moments)

Given $\{w_d\}_{d=1}^D$ and $\{\Psi_d\}_{d=1}^D$, we know $\mathrm{Vr}(\Psi)$ has the unbiased estimator

$$\widehat{\mathrm{Vr}}(\Psi) = \frac{1}{1-\bar{w}}\,\mathrm{SmpVr}(\Psi)\,, \tag{1.7a}$$

where $\bar{w}$ is given by (1.1b) and $\mathrm{SmpVr}(\Psi)$ is the sample variance,

$$\mathrm{SmpVr}(\Psi) = \sum_{d=1}^{D} w_d \left(\Psi_d - \widehat{\mathrm{Ex}}(\Psi)\right)^2 . \tag{1.7b}$$

Below we give estimators for $\mathrm{Cb}(\Psi)$ and $\mathrm{Qr}(\Psi)$ built from $\mathrm{SmpVr}(\Psi)$ and the sample cubic and quartic central moments given by

$$\mathrm{SmpCb}(\Psi) = \sum_{d=1}^{D} w_d \left(\Psi_d - \widehat{\mathrm{Ex}}(\Psi)\right)^3 , \tag{1.8a}$$

$$\mathrm{SmpQr}(\Psi) = \sum_{d=1}^{D} w_d \left(\Psi_d - \widehat{\mathrm{Ex}}(\Psi)\right)^4 . \tag{1.8b}$$

Central Moments (Sample Dispersion Decomposition)

We will also use the *sample dispersion* defined by

$$\mathrm{SmpDs}(\Psi) = \sum_{d=1}^{D} w_d \left(\left(\Psi_d - \widehat{\mathrm{Ex}}(\Psi) \right)^2 - \sum_{d'=1}^{D} w_{d'} \left(\Psi_{d'} - \widehat{\mathrm{Ex}}(\Psi) \right)^2 \right)^2 . \tag{1.9a}$$

The sample quartic central moment $\mathrm{SmpQr}(\Psi)$ defined by (1.8b) then satisfies a discrete analog of the dispersion decomposition (1.5b) given by

$$\mathrm{SmpQr}(\Psi) = \mathrm{SmpDs}(\Psi) + \mathrm{SmpVr}(\Psi)^2 , \tag{1.9b}$$

where that sample variance $\mathrm{SmpVr}(\Psi)$ is defined by (1.7b).

Cubic Central Moment Estimators $(\mathrm{Ex}(\mathrm{SmpCb}(\Psi)))$

Cubic Central Moment Estimators. Let $\widetilde{\Psi}_d = \Psi_d - \mathrm{Ex}(\Psi)$. Then

$$\begin{aligned} \widehat{\mathrm{Ex}}\left(\widetilde{\Psi}\right) &= \widehat{\mathrm{Ex}}(\Psi) - \mathrm{Ex}(\Psi)\,, \\ \Psi_d - \widehat{\mathrm{Ex}}(\Psi) &= \widetilde{\Psi}_d - \widehat{\mathrm{Ex}}\left(\widetilde{\Psi}\right)\,, \end{aligned} \tag{2.10}$$

whereby the sample cubic central moment (1.8a) is

$$\begin{aligned} \mathrm{SmpCb}(\Psi) &= \sum_{d=1}^{D} w_d \left(\Psi_d - \widehat{\mathrm{Ex}}(\Psi)\right)^3 = \sum_{d=1}^{D} w_d \left(\widetilde{\Psi}_d - \widehat{\mathrm{Ex}}\left(\widetilde{\Psi}\right)\right)^3 \\ &= \sum_{d=1}^{D} w_d \widetilde{\Psi}_d^{\,3} - 3 \sum_{d=1}^{D} \sum_{d_1=1}^{D} w_d w_{d_1} \widetilde{\Psi}_d^{\,2} \widetilde{\Psi}_{d_1} \\ &\quad + 2 \sum_{d=1}^{D} \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} w_d w_{d_1} w_{d_2} \widetilde{\Psi}_d \widetilde{\Psi}_{d_1} \widetilde{\Psi}_{d_2}\,. \end{aligned}$$

Cubic Central Moment Estimators $(\mathrm{Ex}(\mathrm{SmpCb}(\Psi)))$

Because $\widetilde{\Psi}_d$ and $\widetilde{\Psi}_{d'}$ are independent when $d \neq d'$, and because $\mathrm{Ex}(\widetilde{\Psi}_d) = 0$, we find that

$$\mathrm{Ex}\left(\widetilde{\Psi}_d^{\,2}\,\widetilde{\Psi}_{d_1}\right) = \delta_{dd_1}\mathrm{Ex}\left(\widetilde{\Psi}^{\,3}\right) \,,$$
$$\mathrm{Ex}\left(\widetilde{\Psi}_d\widetilde{\Psi}_{d_1}\widetilde{\Psi}_{d_2}\right) = \delta_{dd_1}\delta_{dd_2}\mathrm{Ex}\left(\widetilde{\Psi}^{\,3}\right) \,.$$

Because

$$\mathrm{Ex}\left(\widetilde{\Psi}^{\,3}\right) = \mathrm{Cb}(\Psi)\,,$$

we see that the expected value of the sample cubic central moment is

$$\mathrm{Ex}(\mathrm{SmpCb}(\Psi)) = \left(1 - 3\bar{w} + 2\overline{w^2}\right)\mathrm{Cb}(\Psi)\,,$$

where $\bar{w}$ and $\overline{w^2}$ are given by (1.1b).

Cubic Central Moment Estimators (Unbiased)

It can be shown that $\bar{w}^2 \leq \overline{w^2} < \bar{w}$, whereby we have the bounds

$$1 - 3\bar{w} + 2\bar{w}^2 \leq 1 - 3\bar{w} + 2\overline{w^2} < 1 - \bar{w}\,.$$

Because $1 - 3\bar{w} + 2\bar{w}^2 = (1 - 2\bar{w})(1 - \bar{w})$, the lower bound is positive when $\bar{w} < \frac{1}{2}$. In that case we see that an unbiased estimator of $\mathrm{Cb}(\Psi)$ is

$$\widehat{\mathrm{Cb}}(\Psi) = \frac{1}{1 - 3\bar{w} + 2\overline{w^2}}\,\mathrm{SmpCb}(\Psi)\,. \tag{2.11}$$

This is the positive factor $1/(1 - 3\bar{w} + 2\overline{w^2})$ times the sample cubic central moment. The upper bound above shows that this factor is larger than the factor $1/(1 - \bar{w})$ that arises in the unbiased estimator $\widehat{\mathrm{Vr}}(\Psi)$. For uniform weights (2.11) becomes

$$\widehat{\mathrm{Cb}}(\Psi) = \frac{D^2}{(D-1)(D-2)}\,\mathrm{SmpCb}(\Psi)\,. \tag{2.12}$$

Quartic Central Moment Estimators $(\mathrm{Ex}(\mathrm{SmpQr}(\Psi)))$

Quartic Central Moment Estimators. Let $\widetilde{\Psi}_d = \Psi_d - \mathrm{Ex}(\Psi)$. Then by (2.10) the sample quartic central moment (1.8b) is

$$
\begin{aligned}
\mathrm{SmpQr}(\Psi) &= \sum_{d=1}^{D} w_d \Big(\Psi_d - \widehat{\mathrm{Ex}}(\Psi)\Big)^4 = \sum_{d=1}^{D} w_d \left(\widetilde{\Psi}_d - \widehat{\mathrm{Ex}}\big(\widetilde{\Psi}\big)\right)^4 \\
&= \sum_{d=1}^{D} w_d \widetilde{\Psi}_d^{\,4} - 4 \sum_{d=1}^{D} \sum_{d_1=1}^{D} w_d \, w_{d_1} \widetilde{\Psi}_d^{\,3} \, \widetilde{\Psi}_{d_1} \\
&\quad + 6 \sum_{d=1}^{D} \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} w_d \, w_{d_1} w_{d_2} \widetilde{\Psi}_d^{\,2} \, \widetilde{\Psi}_{d_1} \widetilde{\Psi}_{d_2} \\
&\quad - 3 \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} \sum_{d_3=1}^{D} \sum_{d_4=1}^{D} w_{d_1} w_{d_2} w_{d_3} w_{d_4} \widetilde{\Psi}_{d_1} \widetilde{\Psi}_{d_2} \widetilde{\Psi}_{d_3} \widetilde{\Psi}_{d_4} \,.
\end{aligned}
$$

Quartic Central Moment Estimators $(\mathrm{Ex}(\mathrm{SmpQr}(\Psi)))$

Because $\widetilde{\Psi}_d$ and $\widetilde{\Psi}_{d'}$ are independent when $d \neq d'$, and because $\mathrm{Ex}(\widetilde{\Psi}_d) = 0$, we find that

$$
\begin{aligned}
\mathrm{Ex}\left(\widetilde{\Psi}_d^{\,3}\,\widetilde{\Psi}_{d_1}\right) &= \delta_{dd_1}\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,4}\right)\,, \\
\mathrm{Ex}\left(\widetilde{\Psi}_d^{\,2}\,\widetilde{\Psi}_{d_1}\widetilde{\Psi}_{d_2}\right) &= \delta_{d_1d_2}\delta_{dd_1}\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,4}\right) \\
&\quad + \delta_{d_1d_2}(1-\delta_{dd_1})\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,2}\right)^2\,, \\
\mathrm{Ex}\left(\widetilde{\Psi}_{d_1}\widetilde{\Psi}_{d_2}\widetilde{\Psi}_{d_3}\widetilde{\Psi}_{d_4}\right) &= \delta_{d_1d_2}\delta_{d_2d_3}\delta_{d_3d_4}\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,4}\right) \\
&\quad + \delta_{d_1d_2}\delta_{d_3d_4}(1-\delta_{d_1d_3})\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,2}\right)^2 \\
&\quad + \delta_{d_1d_3}\delta_{d_4d_2}(1-\delta_{d_1d_4})\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,2}\right)^2 \\
&\quad + \delta_{d_1d_4}\delta_{d_2d_3}(1-\delta_{d_1d_2})\,\mathrm{Ex}\left(\widetilde{\Psi}^{\,2}\right)^2\,.
\end{aligned}
$$

Quartic Central Moment Estimators $(\mathrm{Ex}(\mathrm{SmpQr}(\Psi)))$

Because

$$\mathrm{Ex}\left(\widetilde{\Psi}^{\,4}\right) = \mathrm{Qr}(\Psi)\,, \qquad \mathrm{Ex}\left(\widetilde{\Psi}^{\,2}\right) = \mathrm{Vr}(\Psi)\,,$$

we have

$$\begin{aligned} \mathrm{Ex}(\mathrm{SmpQr}(\Psi)) = &\left(1 - 4\bar{w} + 6\overline{w^2} - 3\overline{w^3}\right)\,\mathrm{Qr}(\Psi) \\ &+ \left(6\bar{w} - 6\overline{w^2} - 9\bar{w}^2 + 9\overline{w^3}\right)\,\mathrm{Vr}(\Psi)^2\,. \end{aligned} \tag{3.13a}$$

On the other hand, the similar calculation in the first five pages of the proof of **Fact 6** showed that

$$\begin{aligned} \mathrm{Ex}\left(\mathrm{SmpVr}(\Psi)^2\right) = &\left(\bar{w} - 2\overline{w^2} + \overline{w^3}\right)\,\mathrm{Qr}(\Psi) \\ &+ \left(1 - 3\bar{w} + 2\overline{w^2} + 3\bar{w}^2 - 3\overline{w^3}\right)\,\mathrm{Vr}(\Psi)^2\,. \end{aligned} \tag{3.13b}$$

Quartic Central Moment Estimators (Determinant)

Equations (3.13) give $\mathrm{Ex}(\mathrm{SmpQr}(\Psi))$ and $\mathrm{Ex}(\mathrm{SmpVr}(\Psi)^2)$ as linear combinations of $\mathrm{Qr}(\Psi)$ and $\mathrm{Vr}(\Psi)^2$. This linear system can be inverted if and only if its determinant is nonzero. By adding 3 times the second row to the first, factoring out $(1-\bar{w})$ from the first row, and evaluating the resulting determinant, we obtain

$$\begin{aligned}
&\det\begin{pmatrix} 1-4\bar{w}+6\overline{w^2}-3\overline{w^3} & 6\bar{w}-6\overline{w^2}-9\bar{w}^2+9\overline{w^3} \\ \bar{w}-2\overline{w^2}+\overline{w^3} & 1-3\bar{w}+2\overline{w^2}+3\bar{w}^2-3\overline{w^3} \end{pmatrix} \\
&= \det\begin{pmatrix} 1-\bar{w} & 3-3\bar{w} \\ \bar{w}-2\overline{w^2}+\overline{w^3} & 1-3\bar{w}+2\overline{w^2}+3\bar{w}^2-3\overline{w^3} \end{pmatrix} \\
&= (1-\bar{w})\det\begin{pmatrix} 1 & 3 \\ \bar{w}-2\overline{w^2}+\overline{w^3} & 1-3\bar{w}+2\overline{w^2}+3\bar{w}^2-3\overline{w^3} \end{pmatrix} \\
&= (1-\bar{w})\left(1-6\bar{w}+3\bar{w}^2+8\overline{w^2}-6\overline{w^3}\right) .
\end{aligned}$$

Quartic Central Moment Estimators (Determinant)

Because $0 < \bar{w} < 1$ for every $D \geq 2$, we see that $1 - \bar{w} > 0$, so we just have to analyze the sign of the second factor of the determinant.

Because the function $w \mapsto 8w^2 - 6w^3$ is strictly convex over $(-\infty, \frac{4}{9}]$, if $w_d \leq \frac{4}{9}$ for every d then the Jensen inequality bounds this second factor below as

$$\begin{aligned} 1 - 6\bar{w} + 3\bar{w}^2 + 8\overline{w^2} - 6\overline{w^3} &\geq 1 - 6\bar{w} + 11\bar{w}^2 - 6\bar{w}^3 \\ &= (1 - \bar{w})(1 - 2\bar{w})(1 - 3\bar{w}) \,, \end{aligned}$$

which is positive when $\bar{w} < \frac{1}{3}$.

Remark. This lower bound is sharp for uniform weights. It is positive for uniform weights when $D > 3$. In general both the condition $w_d \leq \frac{4}{9}$ for every d and the condition $\bar{w} < \frac{1}{3}$ hold when $\bar{w} \leq \frac{16}{81}$, which is always the case in practice.

Quartic Central Moment Estimators ($\mathrm{Qr}(\Psi)$ & $\mathrm{Vr}(\Psi)^2$)

When the determinant is positive, the system (3.13) can be solved for $\mathrm{Qr}(\Psi)$ and $\mathrm{Vr}(\Psi)^2$ in terms of $\mathrm{Ex}(\mathrm{SmpQr}(\Psi))$ and $\mathrm{Ex}\big(\mathrm{SmpVr}(\Psi)^2\big)$ as

$$
\begin{aligned}
\mathrm{Qr}(\Psi) = {} & \frac{1}{1-\bar{w}}\,\frac{1-3\bar{w}+2\overline{w^2}+3\bar{w}^2-3\overline{w^3}}{1-6\bar{w}+3\bar{w}^2+8\overline{w^2}-6\overline{w^3}}\,\mathrm{Ex}(\mathrm{SmpQr}(\Psi)) \\
& - \frac{1}{1-\bar{w}}\,\frac{6\bar{w}-6\overline{w^2}-9\bar{w}^2+9\overline{w^3}}{1-6\bar{w}+3\bar{w}^2+8\overline{w^2}-6\overline{w^3}}\,\mathrm{Ex}\Big(\mathrm{SmpVr}(\Psi)^2\Big)\,, \\
\mathrm{Vr}(\Psi)^2 = {} & -\frac{1}{1-\bar{w}}\,\frac{\bar{w}-2\overline{w^2}+\overline{w^3}}{1-6\bar{w}+3\bar{w}^2+8\overline{w^2}-6\overline{w^3}}\,\mathrm{Ex}(\mathrm{SmpQr}(\Psi)) \\
& + \frac{1}{1-\bar{w}}\,\frac{1-4\bar{w}+6\overline{w^2}-3\overline{w^3}}{1-6\bar{w}+3\bar{w}^2+8\overline{w^2}-6\overline{w^3}}\,\mathrm{Ex}\Big(\mathrm{SmpVr}(\Psi)^2\Big)\,.
\end{aligned}
$$

Quartic Central Moment Estimators (Unbiased)

Hence, unbiased estimators of $\mathrm{Qr}(\Psi)$ and $\mathrm{Sq}(\Psi) = \mathrm{Vr}(\Psi)^2$ are seen to be

$$\begin{aligned}
\widehat{\mathrm{Qr}}(\Psi) &= \frac{1}{1-\bar{w}}\,\frac{1 - 3\bar{w} + 2\overline{w^2} + 3\bar{w}^2 - 3\overline{w^3}}{1 - 6\bar{w} + 3\bar{w}^2 + 8\overline{w^2} - 6\overline{w^3}}\,\mathrm{SmpQr}(\Psi) \\
&\quad - \frac{1}{1-\bar{w}}\,\frac{6\bar{w} - 6\overline{w^2} - 9\bar{w}^2 + 9\overline{w^3}}{1 - 6\bar{w} + 3\bar{w}^2 + 8\overline{w^2} - 6\overline{w^3}}\,\mathrm{SmpVr}(\Psi)^2\,, \\
\widehat{\mathrm{Sq}}(\Psi) &= -\frac{1}{1-\bar{w}}\,\frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{1 - 6\bar{w} + 3\bar{w}^2 + 8\overline{w^2} - 6\overline{w^3}}\,\mathrm{SmpQr}(\Psi) \\
&\quad + \frac{1}{1-\bar{w}}\,\frac{1 - 4\bar{w} + 6\overline{w^2} - 3\overline{w^3}}{1 - 6\bar{w} + 3\bar{w}^2 + 8\overline{w^2} - 6\overline{w^3}}\,\mathrm{SmpVr}(\Psi)^2\,,
\end{aligned} \tag{3.14}$$

where $\bar{w}$, $\overline{w^2}$ and $\overline{w^3}$ are given by (1.1b). These formulas have none of the simplicity of formula (1.7) for $\widehat{\mathrm{Vr}}(\Psi)$ or formula (2.11) for $\widehat{\mathrm{Cb}}(\Psi)$.

Quartic Central Moment Estimators (Uniform Weights)

For uniform weights the unbiased estimators (3.14) become

$$\widehat{\mathrm{Qr}}(\Psi) = \frac{D(D^2-2D+3)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpQr}(\Psi) - \frac{3D(2D-3)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,, \tag{3.15a}$$

$$\widehat{\mathrm{Sq}}(\Psi) = -\frac{D}{(D-2)(D-3)}\,\mathrm{SmpQr}(\Psi) + \frac{D(D^2-3D+3)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,, \tag{3.15b}$$

and the unbiased estimator of the dispersion is

$$\widehat{\mathrm{Ds}}(\Psi) = \frac{D(D^2-D+2)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpQr}(\Psi) - \frac{D(D^2+3D-6)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,. \tag{3.15c}$$

Quartic Central Moment Estimators (Uniform Weights)

These unbiased estimators can also be expressed in terms of the sample dispersion $\mathrm{SmpDs}(\Psi)$ and sample variance $\mathrm{SmpVr}(\Psi)$ as

$$\begin{aligned}\widehat{\mathrm{Qr}}(\Psi) &= \tfrac{D(D^2-2D+3)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpDs}(\Psi)\\ &\quad+\tfrac{D(D^2-8D+12)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,, \qquad (3.16a)\\ \widehat{\mathrm{Ds}}(\Psi) &= \tfrac{D(D^2-D+2)}{(D-1)(D-2)(D-3)}\,\mathrm{SmpDs}(\Psi)-\tfrac{4D}{(D-1)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,, \qquad (3.16b)\\ \widehat{\mathrm{Sq}}(\Psi) &= -\tfrac{D}{(D-2)(D-3)}\,\mathrm{SmpDs}(\Psi)+\tfrac{D(D-2)}{(D-1)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,. \qquad (3.16c)\end{aligned}$$

These estimators satisfy the dispersion decomposition

$$\widehat{\mathrm{Qr}}(\Psi)=\widehat{\mathrm{Ds}}(\Psi)+\widehat{\mathrm{Sq}}(\Psi)\,.$$

However, while it is clear from (3.16a) that $\widehat{\mathrm{Qr}}(\Psi)$ is positive, it is also clear from (3.16b) and (3.16c) that $\widehat{\mathrm{Ds}}(\Psi)$ or $\widehat{\mathrm{Sq}}(\Psi)$ can be negative!

Central Moment Inequalities (Introduction)

The central moments $\mathrm{Vr}(\Psi)$, $\mathrm{Cb}(\Psi)$ and $\mathrm{Qr}(\Psi)$ must satisfy *moment inequalities* that arise because for every $(p_0, p_1, p_2) \in \mathbb{R}^3$ we have

$$\int \left(p_0 + p_1\, \psi(R) + p_2\, \psi(R)^2 \right)^2 q(R)\, \mathrm{d}R \geq 0\,.$$

This shows that for every $(p_0, p_1, p_2) \in \mathbb{R}^3$ that $\Psi = \psi(R)$ satisfies

$$\begin{pmatrix} p_0 & p_1 & p_2 \end{pmatrix} \begin{pmatrix} 1 & \mathrm{Ex}(\Psi) & \mathrm{Ex}(\Psi^2) \\ \mathrm{Ex}(\Psi) & \mathrm{Ex}(\Psi^2) & \mathrm{Ex}(\Psi^3) \\ \mathrm{Ex}(\Psi^2) & \mathrm{Ex}(\Psi^3) & \mathrm{Ex}(\Psi^4) \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \end{pmatrix} \geq 0\,.$$

This is equivalent to

$$\begin{pmatrix} 1 & \mathrm{Ex}(\Psi) & \mathrm{Ex}(\Psi^2) \\ \mathrm{Ex}(\Psi) & \mathrm{Ex}(\Psi^2) & \mathrm{Ex}(\Psi^3) \\ \mathrm{Ex}(\Psi^2) & \mathrm{Ex}(\Psi^3) & \mathrm{Ex}(\Psi^4) \end{pmatrix} \quad \text{is nonnegative definite}\,. \tag{4.17}$$

Central Moment Inequalities (Central Moments)

Because $\mathrm{Ex}(\Psi)$, $\mathrm{Ex}(\Psi^2)$, $\mathrm{Ex}(\Psi^3)$ and $\mathrm{Ex}(\Psi^4)$, are related to the central moments $\mathrm{Vr}(\Psi)$, $\mathrm{Cb}(\Psi)$ and $\mathrm{Qr}(\Psi)$ by

$$\begin{aligned}
\mathrm{Ex}(\Psi^2) &= \mathrm{Ex}(\Psi)^2 + \mathrm{Vr}(\Psi)\,, \\
\mathrm{Ex}(\Psi^3) &= \mathrm{Ex}(\Psi)^3 + 3\,\mathrm{Ex}(\Psi)\,\mathrm{Vr}(\Psi) + \mathrm{Cb}(\Psi)\,, \\
\mathrm{Ex}(\Psi^4) &= \mathrm{Ex}(\Psi)^4 + 6\,\mathrm{Ex}(\Psi)^2\mathrm{Vr}(\Psi) + 4\,\mathrm{Ex}(\Psi)\,\mathrm{Cb}(\Psi) + \mathrm{Qr}(\Psi)\,,
\end{aligned}$$

a bit of calculation shows that

$$\begin{pmatrix} 1 & \mathrm{Ex}(\Psi) & \mathrm{Ex}(\Psi^2) \\ \mathrm{Ex}(\Psi) & \mathrm{Ex}(\Psi^2) & \mathrm{Ex}(\Psi^3) \\ \mathrm{Ex}(\Psi^2) & \mathrm{Ex}(\Psi^3) & \mathrm{Ex}(\Psi^4) \end{pmatrix} = \mathbf{L} \begin{pmatrix} 1 & 0 & \mathrm{Vr}(\Psi) \\ 0 & \mathrm{Vr}(\Psi) & \mathrm{Cb}(\Psi) \\ \mathrm{Vr}(\Psi) & \mathrm{Cb}(\Psi) & \mathrm{Qr}(\Psi) \end{pmatrix} \mathbf{L}^{\mathrm{T}}\,,$$

where

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \mathrm{Ex}(\Psi) & 1 & 0 \\ \mathrm{Ex}(\Psi)^2 & 2\,\mathrm{Ex}(\Psi) & 1 \end{pmatrix}.$$

Central Moment Inequalities (Central Moments)

We thereby see that condition (4.17) is equivalent to

$$\begin{pmatrix} 1 & 0 & \mathrm{Vr}(\Psi) \\ 0 & \mathrm{Vr}(\Psi) & \mathrm{Cb}(\Psi) \\ \mathrm{Vr}(\Psi) & \mathrm{Cb}(\Psi) & \mathrm{Qr}(\Psi) \end{pmatrix} \quad \text{is nonnegative definite}\,.$$

But this holds if and only if $\mathrm{Vr}(\Psi)$, $\mathrm{Cb}(\Psi)$ and $\mathrm{Qr}(\Psi)$ satisfy

$$\mathrm{Vr}(\Psi) \geq 0\,, \qquad \mathrm{Vr}(\Psi)\,\mathrm{Qr}(\Psi) - \mathrm{Cb}(\Psi)^2 - \mathrm{Vr}(\Psi)^3 \geq 0\,. \tag{4.18}$$

Remark. The moment inequalities (4.18) become strict if we make the reasonable assumption that for every sufficiently nice function $\psi(R)$ the probability density $q(R)$ satisfies

$$\int_{-1}^{\infty} \Big(p_0 + p_1\,\psi(R) + p_2\,\psi(R)^2\Big)^2\, q(R)\,\mathrm{d}R > 0\,,$$

for every $(p_0, p_1, p_2) \in \mathbb{R}^3$ with $(p_0, p_1, p_2) \neq (0, 0, 0)$.

Central Moment Inequalities (Sample Moments)

It might be hoped that our unbiased estimators for $\mathrm{Vr}(\Psi)$, $\mathrm{Cb}(\Psi)$ and $\mathrm{Qr}(\Psi)$ will satisfy an analog of the moment inequalities (4.18).

By mimicing our derivation of (4.18) it is easy to show that the sample moments $\mathrm{SmpVr}(\Psi)$, $\mathrm{SmpCb}(\Psi)$ and $\mathrm{SmpQr}(\Psi)$ given by (1.7b) and (1.8) satisfy

$$\begin{pmatrix} 1 & 0 & \mathrm{SmpVr}(\Psi) \\ 0 & \mathrm{SmpVr}(\Psi) & \mathrm{SmpCb}(\Psi) \\ \mathrm{SmpVr}(\Psi) & \mathrm{SmpCb}(\Psi) & \mathrm{SmpQr}(\Psi) \end{pmatrix} \quad \text{is positive definite},$$

which is equivalent to

$$\begin{aligned} \mathrm{SmpVr}(\Psi) &> 0\,, \\ \mathrm{SmpVr}(\Psi)\,\mathrm{SmpQr}(\Psi) - \mathrm{SmpCb}(\Psi)^2 - \mathrm{SmpVr}(\Psi)^3 &> 0\,. \end{aligned} \tag{4.19}$$

Central Moment Inequalities (Central Moment Estimators)

We would like our central moment estimators to satisfy

$$\begin{pmatrix} 1 & 0 & \widehat{\mathrm{Vr}}(\Psi) \\ 0 & \widehat{\mathrm{Vr}}(\Psi) & \widehat{\mathrm{Cb}}(\Psi) \\ \widehat{\mathrm{Vr}}(\Psi) & \widehat{\mathrm{Cb}}(\Psi) & \widehat{\mathrm{Qr}}(\Psi) \end{pmatrix} \quad \text{is positive definite}\,,$$

These estimators will have the general form

$$\begin{aligned} \widehat{\mathrm{Vr}}(\Psi) = \alpha\, \mathrm{SmpVr}(\Psi)\,, \qquad \widehat{\mathrm{Cb}}(\Psi) = \beta\, \mathrm{SmpCb}(\Psi)\,, \\ \widehat{\mathrm{Qr}}(\Psi) = \gamma\, \mathrm{SmpQr}(\Psi) - \delta\, \mathrm{SmpVr}(\Psi)^2\,, \end{aligned} \tag{4.20}$$

for some positive α, β, γ and δ. Because $\mathrm{SmpVr}(\Psi) > 0$, proving the positive definiteness comes down to showing that

$$\begin{aligned} \alpha\gamma\, \mathrm{SmpVr}(\Psi)\, \mathrm{SmpQr}(\Psi) > \beta^2\, \mathrm{SmpCb}(\Psi)^2 \\ + (\alpha^3 + \alpha\delta)\, \mathrm{SmpVr}(\Psi)^3\,. \end{aligned} \tag{4.21}$$

Central Moment Inequalities (Central Moment Estimators)

Because by (4.19)

$$\mathrm{SmpVr}(\Psi)\,\mathrm{SmpQr}(\Psi) > \mathrm{SmpCb}(\Psi)^2 + \mathrm{SmpVr}(\Psi)^3\,,$$

we see that (4.21) is satisfied when α, β, γ and δ satisfy

$$\alpha\gamma \geq \beta^2\,, \qquad \gamma \geq \alpha^2 + \delta\,. \tag{4.22}$$

For uniform weights the unbiased estimators given by (1.7), (2.12) and (3.15) are specified by

$$\begin{aligned} \alpha &= \frac{D}{D-1}\,, & \beta &= \frac{D^2}{(D-1)(D-2)}\,, \\ \gamma &= \frac{D(D^2-2D+3)}{(D-1)(D-2)(D-3)}\,, & \delta &= \frac{3D(2D-3)}{(D-1)(D-2)(D-3)}\,. \end{aligned} \tag{4.23}$$

These do not satisfy either of the inequalities in (4.22), so it is not clear that our unbiased estimators satisfy the moment inequalities (4.21).

Central Moment Inequalities (New Quartic Estimators)

Both of the inequalities in (4.22) are satisfied for every $D > 3$ by choices

$$\alpha = \frac{D}{D-1}\,, \qquad \beta = \frac{D^2}{(D-1)(D-2)}\,, \qquad \gamma = \frac{D^3}{(D-1)(D-2)(D-3)}\,, \qquad \delta = \frac{4D}{(D-2)(D-3)}\,. \tag{4.24}$$

When these choices are placed into the general form (4.20) they yield

$$\widehat{\mathrm{Vr}}(\Psi) = \frac{D}{D-1}\,\mathrm{SmpVr}(\Psi)\,, \tag{4.25a}$$

$$\widehat{\mathrm{Cb}}(\Psi) = \frac{D^2}{(D-1)(D-2)}\,\mathrm{SmpCb}(\Psi)\,, \tag{4.25b}$$

$$\begin{aligned} \widehat{\mathrm{Qr}}(\Psi) &= \frac{D^3}{(D-1)(D-2)(D-3)}\,\mathrm{SmpQr}(\Psi) - \frac{4D}{(D-2)(D-3)}\,\mathrm{SmpVr}(\Psi)^2 \\ &= \frac{D^3}{(D-1)(D-2)(D-3)}\,\mathrm{SmpDs}(\Psi) + \frac{D(D-2)}{(D-1)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,. \end{aligned} \tag{4.25c}$$

These recover the unbiased estimators for $\widehat{\mathrm{Vr}}(\Psi)$ and $\widehat{\mathrm{Cb}}(\Psi)$, but give a new estimator for $\widehat{\mathrm{Qr}}(\Psi)$.

Central Moment Inequalities (New Quartic Estimators)

We can recast (4.25c) as the dispersion decomposition

$$\widehat{\mathrm{Qr}}(\Psi) = \widehat{\mathrm{Ds}}(\Psi) + \widehat{\mathrm{Sq}}(\Psi)\,, \tag{4.26a}$$

where

$$\widehat{\mathrm{Ds}}(\Psi) = \frac{D^3}{(D-1)(D-2)(D-3)}\,\mathrm{SmpDs}(\Psi)\,, \tag{4.26b}$$

$$\widehat{\mathrm{Sq}}(\Psi) = \frac{D(D-2)}{(D-1)(D-3)}\,\mathrm{SmpVr}(\Psi)^2\,. \tag{4.26c}$$

By comparing these estimators with the unbiased estimators given by (3.16) we see that these are biased estimators that tend to overestimate. However these are better than the unbiased estimators in two ways:

- $\widehat{\mathrm{Vr}}(\Psi)$, $\widehat{\mathrm{Cb}}(\Psi)$ and $\widehat{\mathrm{Qr}}(\Psi)$ satisfy the moment inequalities (4.21);
- $\widehat{\mathrm{Ds}}(\Psi)$ and $\widehat{\mathrm{Sq}}(\Psi)$ are positive whenever $\mathrm{SmpDs}(\Psi)$ and $\mathrm{SmpVr}(\Psi)$ are positive, which is usually the case.

Variance Certainty Metrics (Introduction)

Variance Certainty Metrics. Estimators of $\mathrm{Ds}(\Psi)$ and $\mathrm{Sq}(\Psi)$ lead to an estimator for the variance of $\widehat{\mathrm{Vr}}(\Psi)$. Recall that if $\mathrm{Ex}(\Psi^4) < \infty$ then the unbiased variance estimator $\widehat{\mathrm{Vr}}(\Psi)$ has variance given by (1.6) as

$$\mathrm{Vr}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big) = \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\mathrm{Ds}(\Psi) + 2\frac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\mathrm{Vr}(\Psi)^2\,,$$

where $\bar{w}$, $\overline{w^2}$, and $\overline{w^3}$ are given by (1.1b). Therefore an estimator for the variance of $\widehat{\mathrm{Vr}}(\Psi)$ is

$$\widehat{\mathrm{Vr}}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big) = \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\widehat{\mathrm{Ds}}(\Psi) + 2\,\frac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\widehat{\mathrm{Sq}}(\Psi)\,, \qquad (5.27)$$

where $\widehat{\mathrm{Ds}}(\Psi)$ and $\widehat{\mathrm{Sq}}(\Psi)$ are estimators $\mathrm{Ds}(\Psi)$ and $\mathrm{Sq}(\Psi)$. If they are the unbiased estimators determined by (3.14) then the estimator (5.27) will also be unbiased.

Variance Certainty Metrics (SNR)

The standard deviation of $\widehat{\mathrm{Vr}}(\Psi)$ then has the (biased) estimator

$$\widehat{\mathrm{SD}}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big) = \sqrt{\widehat{\mathrm{Vr}}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big)}\,,$$

where $\widehat{\mathrm{Vr}}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big)$ is given by (5.27).

A signal-to-noise ratio for $\widehat{\mathrm{Vr}}(\Psi)$ is

$$\mathrm{SNR}(\mathrm{Vr}(\Psi)) = \frac{\widehat{\mathrm{Vr}}(\Psi)}{\widehat{\mathrm{SD}}\Big(\widehat{\mathrm{Vr}}(\Psi)\Big)}\,. \tag{5.28}$$

The larger this SNR, the more certainty we have in the estimator $\widehat{\mathrm{Vr}}(\Psi)$.

Variance Certainty Metrics (Metrics)

If a signal-to-noise ratio ratio of at least r_o is desired then this certainty can be scored by the metric

$$\omega^{\widehat{\mathrm{Vr}}(\Psi)} = \frac{r_o^2}{r_o^2 + \mathrm{SNR}(\mathrm{Vr}(\Psi))^2} = \frac{r_o^2 \widehat{\mathrm{Vr}}\left(\widehat{\mathrm{Vr}}(\Psi)\right)}{r_o^2 \widehat{\mathrm{Vr}}\left(\widehat{\mathrm{Vr}}(\Psi)\right) + \widehat{\mathrm{Vr}}(\Psi)^2} \tag{5.29}$$

$$= \frac{\dfrac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\widehat{\mathrm{Ds}}(\Psi) + 2\,\dfrac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\widehat{\mathrm{Sq}}(\Psi)}{\dfrac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\widehat{\mathrm{Ds}}(\Psi) + 2\,\dfrac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\widehat{\mathrm{Sq}}(\Psi) + \dfrac{\widehat{\mathrm{Vr}}(\Psi)^2}{r_o^2}}.$$

The smaller this is, the more certainty we have in the estimator $\widehat{\mathrm{Vr}}(\Psi)$. The closer it is to 1, the less certainty we have in the estimator $\widehat{\mathrm{Vr}}(\Psi)$.

Variance Certainty Metrics (Uniform)

For unifrom weights this metric becomes

$$\omega^{\widehat{\mathrm{Vr}}(\Psi)} = \frac{\frac{1}{D}\,\widehat{\mathrm{Ds}}(\Psi) + \frac{2}{D(D-1)}\,\widehat{\mathrm{Sq}}(\Psi)}{\frac{1}{D}\,\widehat{\mathrm{Ds}}(\Psi) + \frac{2}{D(D-1)}\,\widehat{\mathrm{Sq}}(\Psi) + \frac{\widehat{\mathrm{Vr}}(\Psi)^2}{r_o^2}}\,, \tag{5.30a}$$

where $\widehat{\mathrm{Vr}}(\Psi)$ is the unbiased variance estimator

$$\widehat{\mathrm{Vr}}(\Psi) = \frac{D}{D-1}\,\mathrm{SmpVr}(\Psi)\,, \tag{5.30b}$$

and $\widehat{\mathrm{Ds}}(\Psi)$ and $\widehat{\mathrm{Sq}}(\Psi)$ are either

- the unbiased estimators given by (3.16b) and (3.16c), or
- the biased estimators given by (4.26b) and (4.26c).

Variance Certainty Metrics (Example)

Example. Let $\{r_i(d)\}_{d=1}^D$ be a return history for asset i. Set

$$m_i = \widehat{\mathrm{Ex}}(r_i) = \frac{1}{D}\sum_{d=1}^{D} r_i(d)\,,$$

$$v_i = \mathrm{SmpVr}(r_i) = \frac{1}{D}\sum_{d=1}^{D} \big(r_i(d) - m_i\big)^2\,,$$

$$q_i = \mathrm{SmpDs}(r_i) = \frac{1}{D}\sum_{d=1}^{D} \Big(\big(r_i(d) - m_i\big)^2 - v_i\Big)^2\,.$$

Using (5.30b) and the biased estimators (4.26b) and (4.26c) gives

$$\hat{\xi}_i = \widehat{\mathrm{Vr}}(r_i) = \frac{D}{D-1}\,v_i\,,$$

$$\widehat{\mathrm{Ds}}(r_i) = \frac{D^3}{(D-1)(D-2)(D-3)}\,q_i\,, \qquad \widehat{\mathrm{Sq}}(r_i) = \frac{D(D-2)}{(D-1)(D-3)}\,{v_i}^2\,.$$

Variance Certainty Metrics (Example)

Then because

$$\widehat{\mathrm{Vr}}\left(\hat{\xi}_i\right) = \frac{1}{D} \frac{D^3}{(D-1)(D-2)(D-3)} q_i + \frac{2}{D(D-1)} \frac{D(D-2)}{(D-1)(D-3)} v_i^2$$
$$= \frac{D^2}{(D-1)(D-2)(D-3)} q_i + \frac{2(D-2)}{(D-1)^2(D-3)} v_i^2 ,$$

the signal-to-noise ratio (5.28) becomes

$$\mathrm{SNR}(\xi_i) = \frac{\frac{D}{D-1} v_i}{\sqrt{\frac{D^2}{(D-1)(D-2)(D-3)} q_i + \frac{2(D-2)}{(D-1)^2(D-3)} v_i^2}} = \frac{v_i}{\sqrt{\frac{D-1}{(D-2)(D-3)} q_i + \frac{2(D-2)}{D^2(D-3)} v_i^2}} . \tag{5.31}$$

Variance Certainty Metrics (Example)

By placing $\mathrm{SNR}(\xi_i)$ given by (5.31) into the general metric formula (5.29) we arrive at the metric

$$\omega_i^{\hat{\xi}} = \frac{\frac{D-1}{(D-2)(D-3)}\, q_i + \frac{2(D-2)}{D^2(D-3)}\, v_i^2}{\frac{D-1}{(D-2)(D-3)}\, q_i + \frac{2(D-2)}{D^2(D-3)}\, v_i^2 + \frac{1}{r_o^2}\, v_i^2} . \tag{5.32}$$

This is the analog for the return variance estimator $\hat{\xi}_i$ of the metric for the return mean estimator $\hat{\mu}_i$ given by

$$\omega_i^{\hat{\mu}} = \frac{\frac{1}{D-1} v_i}{\frac{1}{D-1} v_i + \frac{1}{r_o^2}\, m_i^2} . \tag{5.33}$$

The smaller these metrics are, the greater certainty we have in their associated estimator.