Portfolios that Contain Risky Assets
5.4. Limited-Leverage and Return Bounds

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Part I: Portfolio Models

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5. Models for Limited-Leverage Portfolios

Limited-Leverage and Return Bounds

Downside and Upside Potentials: Λ, Ω, $\mathcal{M}$ & Π^ℓ

Recall that we have the containments

$$\Lambda \subset \Omega \subset \mathcal{M}\,, \qquad \Lambda \subset \Pi^\ell \subset \mathcal{M} \quad \text{for every } \ell \in (0,\infty)\,, \tag{1.1}$$

where

- Λ is the set of long Markowitz allocations,
- Ω is the set of solvent Markowitz allocations,
- $\mathcal{M}$ is the set of all Markowitz allocations,
- Π^ℓ is the set of ℓ-limited leverage Markowitz allocations.

Here we quantify the relationship between Π^ℓ and Ω. Specifically, we will characterize those $\ell \in (0,\infty)$ for which $\Pi^\ell \subset \Omega$. This gives the leverage ratio limit ℓ that insures we are working with solvent Markowitz allocations, which is required for increasing returns to align with increasing reward!

Downside and Upside Potentials: $\delta(\mathbf{f})$ & $\upsilon(\mathbf{f})$

Key tools for proving this characterization are the downside and upside potentials. Recall that given a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ we defined the downside and upside potentials for every Markowitz allocation $\mathbf{f} \in \mathcal{M}$ by

$$\delta(\mathbf{f}) = \max_d \Big\{ -\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \Big\}, \qquad \upsilon(\mathbf{f}) = \max_d \Big\{ \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \Big\}. \tag{1.2}$$

We showed that these satisfy the inequalities

$$0 < \delta(\mathbf{f}) + \mu(\mathbf{f}) < \delta(\mathbf{f}) + \upsilon(\mathbf{f}), \tag{1.3}$$

where $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$ is the return mean. We proved $\mathbf{f} \in \Omega$ if and only if $\delta(\mathbf{f}) < 1$. This motivated the developement of the liquidity function

$$\omega^{\delta}(\mathbf{f}) = \begin{cases} \dfrac{\delta(\mathbf{f}) + \mu(\mathbf{f})}{1 + \mu(\mathbf{f})} & \text{if } \delta(\mathbf{f}) < 1, \\ 1 & \text{if } \delta(\mathbf{f}) \geq 1. \end{cases} \tag{1.4}$$

It takes values in $(0, 1]$. As it approaches 1, $\mathbf{f}$ comes closer to leaving Ω.

Downside and Upside Potentials: Λ, δ_{mx} & υ_{mx}

We define the downside and upside potentials of the set Λ by

$$\delta_{\mathrm{mx}} = \max\Big\{\delta(\mathbf{f}) \,:\, \mathbf{f} \in \Lambda\Big\}, \qquad \upsilon_{\mathrm{mx}} = \max\Big\{\upsilon(\mathbf{f}) \,:\, \mathbf{f} \in \Lambda\Big\}. \tag{1.5}$$

Because $\delta(\mathbf{f})$ and $\upsilon(\mathbf{f})$ given by (1.2) are each the maximum of a finite set of linear functions, they each must be convex over the convex set Λ. Furthermore, we know $\Lambda = \mathrm{Hull}(\mathcal{E})$, where

$$\mathcal{E} = \Big\{\mathbf{e}_i \,:\, i = 1, \cdots, N\Big\}. \tag{1.6}$$

Therefore the maximums over Λ in (1.5) reduce to

$$\delta_{\mathrm{mx}} = \max\Big\{\delta(\mathbf{f}) \,:\, \mathbf{f} \in \mathcal{E}\Big\}, \qquad \upsilon_{\mathrm{mx}} = \max\Big\{\upsilon(\mathbf{f}) \,:\, \mathbf{f} \in \mathcal{E}\Big\}. \tag{1.7}$$

Downside and Upside Potentials: Λ, δ_{mx} & υ_{mx}

Because

$$\mathbf{r}(d)^{\mathrm{T}}\mathbf{e}_i = r_i(d)\,,$$

we see that

$$\delta(\mathbf{e}_i) = \max_d\{-r_i(d)\}\,, \qquad \upsilon(\mathbf{e}_i) = \max_d\{r_i(d)\}\,.$$

Therefore (1.7) becomes

$$\delta_{\mathrm{mx}} = \max_{i,d}\{-r_i(d)\}\,, \qquad \upsilon_{\mathrm{mx}} = \max_{i,d}\{r_i(d)\}\,. \tag{1.8}$$

Because each individual asset is solvent, we have

$$0 < 1 + r_i(d) \qquad \text{for every } i \text{ and } d\,,$$

whereby we see from (1.8) that

$$0 < 1 - \delta_{\mathrm{mx}} \leq 1 + r_i(d) \leq 1 + \upsilon_{\mathrm{mx}} \quad \text{for every } i \text{ and } d\,. \tag{1.9}$$

Downside and Upside Potentials: Π^ℓ, $\delta^\ell_{\rm mx}$ & $\upsilon^\ell_{\rm mx}$

We define the downside and upside potentials of the set Π^ℓ by

$$\delta^\ell_{\rm mx} = \max\Big\{\delta(\mathbf{f}) \,:\, \mathbf{f} \in \Pi^\ell\Big\}\,, \qquad \upsilon^\ell_{\rm mx} = \max\Big\{\upsilon(\mathbf{f}) \,:\, \mathbf{f} \in \Pi^\ell\Big\}\,. \qquad (1.10)$$

Because $\delta(\mathbf{f})$ and $\upsilon(\mathbf{f})$ given by (1.2) are each the maximum of a finite set of linear functions, they each must be convex over the convex set Π^ℓ. Furthermore, we know $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$, where

$$\mathcal{E}^\ell = \Big\{\mathbf{e}^\ell_{ij} \,:\, i,j = 1, \cdots, N\,,\, j \neq i\Big\}\,. \qquad (1.11)$$

Therefore the maximums over Π^ℓ in (1.10) reduce to

$$\delta^\ell_{\rm mx} = \max\Big\{\delta(\mathbf{f}) \,:\, \mathbf{f} \in \mathcal{E}^\ell\Big\}\,, \qquad \upsilon^\ell_{\rm mx} = \max\Big\{\upsilon(\mathbf{f}) \,:\, \mathbf{f} \in \mathcal{E}^\ell\Big\}\,. \qquad (1.12)$$

Downside and Upside Potentials: Π^ℓ, $\delta^\ell_{\mathrm{mx}}$ & $\upsilon^\ell_{\mathrm{mx}}$

Because

$$\mathbf{r}(d)^{\mathrm{T}}\mathbf{e}^\ell_{ij} = (1+\ell)\, r_i(d) - \ell\, r_j(d)\,,$$

we see that

$$\delta(\mathbf{e}^\ell_{ij}) = \max_d \Big\{ \ell\, r_j(d) - (1+\ell)\, r_i(d) \Big\}\,,$$
$$\upsilon(\mathbf{e}^\ell_{ij}) = \max_d \Big\{ (1+\ell)\, r_i(d) - \ell\, r_i(d) \Big\}\,.$$

Therefore (1.12) becomes

$$\delta^\ell_{\mathrm{mx}} = \max_{i,j\neq i} \left\{ \max_d \Big\{ \ell\, r_j(d) - (1+\ell)\, r_i(d) \Big\} \right\}\,,$$
$$\upsilon^\ell_{\mathrm{mx}} = \max_{i,j\neq i} \left\{ \max_d \Big\{ (1+\ell)\, r_i(d) - \ell\, r_i(d) \Big\} \right\}\,.$$

Downside and Upside Potentials: Π^ℓ, $\delta^\ell_{\rm mx}$ & $\upsilon^\ell_{\rm mx}$

By observing that the $j = i$ terms can be included (Do you see why?) and by exchanging the order of the maximizations we have

$$\delta^\ell_{\rm mx} = \max_d \left\{ \max_{i,j} \left\{ \ell\, r_j(d) - (1+\ell)\, r_i(d) \right\} \right\} ,$$
$$\upsilon^\ell_{\rm mx} = \max_d \left\{ \max_{i,j} \left\{ (1+\ell)\, r_i(d) - \ell\, r_j(d) \right\} \right\} .$$

Therefore

$$\begin{aligned} \delta^\ell_{\rm mx} &= \max_d \left\{ \ell\, r_{\rm mx}(d) - (1+\ell)\, r_{\rm mn}(d) \right\} , \\ \upsilon^\ell_{\rm mx} &= \max_d \left\{ (1+\ell)\, r_{\rm mx}(d) - \ell\, r_{\rm mn}(d) \right\} , \end{aligned} \tag{1.13}$$

where $r_{\rm mn}(d)$ and $r_{\rm mx}(d)$ are the *extreme returns* on day d defined by

$$r_{\rm mn}(d) = \min_i \{ r_i(d) \} , \qquad r_{\rm mx}(d) = \max_i \{ r_i(d) \} . \tag{1.14}$$

Downside and Upside Potentials: $r_{\rm mn}(d)$ & $r_{\rm mx}(d)$

Remark. The extreme returns $r_{\rm mn}(d)$ and $r_{\rm mx}(d)$ defined by (1.14) represent the returns of the worst and best performing assets on day d. This data is not captured by the mean-variance statistics of **m** and **V**. They arose naturally when computing formulas (1.12) for $\delta_{\rm mx}^{\ell}$ and $\upsilon_{\rm mx}^{\ell}$. They play a central role in our subsequent analysis.

By using the fact that $\Lambda = \mathrm{Hull}(\mathcal{E})$, it can be shown for every d that

$$\big[r_{\rm mn}(d), r_{\rm mx}(d)\big] = \Big\{\mathbf{r}(d)^{\rm T}\mathbf{f} \,:\, \mathbf{f} \in \Lambda\Big\}\,.$$

Because (1.8) and (1.14) imply that

$$-\delta_{\rm mx} = \min_{d}\{r_{\rm mn}(d)\}\,, \qquad \upsilon_{\rm mx} = \max_{d}\{r_{\rm mx}(d)\}\,,$$

we see that

$$\bigcup_{d}\big[r_{\rm mn}(d), r_{\rm mx}(d)\big] \subset \big[-\delta_{\rm mx}, \upsilon_{\rm mx}\big]\,.$$

We get equality when the intervals in the union leave no gaps.

Downside and Upside Potentials: $r^{\ell}_{\mathrm{mn}}(d)$ & $r^{\ell}_{\mathrm{mx}}(d)$

$$
\begin{aligned}
r^{\ell}_{\mathrm{mn}}(d) &= r_{\mathrm{mn}}(d) - \ell\left(r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d)\right), \\
r^{\ell}_{\mathrm{mx}}(d) &= r_{\mathrm{mx}}(d) + \ell\left(r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d)\right),
\end{aligned}
$$

By using the fact that $\Pi^{\ell} = \mathrm{Hull}(\mathcal{E}^{\ell})$, it can be shown for every d that

$$
\left[r^{\ell}_{\mathrm{mn}}(d), r^{\ell}_{\mathrm{mx}}(d)\right] = \left\{\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \,:\, \mathbf{f} \in \Pi^{\ell}\right\}.
$$

Because (1.8) and (1.14) imply that

$$
-\delta^{\ell}_{\mathrm{mx}} = \min_{d}\left\{r^{\ell}_{\mathrm{mn}}(d)\right\}, \qquad \upsilon^{\ell}_{\mathrm{mx}} = \max_{d}\left\{r^{\ell}_{\mathrm{mx}}(d)\right\},
$$

we see that

$$
\bigcup_{d}\left[r^{\ell}_{\mathrm{mn}}(d), r^{\ell}_{\mathrm{mx}}(d)\right] \subset \left[-\delta^{\ell}_{\mathrm{mx}}, \upsilon^{\ell}_{\mathrm{mx}}\right].
$$

We get equality when the intervals in the union leave no gaps.

Downside and Upside Potentials: $r^{\ell}_{\mathrm{mn}}(d)$ & $r^{\ell}_{\mathrm{mx}}(d)$

Because each individual asset is solvent, we know that

$$-1 < r_{\mathrm{mn}}(d) \qquad \text{for every } d\,. \tag{1.15a}$$

We will make the assumption that

$$r_{\mathrm{mn}}(d) < r_{\mathrm{mx}}(d) \qquad \text{for every } d\,. \tag{1.15b}$$

This excludes only the unlikely event that there is a day in which every asset has the same return!

Downside and Upside Potentials: $r_{\rm mn}(d)$ & $r_{\rm mx}(d)$

It is clear from (1.9) and (1.15) that $r_{\rm mn}(d)$ and $r_{\rm mx}(d)$ satisfy the bounds

$$-1 < -\delta_{\rm mx} \leq r_{\rm mn}(d) < r_{\rm mx}(d) \leq \upsilon_{\rm mx} \,,$$

where $\delta_{\rm mx}$ and $\upsilon_{\rm mx}$ are given by (1.8). Then $\delta^{\ell}_{\rm mx}$ and $\upsilon^{\ell}_{\rm mx}$ given by (1.13) are bounded as

$$\begin{aligned} \delta^{\ell}_{\rm mx} &\leq \delta_{\rm mx} + \ell\,(\delta_{\rm mx} + \upsilon_{\rm mx})\,, \\ \upsilon^{\ell}_{\rm mx} &\leq \upsilon_{\rm mx} + \ell\,(\delta_{\rm mx} + \upsilon_{\rm mx})\,. \end{aligned} \tag{1.16}$$

These bounds are rough. They will be improved soon.

Remark. On most trading days a well-balanced portfolio will have assets that decrease and assets that increase in value. For such days we will have

$$-1 < r_{\rm mn}(d) < 0 < r_{\rm mx}(d)\,.$$

For small portfolios it is not uncommon to have

- $-1 < r_{\rm mn}(d) < r_{\rm mx}(d) < 0$ on days when the market goes down, or
- $0 < r_{\rm mn}(d) < r_{\rm mx}(d)$ on days when the market goes up.

Containments: Ω, $\Omega_{\bar\delta}$ & $\Omega_{\bar\delta}^{\bar\upsilon}$

Recall that the set of solvent Markowitz allocations is

$$\Omega = \Big\{ \mathbf{f} \in \mathcal{M} \,:\, \delta(\mathbf{f}) < 1 \Big\} . \tag{2.17a}$$

We now introduce the set of Markowitz allocations with downside potential no greater than $\bar\delta \in (0, \infty)$ as

$$\Omega_{\bar\delta} = \Big\{ \mathbf{f} \in \mathcal{M} \,:\, \delta(\mathbf{f}) \leq \bar\delta \Big\} , \tag{2.17b}$$

and the set of Markowitz allocations with downside potential no greater than $\bar\delta \in (0, \infty)$ and upside potential on greater than $\bar\upsilon \in (0, \infty)$ as

$$\Omega_{\bar\delta}^{\bar\upsilon} = \Big\{ \mathbf{f} \in \mathcal{M} \,:\, \delta(\mathbf{f}) \leq \bar\delta \,,\, \upsilon(\mathbf{f}) \leq \bar\upsilon \Big\} . \tag{2.17c}$$

Here we will give bounds on the leverage limit ℓ that will characterize when $\Pi^\ell \subset \Omega_{\bar\delta}^{\bar\upsilon}$, when $\Pi^\ell \subset \Omega_{\bar\delta}$, and when $\Pi^\ell \subset \Omega$.

Containments: $\delta^\ell_{\rm mx}$ & $\upsilon^\ell_{\rm mx}$ Bound Characterizations

We begin with the following characterizations.

Fact 1. For every $\bar\delta > \delta_{\rm mx}$ we have

$$\delta^\ell_{\rm mx} \leq \bar\delta \quad \iff \quad \ell \leq \min_d \left\{ \frac{\bar\delta + r_{\rm mn}(d)}{r_{\rm mx}(d) - r_{\rm mn}(d)} \right\} . \tag{2.18a}$$

For every $\bar\upsilon > \upsilon_{\rm mx}$ we have

$$\upsilon^\ell_{\rm mx} \leq \bar\upsilon \quad \iff \quad \ell \leq \min_d \left\{ \frac{\bar\upsilon - r_{\rm mx}(d)}{r_{\rm mx}(d) - r_{\rm mn}(d)} \right\} . \tag{2.18b}$$

Remark. Any day that assumption (1.15b) breaks down would simply be excluded from the minimizations in (2.18). Our previous asssumption that **m** and **1** are not proportional already excludes the possability that assumption (1.15b) breaks down every day. Assumption (1.15b) avoids having to handle exceptional cases in formulas (2.18).

Containments: $\delta^{\ell}_{\mathrm{mx}}$ Bound Characterization Proof

Proof of (2.19a). Let $\bar{\delta} > \delta_{\mathrm{mx}}$. From (1.13) we have

$$\delta^{\ell}_{\mathrm{mx}} = \max_{d}\Big\{\ell\, r_{\mathrm{mx}}(d) - (1+\ell)\, r_{\mathrm{mn}}(d)\Big\}\,.$$

It is clear from this that $\delta^{\ell}_{\mathrm{mx}} \leq \bar{\delta}$ if and only if

$$\ell\, r_{\mathrm{mx}}(d) - (1+\ell)\, r_{\mathrm{mn}}(d) \leq \bar{\delta} \qquad \text{for every } d\,,$$

which holds if and only if

$$\ell \leq \frac{\bar{\delta} + r_{\mathrm{mn}}(d)}{r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d)} \qquad \text{for every } d\,,$$

which is equivalent to the minimum condition in (2.18a). □

Containments: $v_{\rm mx}^{\ell}$ Bound Characterization Proof

Proof of (2.19b). Let $\bar{v} > v_{\rm mx}$. From (1.13) we have

$$v_{\rm mx}^{\ell} = \max_{d} \Big\{ (1+\ell)\, r_{\rm mx}(d) - \ell\, r_{\rm mn}(d) \Big\} .$$

It is clear from this that $v_{\rm mx}^{\ell} \leq \bar{v}$ if and only if

$$(1+\ell)\, r_{\rm mx}(d) - \ell\, r_{\rm mn}(d) \leq \bar{v} \qquad \text{for every } d ,$$

which holds if and only if

$$\ell \leq \frac{\bar{v} - r_{\rm mx}(d)}{r_{\rm mx}(d) - r_{\rm mn}(d)} \qquad \text{for every } d ,$$

which is equivalent to the minimum condition in (2.18b). □

Containments: $\Pi^\ell \subset \Omega_{\bar\delta}^{\bar\upsilon}$

We are now ready to state and prove our containment characterizations.

Fact 2. For every $\bar\delta > \delta_{\rm mx}$ and $\bar\upsilon > \upsilon_{\rm mx}$ we have

$$\Pi^\ell \subset \Omega_{\bar\delta}^{\bar\upsilon} \quad \Longleftrightarrow \quad \ell \leq \ell_{\bar\delta}^{\bar\upsilon}\,, \tag{2.19a}$$

where $\ell_{\bar\delta}^{\bar\upsilon}$ is defined by

$$\ell_{\bar\delta}^{\bar\upsilon} = \min_d \left\{ \frac{\bar\delta + r_{\rm mn}(d)}{r_{\rm mx}(d) - r_{\rm mn}(d)}\,, \frac{\bar\upsilon - r_{\rm mx}(d)}{r_{\rm mx}(d) - r_{\rm mn}(d)} \right\}. \tag{2.19b}$$

Proof. Let $\bar\delta > \delta_{\rm mx}$ and $\bar\upsilon > \upsilon_{\rm mx}$. **Fact 1** shows that $\ell \leq \ell_{\bar\delta}^{\bar\upsilon}$ if and only if

$$\delta_{\rm mx}^\ell \leq \bar\delta \quad \text{and} \quad \upsilon_{\rm mx}^\ell \leq \bar\upsilon\,,$$

which by definitions (1.10) and (2.17c) holds if and only if $\Pi^\ell \subset \Omega_{\bar\delta}^{\bar\upsilon}$. □

Containments: $\Pi^\ell \subset \Omega_{\bar{\delta}}$

Fact 3. For every $\bar{\delta} > \delta_{\mathrm{mx}}$ we have

$$\Pi^\ell \subset \Omega_{\bar{\delta}} \quad \Longleftrightarrow \quad \ell \le \ell_{\bar{\delta}}\,, \tag{2.20a}$$

where $\ell_{\bar{\delta}}$ is defined by

$$\ell_{\bar{\delta}} = \min_{d} \left\{ \frac{\bar{\delta} + r_{\mathrm{mn}}(d)}{r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d)} \right\} . \tag{2.20b}$$

Proof. Let $\bar{\delta} > \delta_{\mathrm{mx}}$. **Fact 1** shows that $\ell \le \ell_{\bar{\delta}}$ if and only if

$$\delta_{\mathrm{mx}}^{\ell} \le \bar{\delta}\,,$$

which by definitions (1.10) and (2.17b) holds if and only if $\Pi^\ell \subset \Omega_{\bar{\delta}}$. □

Containments: $\Pi^\ell \subset \Omega$

Fact 4. We have

$$\Pi^\ell \subset \Omega \quad \Longleftrightarrow \quad \ell < \ell_1\,, \tag{2.21}$$

where ℓ_1 is given by (2.20b).

Proof. Because

$$\Omega = \bigcup \left\{ \Omega_{\bar{\delta}} \,:\, \bar{\delta} \in (\delta_{\mathrm{mx}}, 1) \right\},$$

we see that $\Pi^\ell \subset \Omega$ if and only if there exists $\bar{\delta} \in (\delta_{\mathrm{mx}}, 1)$ such that $\Pi^\ell \subset \Omega_{\bar{\delta}}$, which by **Fact 3** is the case if and only if $\ell \leq \ell_{\bar{\delta}}$. But this will be the case if and only if $\ell < \ell_1$. □

Containments: $\Pi^\ell \subset \Omega$

We can restate **Fact 4** as

$$\Pi^\ell \subset \Omega \quad \Longleftrightarrow \quad \ell < \ell_\Omega \,, \tag{2.22a}$$

where ℓ_Ω is given by

$$\ell_\Omega = \min_d \left\{ \frac{1 + r_{\mathrm{mn}}(d)}{r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d)} \right\} = \frac{\rho_\Omega}{1 - \rho_\Omega} \,, \tag{2.22b}$$

with ρ_Ω defined by

$$\rho_\Omega = \min_d \left\{ \frac{1 + r_{\mathrm{mn}}(d)}{1 + r_{\mathrm{mx}}(d)} \right\} . \tag{2.22c}$$

Notice that $\rho_\Omega \in (0, 1)$. The ratios upon which it depends can be near 1 on days when the market moves down or up substantially, and can be smallest on days when the market does not make a major move.

Leverage Limit Selection: Uses of ℓ_Ω & ρ_Ω

There are at least two potential uses of ℓ_Ω and ρ_Ω.

- It might be used to monitor market stress. The closer ρ_Ω gets to 1, the larger ℓ_Ω gets, which means it gets less likely that margins of leveraged portfolios will be called. The closer ρ_Ω gets to 0, the more stress the market is under. This use suggests introducing the metric

$$\omega_\Omega = 1 - \rho_\Omega \,.$$

  It is wise to use a large well-diversified portfolio when computing ρ_Ω for this purpose.

- It might be used to select a leverage ratio limit for your own portfolio. It is wise to use a long history when computing ρ_Ω for this purpose.

It is the second use that we will explore here.

Leverage Limit Selection: Uses of ℓ_Ω

For a Markowitz allocation $\mathbf{f} \in \mathcal{M}$ the general ideas are as follows.

- The leverage ratio $\lambda(\mathbf{f})$ must be maintain below a certain limit that is determined by the lender to protect the lender's interests.
- The downside potential $\delta(\mathbf{f})$ quantifies the potential loss in value of the portfolio in a single day. The larger it is, the greater the risk to the lender's interests, so the more likely the leverage ratio limit will be decreased.
- $\delta(\mathbf{f})$ often increases with $\lambda(\mathbf{f})$.
- Both $\lambda(\mathbf{f})$ and $\delta(\mathbf{f})$ have uncertainty associated with the choice of $\mathbf{f}$, and $\delta(\mathbf{f})$ has the additional uncertainty associated with the relevance to the current situation of the return history used to compute it.

Leverage Limit Selection: Uses of ℓ_Ω

Recall from (1.13) that $\delta^{\ell}_{\mathrm{mx}}$ is given by

$$\delta^{\ell}_{\mathrm{mx}} = \max_{d} \Big\{ \ell \, r_{\mathrm{mx}}(d) - (1+\ell)\, r_{\mathrm{mn}}(d) \Big\} . \tag{3.23}$$

Because we see that

- $\delta^{\ell}_{\mathrm{mx}} = \delta_{\mathrm{mx}}$ when $\ell = 0$,
- $\delta^{\ell}_{\mathrm{mx}} = 1$ when $\ell = \ell_\Omega$,
- the mapping $\ell \mapsto \delta^{\ell}_{\mathrm{mx}}$ is convex.

Therefore we can bound $\delta^{\ell}_{\mathrm{mx}}$ above over $[0, \ell_\Omega]$ by the linear interpolant

$$\delta^{\ell}_{\mathrm{mx}} \leq \bar{\delta}^{\ell}_{\Omega} = \left(1 - \frac{\ell}{\ell_\Omega}\right) \delta_{\mathrm{mx}} + \frac{\ell}{\ell_\Omega} . \tag{3.24}$$

By construction this is the best upper bound for $\delta^{\ell}_{\mathrm{mx}}$ over $[0, \ell_\Omega]$ that is linear in ℓ. In particular, it is better than the rough upper bound in (1.16).

Leverage Limit Selection:

We see from (2.22b) that

$$1 + \ell_\Omega = \min_d \left\{ \frac{1 + r_{\mathrm{mx}}(d)}{r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d)} \right\} .$$

We can use this identity to bound v_{mx}^ℓ above by

$$\begin{aligned} v_{\mathrm{mx}}^\ell &= \max_d \Big\{ r_{\mathrm{mx}}(d) + \ell \left(r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d) \right) \Big\} \\ &= \max_d \left\{ r_{\mathrm{mx}}(d) + \frac{\ell}{1 + \ell_\Omega} \left(1 + \ell_\Omega \right) \left(r_{\mathrm{mx}}(d) - r_{\mathrm{mn}}(d) \right) \right\} \\ &\leq \max_d \left\{ r_{\mathrm{mx}}(d) + \frac{\ell}{1 + \ell_\Omega} \left(1 + r_{\mathrm{mx}}(d) \right) \right\} \\ &= v_{\mathrm{mx}} + \frac{\ell}{1 + \ell_\Omega} \left(1 + v_{\mathrm{mx}} \right) . \end{aligned}$$

Leverage Limit Selection:

Therefore an upper bound for v_{mx}^{ℓ} is

$$v_{\mathrm{mx}}^{\ell} \leq v_{\Omega}^{\ell} = v_{\mathrm{mx}} + \ell\, \frac{1 + v_{\mathrm{mx}}}{1 + \ell_{\Omega}} \,. \tag{3.25}$$

Leverage Limit Selection:

By evaluating the rough upper bound for $\delta^{\ell}_{\mathrm{mx}}$ in (1.16) at $\ell = \ell_\Omega$ we obtain

$$1 \leq \delta_{\mathrm{mx}} + \ell_\Omega \left(\delta_{\mathrm{mx}} + \upsilon_{\mathrm{mx}} \right),$$

from which we can add υ_{mx} to both sides and derive

$$\frac{1 + \upsilon_{\mathrm{mx}}}{1 + \ell_\Omega} \leq \left(\delta_{\mathrm{mx}} + \upsilon_{\mathrm{mx}} \right).$$

We thereby see that

$$\bar{\upsilon}^{\ell}_{\Omega} = \upsilon_{\mathrm{mx}} + \ell \, \frac{1 + \upsilon_{\mathrm{mx}}}{1 + \ell_\Omega} \leq \upsilon_{\mathrm{mx}} + \ell \left(\delta_{\mathrm{mx}} + \upsilon_{\mathrm{mx}} \right).$$

This confirms that $\bar{\upsilon}^{\ell}_{\Omega}$ is a better upper bound for $\upsilon^{\ell}_{\mathrm{mx}}$ than the rough upper bound for it in (1.16).

Leverage Limit Selection:

Remark. It is natural to ask why an investor who maintains a long portfolio should care about bounds on leverage limits. The answer is that bounds on leverage limits can fall well before a market bubble collapses. During a bubble some investors will succumb to the temptation of taking highly leveraged positions. The most highly leveraged investors will be stressed when bounds on leverage limits fall. They may have to shed some of their position to cover their margins. This creates market volatility, which in turn can drive bounds on leverage limits down further. This can go on for quite a while before the market turns down — if it turns down. Observant long investors can use this time move into a more conservative position. It is wise to use short histories when computing these bounds for this purpose.