Portfolios that Contain Risky Assets
5.3. Limited-Leverage with Risk-Free Assets

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

5. Models for Limited-Leverage Portfolios

Limited-Leverage with Risk-Free Assets

Limited-Leverage One-Rate Model: $\mathcal{M}_1$ and Π_1^ℓ

The set of Markowitz allocations for the one-rate model is

$$\mathcal{M}_1 = \left\{ (\mathbf{f}, f_{\rm rf}) \in \mathbb{R}^{N+1} \; : \; \mathbf{1}^{\rm T}\mathbf{f} + f_{\rm rf} = 1 \right\}. \tag{1.1}$$

The return mean and volatility for a portfolio with allocation $(\mathbf{f}, f_{\rm rf})$ are

$$\mu(\mathbf{f}, f_{\rm rf}) = \mathbf{m}^{\rm T}\mathbf{f} + \mu_{\rm rf} f_{\rm rf} \,, \qquad \sigma(\mathbf{f}, f_{\rm rf}) = \sqrt{\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}} \,. \tag{1.2}$$

The leverage ratio of this portfolio can be expressed as

$$\lambda(\mathbf{f}, f_{\rm rf}) = \tfrac{1}{2}\left(\|\mathbf{f}\|_1 + |f_{\rm rf}| - 1 \right). \tag{1.3}$$

Therefore for every $\ell \geq 0$ the set of Markowitz allocations with leverage ratio no greater than ℓ is

$$\Pi_1^\ell = \left\{ (\mathbf{f}, f_{\rm rf}) \in \mathcal{M}_1 \; : \; \|\mathbf{f}\|_1 + |f_{\rm rf}| \leq 1 + 2\,\ell \right\}. \tag{1.4}$$

Limited-Leverage One-Rate Model: Tobin Frontier

The efficient Tobin frontier for $\mathcal{M}_1$ is given by

$$\mu = \mu_{\mathrm{rf}} + \nu_{\mathrm{rf}}\sigma \qquad \text{for } \sigma \geq 0\,, \tag{1.5}$$

where $\nu_{\mathrm{rf}} > 0$ is determined by

$$\nu_{\mathrm{rf}}^2 = \left(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}\right)^{\mathrm{T}}\mathbf{V}\left(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}\right)\,,$$

and the Tobin frontier allocation is given by

$$\mathbf{f}_{\mathrm{tf}}(\mu) = \frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\,\mathbf{V}^{-1}\left(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}\right)\,. \tag{1.6}$$

The asssociated risk-free frontier allocation is

$$f_{\mathrm{rff}}(\mu) = 1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}_{\mathrm{tf}}(\mu) = 1 - \frac{(\mu - \mu_{\mathrm{rf}})(\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})}{\nu_{\mathrm{rf}}^2\,\sigma_{\mathrm{mv}}^2}\,. \tag{1.7}$$

Limited-Leverage One-Rate Model: Tangent Portfolio

When $\mu_{\rm rf} \neq \mu_{\rm mv}$ the tangent portfolio exists and is given by

$$\mathbf{f}_{\rm tg} = \mathbf{f}_{\rm tf}(\mu_{\rm tg}) = \frac{\mu_{\rm tg} - \mu_{\rm rf}}{\nu_{\rm rf}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm rf}\mathbf{1}) \,,$$

where $\mu_{\rm tg}$ is determined by

$$\frac{(\mu_{\rm tg} - \mu_{\rm rf})(\mu_{\rm mv} - \mu_{\rm rf})}{\nu_{\rm rf}^2\, \sigma_{\rm mv}^2} = 1 \,.$$

The analysis of the Tobin frontier allocations falls into three cases:

$$\begin{aligned} \mu_{\rm rf} &< \mu_{\rm mv} \,, && \text{tangent portfolio is efficient}\,; \\ \mu_{\rm mv} &= \mu_{\rm rf} \,, && \text{tangent portfolio does not exist}\,; \\ \mu_{\rm mv} &< \mu_{\rm rf} \,, && \text{tangent portfolio is inefficient}\,. \end{aligned}$$

Limited-Leverage One-Rate Model: Tangent Efficient

When $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ the tangent portfolio is efficient. The efficient frontier allocation is

$$\mathbf{f}_{\mathrm{etf}}(\sigma) = \frac{\sigma}{\sigma_{\mathrm{tg}}} \, \mathbf{f}_{\mathrm{tg}} \,, \qquad f_{\mathrm{rfef}}(\sigma) = 1 - \frac{\sigma}{\sigma_{\mathrm{tg}}} \,.$$

The leverage ratio is

$$\begin{aligned} \lambda_{\mathrm{etf}}(\sigma) &= \lambda(\mathbf{f}_{\mathrm{etf}}(\sigma)\,,\, f_{\mathrm{rfef}}(\sigma)) \\ &= \begin{cases} \frac{1}{2}\,\dfrac{\sigma}{\sigma_{\mathrm{tg}}}\left(\|\mathbf{f}_{\mathrm{tg}}\|_1 - 1\right) & \text{for } \sigma \in [0, \sigma_{\mathrm{tg}}]\,, \\ \frac{1}{2}\left(\dfrac{\sigma}{\sigma_{\mathrm{tg}}}\left(\|\mathbf{f}_{\mathrm{tg}}\|_1 + 1\right) - 2\right) & \text{for } \sigma \in (\sigma_{\mathrm{tg}}, \infty)\,. \end{cases} \end{aligned}$$

Notice that $\lambda_{\mathrm{etf}}(\sigma) \leq \ell$ for sufficiently small σ.

Limited-Leverage One-Rate Model: Tangent Inefficient

When $\mu_{\rm mv} < \mu_{\rm rf}$ the tangent portfolio is inefficient. The efficient frontier allocation is

$$\mathbf{f}_{\rm etf}(\sigma) = -\frac{\sigma}{\sigma_{\rm tg}}\,\mathbf{f}_{\rm tg}\,, \qquad f_{\rm rfef}(\sigma) = 1 + \frac{\sigma}{\sigma_{\rm tg}}\,.$$

The leverage ratio is

$$\lambda_{\rm etf}(\sigma) = \lambda(\mathbf{f}_{\rm etf}(\sigma)\,,\, f_{\rm rfef}(\sigma)) = \tfrac{1}{2}\,\frac{\sigma}{\sigma_{\rm tg}}\left(\|\mathbf{f}_{\rm tg}\|_1 + 1\right), \qquad \text{for } \sigma \in [0,\infty)\,.$$

Notice that $\lambda_{\rm etf}(\sigma) \leq \ell$ for sufficiently small σ.

Remark. This shows that for sufficiently small σ the efficient frontier for Π_1^ℓ coincides with the efficient Tobin frontier for $\mathcal{M}_1$. In particular, it is linear with slope $\nu_{\rm rf}$ for small σ.

Limited-Leverage One-Rate Model: μ^ℓ_{mx}

It can be shown that $\Pi^\ell_1 = \mathrm{Hull}(\mathcal{E}^\ell_1)$ where

$$\mathcal{E}^\ell_1 = \Big\{ (\mathbf{e}^\ell_{ij}\,,\, 0)\;,\; ((1+\ell)\,\mathbf{e}_i\,,\, -\ell)\;,\; (-\,\ell\,\mathbf{e}_j\,,\, 1+\ell) \Big\}\,.$$

Therefore

$$\begin{aligned}
\mu^\ell_{\mathrm{mx}} &= \max\Big\{ \mu(\mathbf{f}\,,\, f_{\mathrm{rf}}) \;:\; (\mathbf{f}\,,\, f_{\mathrm{rf}}) \in \Pi^\ell_1 \Big\} \\
&= \max\Big\{ \mu(\mathbf{f}\,,\, f_{\mathrm{rf}}) \;:\; (\mathbf{f}\,,\, f_{\mathrm{rf}}) \in \mathcal{E}^\ell_1 \Big\} \\
&= \max\Big\{ (1+\ell)\,\mu_{\mathrm{mx}} - \ell\,\mu_{\mathrm{mn}}\,, \\
&\qquad\qquad (1+\ell)\,\mu_{\mathrm{mx}} - \ell\,\mu_{\mathrm{rf}}\,, \\
&\qquad\qquad (1+\ell)\,\mu_{\mathrm{rf}} - \ell\,\mu_{\mathrm{mn}} \Big\}\,,.
\end{aligned}$$

where

$$\mu_{\mathrm{mn}} = \min_i\{m_i\}\,, \qquad \mu_{\mathrm{mx}} = \max_i\{m_i\}\,.$$

Limited-Leverage One-Rate Model: Tangent Inefficient

Because

$$\frac{\ell}{1+2\ell}\Big((1+\ell)\,\mu_{\rm mx} - \ell\,\mu_{\rm rf}\Big) + \frac{1+\ell}{1+2\ell}\Big((1+\ell)\,\mu_{\rm rf} - \ell\,\mu_{\rm mn}\Big) = \mu_{\rm rf}\,,$$

and because

$$(1+\ell)\,\mu_{\rm mx} - \ell\,\mu_{\rm rf} > (1+\ell)\,\mu_{\rm rf} - \ell\,\mu_{\rm mn}\,,$$

we see that $\mu_{\rm rf} < (1+\ell)\,\mu_{\rm mx} - \ell\,\mu_{\rm rf} \leq \mu^{\ell}_{\rm mx}$. Therefore the efficient frontier of Π^{ℓ}_1 exists for $\mu \in \big[\mu_{\rm rf}, \mu^{\ell}_{\rm mx}\big]$. We can approximate it numerically using the Malab "quadprog" command.

Computng the One-Rate Model: Quadratic Programming

Because the function being minimized is quadratic in **f** while the constraints are linear in **f**, this is called a *quadratic programming problem*. It can be solved for a particular **V**, **m**, and μ by using either the Matlab command "quadprog" or an equivalent command in some other language.

Recall that the Matlab command quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\mathrm{eq}}, \mathbf{d}_{\mathrm{eq}})$ returns the solution of a quadratic programming problem in the *standard form*

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^{M},\ \mathbf{C}\mathbf{x} \le \mathbf{d},\ \mathbf{C}_{\mathrm{eq}}\mathbf{x} = \mathbf{d}_{\mathrm{eq}} \right\},$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^{M}$, $\mathbf{C} \in \mathbb{R}^{K\times M}$, $\mathbf{d} \in \mathbb{R}^{K}$, $\mathbf{C}_{\mathrm{eq}} \in \mathbb{R}^{K_{\mathrm{eq}}\times M}$, and $\mathbf{d}_{\mathrm{eq}} \in \mathbb{R}^{K_{\mathrm{eq}}}$. Here K and K_{eq} are the number of inequality and equality constraints respectively.

Computng the One-Rate Model: Standard Form

Given **V**, **m**, and $\mu \in [\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell}]$, the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ and $f_{\mathrm{rff}}(\mu)$ is

$$\arg\min\Big\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{f}, f_{\mathrm{rf}}) \in \mathbb{R}^{N+1}\,,\ \|\mathbf{f}\|_1 + |f_{\mathrm{rf}}| \leq 1 + 2\,\ell\,, \\ \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{rf}} = 1\,,\ \mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{rf}} f_{\mathrm{rf}} = \mu \Big\}\,.$$

By comparing this with the standard form on the previous slide we see that if we set $\mathbf{x}^{\mathrm{T}} = (\mathbf{f}\ \ f_{\mathrm{rf}})$ then $M = N + 1$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ 0 \end{pmatrix}, \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} & 1 \\ \mathbf{m}^{\mathrm{T}} & \mu_{\mathrm{rf}} \end{pmatrix}, \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

However, the inequality constraint $\|\mathbf{f}\|_1 + |f_{\mathrm{rf}}| \leq 1 + 2\,\ell$ is not in the standard form $\mathbf{C}\mathbf{x} \leq \mathbf{d}$.

Computng the One-Rate Model: Two Reformulations

As was done for the limited-leverage allocations Π^ℓ without risk-free assets, we enlarge the dimension of $\mathbf{x}$ and consider the following reformulations Π^ℓ_1.

$$
\begin{aligned}
\Pi^\ell_1 &= \Big\{ (\mathbf{f}, f_{\mathrm{rf}}) \in \mathbb{R}^{N+1} \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{rf}} = 1\,,\; \|\mathbf{f}\|_1 + |f_{\mathrm{rf}}| \leq 1 + 2\ell \Big\} \\
&= \Big\{ (\mathbf{f}, f_{\mathrm{rf}}) \in \mathbb{R}^{N+1} \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{rf}} = 1\,,\; \exists (\mathbf{s}, s_{\mathrm{rf}}) \in \mathbb{R}^{N+1} \;: \\
&\qquad \mathbf{s} \geq \mathbf{0}\,,\; (\mathbf{f} + \mathbf{s}) \geq \mathbf{0}\,,\; s_{\mathrm{rf}} \geq 0\,,\; (f_{\mathrm{rf}} + s_{\mathrm{rf}}) \geq 0\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{s} + s_{\mathrm{rf}} \leq \ell \Big\} \\
&= \Big\{ (\mathbf{f}, f_{\mathrm{rf}}) \in \mathbb{R}^{N+1} \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{rf}} = 1\,,\; \exists (\mathbf{g}, g_{\mathrm{rf}}) \in \mathbb{R}^{N+1} \;: \\
&\qquad (\mathbf{g} \pm \mathbf{f}) \geq \mathbf{0}\,,\; (g_{\mathrm{rf}} \pm f_{\mathrm{rf}}) \geq 0\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{g} + g_{\mathrm{rf}} \leq 1 + 2\ell \Big\}\,.
\end{aligned}
$$

The proofs of these reformulations are left as exercises.

Computng the One-Rate Model: First Reformulation

If we use the first reformulation then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ and $f_{\mathrm{rff}}(\mu)$ is

$$\arg\min\Big\{\tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{f},\mathbf{s},f_{\mathrm{rf}},s_{\mathrm{rf}})\in\mathbb{R}^{2N+2},\ \mathbf{1}^{\mathrm{T}}\mathbf{f}+f_{\mathrm{rf}}=1,\ \mathbf{m}^{\mathrm{T}}\mathbf{f}+\mu_{\mathrm{rf}}f_{\mathrm{rf}}=\mu,$$
$$\mathbf{s}\geq\mathbf{0}\,,\ (\mathbf{f}+\mathbf{s})\geq\mathbf{0}\,,\ s_{\mathrm{rf}}\geq 0\,,\ (f_{\mathrm{rf}}+s_{\mathrm{rf}})\geq 0\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{s}+s_{\mathrm{rf}}\leq\ell\Big\}\,.$$

By comparing this with the standard form we see that if we set $\mathbf{x}=(\mathbf{f}^{\mathrm{T}}\ \mathbf{s}^{\mathrm{T}}\ f_{\mathrm{rf}}\ s_{\mathrm{rf}})^{\mathrm{T}}$ then $M=2N+2$, $K=2N+3$, $K_{\mathrm{eq}}=2$, and

$$\mathbf{A}=\begin{pmatrix}\mathbf{V} & \mathbf{O} & \mathbf{0} & \mathbf{0}\\ \mathbf{O} & \mathbf{O} & \mathbf{0} & \mathbf{0}\\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & 0 & 0\\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & 0 & 0\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ 0\\ 0\end{pmatrix},\ \mathbf{C}=\begin{pmatrix}-\mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{O} & -\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} & 0 & 1\end{pmatrix},\ \mathbf{d}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \ell\end{pmatrix},$$

$$\mathbf{C}_{\mathrm{eq}}=\begin{pmatrix}\mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & 1 & 0\\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mu_{\mathrm{rf}} & 0\end{pmatrix},\quad \mathbf{d}_{\mathrm{eq}}=\begin{pmatrix}1\\ \mu\end{pmatrix}.$$

Computng the One-Rate Model: Second Reformulation

If we use the second reformulation then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ and $f_{\mathrm{rff}}(\mu)$ is

$$\arg\min\Big\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{f},\mathbf{g},f_{\mathrm{rf}},g_{\mathrm{rf}}) \in \mathbb{R}^{2N+2},\ \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{rf}} = 1,\ \mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{rf}} f_{\mathrm{rf}} = \mu,$$
$$(\mathbf{g} \pm \mathbf{f}) \geq \mathbf{0}\,,\ (g_{\mathrm{rf}} \pm f_{\mathrm{rf}}) \geq 0\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{g} + g_{\mathrm{rf}} \leq 1 + 2\,\ell \Big\}\,.$$

By comparing this with the standard form we see that if we set $\mathbf{x} = (\mathbf{f}^{\mathrm{T}}\ \mathbf{g}^{\mathrm{T}}\ f_{\mathrm{rf}}\ g_{\mathrm{rf}})^{\mathrm{T}}$ then $M = 2N+2$, $K = 2N+3$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{O} & \mathbf{O} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & 0 & 0 \\ \mathbf{0}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & 0 & 0 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{C} = \begin{pmatrix} -\mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} & 0 & 1 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 1 + 2\ell \end{pmatrix},$$

$$\mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & 1 & 0 \\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} & \mu_{\mathrm{rf}} & 0 \end{pmatrix}, \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

Computng the One-Rate Model: quadprog Command

In either case $\mathbf{f}^{\ell}_{\mathrm{f}}(\mu)$ and $f^{\ell}_{\mathrm{rff}}(\mu)$ are the first N entries and the $2N+1$ entry of the output x of a quadprog command that is formated as

$$\mathrm{x} = \text{quadprog}(\mathrm{A}, \mathrm{b}, \mathrm{C}, \mathrm{d}, \mathrm{Ceq}, \mathrm{deq}) \, ,$$

where the matrices A, C, and Ceq, and the vectors b, d, and deq are given on the previous slides.

Remark. By doubling the dimension of the vector $\mathbf{x}$ from $N+1$ to $2N+2$, the number of inequality constraints becomes $2N+3$. If $N=9$ then this is 21!

Remark. There are other ways to use quadprog to find $\mathbf{f}^{\ell}_{\mathrm{f}}(\mu)$ and $f^{\ell}_{\mathrm{rff}}(\mu)$. Documentation for this command is easy to find on the web. The similar command in R is also called "quadprog".

Computng the One-Rate Model: Properties of $\sigma_{\mathrm{f}}^{\ell}(\mu)$

When computing an ℓ-limited efficient frontier, it helps to know some general properties of the function $\sigma_{\mathrm{f}}^{\ell}(\mu)$ over the interval $[\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell}]$. These include:

- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is continuous over $[\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is increasing and convex over $[\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is piecewise hyperbolic over $[\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell}]$.

This means that $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is built up from segments of a line and hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell})$ where $\sigma_{\mathrm{f}}^{\ell}(\mu)$ has either

- *a jump discontinuity in its first derivative* or
- *a jump discontinuity in its second derivative.*

Guided by these facts we now show how an ℓ-limited efficient frontier can be approximated numerically with the Matlab “quadprog” command.

Computing the One-Rate Model: Appproximating $\sigma_{\mathrm{f}}^{\ell}(\mu)$

First, partition the interval $[\mu_{\mathrm{rf}}, \mu_{\mathrm{mx}}^{\ell}]$ as

$$\mu_{\mathrm{rf}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathrm{mx}}^{\ell} .$$

For example, set $\mu_k = \mu_{\mathrm{mn}}^{\ell} + k(\mu_{\mathrm{mx}}^{\ell} - \mu_{\mathrm{rf}})/n$ for a uniform partition. Pick n large enough to resolve all the features of the ℓ-limited efficient frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

Second, for every $k = 1, \cdots, n-1$ use quadprog to compute $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)$ and $f_{\mathrm{rff}}^{\ell}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)\}_{k=0}^{n}$ should be saved.

Third, for every $k = 1, \cdots, n-1$ compute σ_k by

$$\sigma_k = \sigma_{\mathrm{f}}^{\ell}(\mu_k) = \sqrt{\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)} .$$

Computing the One-Rate Model: Interpolation in $\sigma\mu$-Plane

Fourth. Set

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_0) = \mathbf{0}\,, \quad f_{\mathrm{rff}}^{\ell}(\mu_0) = 1\,, \quad \sigma_0 = 0\,.$$

There is typically a unique long-short pair allocation $\mathbf{e}_{ij}^{\ell}$ such that $\mu_{\mathrm{mx}}^{\ell} = (1+\ell)\,m_i - \ell\,m_j$ that is most efficient, in which case we have

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_n) = \mathbf{e}_{ij}^{\ell}\,, \quad f_{\mathrm{rff}}^{\ell}(\mu_n) = 0\,, \quad \sigma_n = \sigma_{ij}^{\ell} = \sqrt{(\mathbf{e}_{ij}^{\ell})^{\mathrm{T}}\mathbf{V}\mathbf{e}_{ij}^{\ell}}\,.$$

Finally, we "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the ℓ-limited frontier in the $\sigma\mu$-plane. This can be done by linear interpolation. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\,\sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\,\sigma_k\,.$$

Computing the One-Rate Model: Linear Interpolation in Π_1^ℓ

A better way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathrm{f}}^\ell(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{f}}^\ell(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{f}}^\ell(\mu_k)\,,$$
$$\tilde{f}_{\mathrm{rff}}^\ell(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, f_{\mathrm{rff}}^\ell(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, f_{\mathrm{rff}}^\ell(\mu_k)\,,$$

and then set

$$\tilde{\sigma}_{\mathrm{f}}^\ell(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{f}}^\ell(\mu)^{\mathrm{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathrm{f}}^\ell(\mu)}\,.$$

Remark. This will be a very good approximation if n is large enough. Over each interval (μ_{k-1}, μ_k) it generally approximates $\sigma_{\mathrm{f}}^\ell(\mu)$ with a hyperbola rather than with a line.

Computing the One-Rate Model: Linear Interpolation in Π_1^ℓ

Remark. Because $\mathbf{f}_\mathrm{f}^\ell(\mu_k) \in \Pi^\ell(\mu_k)$ and $\mathbf{f}_\mathrm{f}^\ell(\mu_{k-1}) \in \Pi^\ell(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_\mathrm{f}^\ell(\mu) \in \Pi^\ell(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k)\,.$$

Therefore $\tilde{\sigma}_\mathrm{f}^\ell(\mu)$ gives an approximation to the ℓ-limited frontier that lies on or to the right of the ℓ-limited frontier in the $\sigma\mu$-plane.

Remark. When there are no nodes in the interval (μ_{k-1}, μ_k) then we can use the two-fund property to show that $\tilde{\sigma}_\mathrm{f}^\ell(\mu) = \sigma_\mathrm{f}^\ell(\mu)$.