Portfolios that Contain Risky Assets
5.2. Limited-Leverage Frontiers

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Part I: Portfolio Models

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5. Models for Limited-Leverage Portfolios

Limited-Leverage Frontiers

Vizualizing Frontiers: $\Sigma(\Pi^\ell)$ in the $\sigma\mu$-Plane

The set Π^ℓ in $\mathbb{R}^N$ of ℓ-limited portfolio allocations is associated with the set $\Sigma(\Pi^\ell)$ in the $\sigma\mu$-plane of volatilities and return means given by

$$\Sigma(\Pi^\ell) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \ : \ \sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \ \mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \ \mathbf{f} \in \Pi^\ell \right\}.$$

For every $\ell > 0$ we have shown that $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$, the convex hull in $\mathbb{R}^N$ of the set $\mathcal{E}^\ell = \{\mathbf{e}_{ij}^\ell\}$ of the $N(N-1)$ long-short pair allocations. As such, the set Π^ℓ is convex and compact (closed and bounded).

Because the set $\Sigma(\Pi^\ell)$ is the image in $\mathbb{R}^2$ of Π^ℓ under the continuous mapping

$$\mathbf{f} \mapsto (\sigma(\mathbf{f})\,,\, \mu(\mathbf{f})) = \left(\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,,\, \mathbf{m}^{\mathrm{T}}\mathbf{f}\right),$$

the set $\Sigma(\Pi^\ell)$ is pathwise connected and compact.

Vizualizing Frontiers: Bounding μ and σ for $\mathbf{f} \in \Pi^\ell$

We already have some easy bounds on μ and σ for $\mathbf{f} \in \Pi^\ell$.

- Because the mapping $\mathbf{f} \mapsto \mathbf{m}^{\mathrm{T}}\mathbf{f}$ is linear over $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$, its image is the interval $[\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$, where

$$\mu^\ell_{\mathrm{mn}} = \min\left\{\mathbf{m}^{\mathrm{T}}\mathbf{f} \,:\, \mathbf{f} \in \mathcal{E}^\ell\right\}, \qquad \mu^\ell_{\mathrm{mx}} = \max\left\{\mathbf{m}^{\mathrm{T}}\mathbf{f} \,:\, \mathbf{f} \in \mathcal{E}^\ell\right\}.$$

- Because the mapping $\mathbf{f} \mapsto \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$ is convex over $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$, its image is the interval $[\sigma^\ell_{\mathrm{mn}}, \sigma^\ell_{\mathrm{mx}}]$, where

$$\sigma^\ell_{\mathrm{mv}} = \min\left\{\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \,:\, \mathbf{f} \in \Pi^\ell\right\}, \qquad \sigma^\ell_{\mathrm{mx}} = \max\left\{\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \,:\, \mathbf{f} \in \mathcal{E}^\ell\right\}.$$

  Notice that $\sigma^\ell_{\mathrm{mv}} \geq \sigma_{\mathrm{mv}}$ with equality if and only if $\mathbf{f}_{\mathrm{mv}} \in \Pi^\ell$.

- Because $\Pi^\ell \subset \mathcal{M}$, we know that $\Sigma(\Pi^\ell) \subset \Sigma(\mathcal{M})$, where

$$\Sigma(\mathcal{M}) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,,\ \mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,,\ \mathbf{f} \in \mathcal{M} \right\}.$$

Vizualizing Frontiers: Bounding $\Sigma(\Pi^\ell)$

Therefore $\Sigma(\Pi^\ell)$ is contained within the intersction of the Markowitz region $\Sigma(\mathcal{M})$ with the closed box $[\sigma^\ell_{\mathrm{mv}}, \sigma^\ell_{\mathrm{mx}}] \times [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$. Earlier we derived the exact formulas

$$\mu^\ell_{\mathrm{mn}} = \mu_{\mathrm{mn}} - \ell\left(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}\right), \qquad \mu^\ell_{\mathrm{mx}} = \mu_{\mathrm{mn}} + \ell\left(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}\right),$$

where

$$\mu_{\mathrm{mn}} = \min_i\{m_i\}, \qquad \mu_{\mathrm{mx}} = \max_i\{m_i\},$$

and the rough bounds

$$\sigma_{\mathrm{mv}} \leq \sigma^\ell_{\mathrm{mv}} \leq \sigma_{\mathrm{mn}},$$
$$\sigma_{\mathrm{mx}} + \ell\left(\sigma_{\mathrm{mx}} - \sigma_{\mathrm{mn}}\right) \leq \sigma^\ell_{\mathrm{mx}} \leq (1 + 2\,\ell)\,\sigma_{\mathrm{mx}},$$

where

$$\sigma_{\mathrm{mn}} = \min_i\{\sigma_i\}, \qquad \sigma_{\mathrm{mx}} = \max_i\{\sigma_i\}.$$

Vizualizing Frontiers: Definition of the ℓ-Limited Frontier

The set $\Pi^\ell(\mu)$ of ℓ-limited portfolio allocations with return mean μ is defined for every $\mu \in \mathbb{R}$ by

$$\Pi^\ell(\mu) = \left\{ \mathbf{f} \in \Pi^\ell \,:\, \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}.$$

This set is nonempty if and only if $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$. Hence, $\Sigma(\Pi^\ell)$ can be expressed as

$$\Sigma(\Pi^\ell) = \left\{ \left(\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,,\, \mu \right) \,:\, \mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]\,,\ \mathbf{f} \in \Pi^\ell(\mu) \right\}.$$

Definition. The points on the boundary of $\Sigma(\Pi^\ell)$ that correspond to those ℓ-limited portfolios that have less volatility than every other ℓ-limited portfolio with the same return mean is called the *ℓ-limited frontier.*

Vizualizing Frontiers: Definition of the ℓ-Limited Frontier

Remark. The rest of the boundary of $\Sigma(\Pi^\ell)$ is contained within the image of all convex combinations of pairs of long-short allocations in $\mathcal{E}^\ell$ that are vertices of an edge of Π^ℓ. Recall that $\mathbf{e}_{ij}^\ell$, $\mathbf{e}_{kl}^\ell \in \mathcal{E}^\ell$ are vertices of an edge of Π^ℓ if and only if $(i,j) \neq (k,l)$ with $k = i$ or $l = j$. For each such pair set

$$\sigma_{ij}^\ell = \sigma\big(\mathbf{e}_{ij}^\ell\big)\,, \quad \mu_{ij}^\ell = \mu\big(\mathbf{e}_{ij}^\ell\big)\,, \qquad \sigma_{kl}^\ell = \sigma\big(\mathbf{e}_{kl}^\ell\big)\,, \quad \mu_{kl}^\ell = \mu\big(\mathbf{e}_{kl}^\ell\big)\,.$$

If $\mu_{ij}^\ell = \mu_{kl}^\ell$ then plot the line segment connecting $(\sigma_{ij}^\ell, \mu_{ij}^\ell)$ and $(\sigma_{kl}^\ell, \mu_{kl}^\ell)$. If $\mu_{ij}^\ell \neq \mu_{kl}^\ell$ then without loss of generality suppose that $\mu_{ij}^\ell < \mu_{kl}^\ell$, set

$$\mathbf{f}_{ij,kl}^\ell(\mu) = \frac{\mu_{kl}^\ell - \mu}{\mu_{kl}^\ell - \mu_{ij}^\ell}\,\mathbf{e}_{ij}^\ell + \frac{\mu - \mu_{ij}^\ell}{\mu_{kl}^\ell - \mu_{ij}^\ell}\,\mathbf{e}_{kl}^\ell\,,$$

and plot the hyperbola segment

$$\sigma = \sigma\Big(\mathbf{f}_{ij,kl}^\ell(\mu)\Big) \qquad \text{over } \mu \in [\mu_{ij}^\ell, \mu_{kl}^\ell]\,.$$

There are $N(N-1)(N-2)$ such pairs of pairs, one for each edge of Π^ℓ.

Vizualizing Frontiers: Definition of $\sigma_{\mathrm{f}}^{\ell}(\mu)$

The ℓ-limited frontier is the curve in the $\sigma\mu$-plane given by the equation

$$\sigma = \sigma_{\mathrm{f}}^{\ell}(\mu) \quad \text{over} \quad \mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}] \,,$$

where the value of $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is obtained for each $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ by solving the constrained minimization problem

$$\sigma_{\mathrm{f}}^{\ell}(\mu)^2 = \min\left\{ \sigma^2 \ : \ (\sigma, \mu) \in \Sigma(\Pi^{\ell}) \right\} = \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \ : \ \mathbf{f} \in \Pi^{\ell}(\mu) \right\} .$$

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is continuous over the compact set $\Pi^{\ell}(\mu)$, *a minimizer exists.*

Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Pi^{\ell}(\mu)$, whereby *the minimizer is unique.*

Vizualizing Frontiers: Definition of $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$

If we denote this unique minimizer by $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ then for every $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ the function $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is given by

$$\sigma_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{f}}^{\ell}(\mu)}\,,$$

where $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu) = \arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Pi^{\ell}(\mu) \right\}.$$

Here $\arg\min$ is read *"the argument that minimizes"*. It means that $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ subject to the given constraints.
Remark. This problem cannot be solved by Lagrange multipliers because the set $\Pi^{\ell}(\mu)$ is defined by inequality constraints. It is harder to solve analytically than the analogous minimization problem for long portfolios. Therefore we take a numerical approach that can be applied generally.

Quadratic Programming: Standard Form

Because the function being minimized is quadratic in **f** while the constraints are linear in **f**, this is called a *quadratic programming problem*. It can be solved for a particular **V**, **m**, and μ by using either the Matlab command "quadprog" or an equivalent command in some other language.

Recall that the Matlab command quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\rm eq}, \mathbf{d}_{\rm eq})$ returns the solution of a quadratic programming problem in the *standard form*

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{x}^{\rm T}\mathbf{A}\mathbf{x} + \mathbf{b}^{\rm T}\mathbf{x} \;:\; \mathbf{x} \in \mathbb{R}^M,\; \mathbf{C}\mathbf{x} \leq \mathbf{d},\; \mathbf{C}_{\rm eq}\mathbf{x} = \mathbf{d}_{\rm eq} \right\},$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^M$, $\mathbf{C} \in \mathbb{R}^{K\times M}$, $\mathbf{d} \in \mathbb{R}^K$, $\mathbf{C}_{\rm eq} \in \mathbb{R}^{K_{\rm eq}\times M}$, and $\mathbf{d}_{\rm eq} \in \mathbb{R}^{K_{\rm eq}}$. Here K and $K_{\rm eq}$ are the number of inequality and equality constraints respectively.

Quadratic Programming: A Complication

Given $\mathbf{V}$, $\mathbf{m}$, and $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$, the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N ,\; \|\mathbf{f}\|_1 \leq 1 + 2\ell ,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 ,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\} .$$

By comparing this with the standard quadratic programming problem on the previous slide we see that if we set $\mathbf{x} = \mathbf{f}$ then $M = N$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \mathbf{V} , \quad \mathbf{b} = \mathbf{0} , \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix} , \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} .$$

However, it is less clear how the inequality constraint $\|\mathbf{f}\|_1 \leq 1 + 2\ell$ can be expressed in the standard form $\mathbf{C}\mathbf{f} \leq \mathbf{d}$.

Quadratic Programming: Converting $\|\mathbf{f}\|_1 \leq 1 + 2\ell$

The inequality $\|\mathbf{f}\|_1 \leq 1 + 2\ell$ can be expressed as the inequality constraints

$$\pm f_1 \pm f_2 \pm \cdots \pm f_{N-1} \pm f_N \leq 1 + 2\ell\,,$$

where there are 2^N choices of $\pm$ signs. When the $\pm$ are chosen to be the same sign then the inequality constraint is always satisfied because of the the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$. That leaves $2^N - 2$ inequality constraints that still need to be imposed.

The number $2^N - 2$ grows too fast with N for this approach to be useful for all but small values of N. For example, if $N = 9$ then $2^N - 2 = 510$. With this many inequality constraints quadprog could suffer numerical difficulties. This raises the following question.

Are all of these $2^N - 2$ inequality constraints needed?

Quadratic Programming: Index Subsets Constraints

The answer is yes if we insist on setting $\mathbf{x} = \mathbf{f}$. However, the answer is no if we enlarge the dimension of $\mathbf{x}$.

To understand why the answer is yes if we insist on setting $\mathbf{x} = \mathbf{f}$, consider any of these inequality constraints written along with the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ as

$$\pm f_1 \pm f_2 \pm \cdots \pm f_{N-1} \pm f_N \leq 1 + 2\ell \,,$$
$$f_1 + f_2 + \cdots + f_{N-1} + f_N = 1 \,.$$

By adding these and dividing by 2 we obtain

$$\sum_{i \in S} f_i \leq 1 + \ell \,,$$

where S is the subset of indices i with a plus in the inequality constraint.

Quadratic Programming: Index Subsets Constraints

For every $S \subset \{1, 2, \cdots, N\}$ define the i^{th} entry of $\mathbf{1}_S \in \mathbb{R}^N$ by

$$\mathrm{ent}_i(\mathbf{1}_S) = \begin{cases} 1 & \text{if } i \in S\,, \\ 0 & \text{if } i \notin S\,. \end{cases}$$

Then the $2^N - 2$ inequality conatraints can be expressed as

$$\mathbf{1}_S^{\mathrm{T}}\mathbf{f} \leq 1 + \ell \quad \text{for every nonempty, proper } S \subset \{1, 2, \cdots, N\}\,. \tag{2.1a}$$

The equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ can be used to show that these $2^N - 2$ inequality conatraints can also be expressed as

$$-\ell \leq \mathbf{1}_S^{\mathrm{T}}\mathbf{f} \quad \text{for every nonempty, proper } S \subset \{1, 2, \cdots, N\}\,. \tag{2.1b}$$

Quadratic Programming: Two Reformulations

To understand why the answer is no if we enlarge the dimension of **x**, consider the following reformulations.

$$\begin{aligned}\Pi^\ell &\equiv \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \|\mathbf{f}\|_1 \leq 1 + 2\ell \right\} \\ &= \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \exists \mathbf{s} \in \mathbb{R}^N \;:\; \mathbf{s} \geq \mathbf{0}\,,\; (\mathbf{f} + \mathbf{s}) \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{s} \leq \ell \right\} \\ &= \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \exists \mathbf{g} \in \mathbb{R}^N \;:\; (\mathbf{g} \pm \mathbf{f}) \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{g} \leq 1 + 2\ell \right\}.\end{aligned}$$

The fact that the last two sets contain Π^ℓ is seen by taking

$$\mathbf{s} = \mathbf{f}^-\,, \qquad \mathbf{g} = \mathbf{f}^+ + \mathbf{f}^-\,.$$

We must show that they are equal to Π^ℓ. This is left as an exercise.

Quadratic Programming: First Reformulation

If we use the first reformulation then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{s} \geq \mathbf{0}\,,\ (\mathbf{f}+\mathbf{s}) \geq \mathbf{0}\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{s} \leq \ell\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}.$$

By comparing this with the standard quadratic programming problem we see that if we set $\mathbf{x} = (\mathbf{f}^{\mathrm{T}}\ \mathbf{s}^{\mathrm{T}})^{\mathrm{T}}$ then $M = 2N$, $K = 2N+1$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -\mathbf{I} & -\mathbf{I} \\ \mathbf{O} & -\mathbf{I} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \ell \end{pmatrix}, \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where $\mathbf{O}$ and $\mathbf{I}$ are the $N\times N$ zero and identity matrices.

Quadratic Programming: Second Reformulation

If we use the second reformulation then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{g}\pm\mathbf{f}) \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{g} \leq 1+2\ell\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}.$$

By comparing this with the standard quadratic programming problem we see that if we set $\mathbf{x} = (\mathbf{f}^{\mathrm{T}}\ \mathbf{g}^{\mathrm{T}})^{\mathrm{T}}$ then $M = 2N$, $K = 2N+1$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -\mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 1+2\ell \end{pmatrix}, \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where $\mathbf{O}$ and $\mathbf{I}$ are the $N\times N$ zero and identity matrices.

Quadratic Programming: Matlab "quadprog" Command

In either case $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ can be obtained as the first N entries of the output x of a quadprog command that is formated as

$$\mathrm{x} = \text{quadprog}(\mathrm{A}, \mathrm{b}, \mathrm{C}, \mathrm{d}, \mathrm{Ceq}, \mathrm{deq}) ,$$

where the matrices A, C, and Ceq, and the vectors b, d, and deq are given on the previous slides.

Remark. By doubling the dimension of the vector $\mathbf{x}$ from N to $2N$ we have reduced the number of inequality constraints from $2^N - 2$ to $2N + 1$. If $N = 9$ then reduction is from 510 to 19!

Remark. There are other ways to use quadprog to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$. Documentation for this command is easy to find on the web. The similar command in R is also called "quadprog".

Computing Frontiers: Properties of $\sigma_{\mathrm{f}}^{\ell}(\mu)$

When computing an ℓ-limited frontier, it helps to know some general properties of the function $\sigma_{\mathrm{f}}^{\ell}(\mu)$. These include:

- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is continuous over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is strictly convex over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is piecewise hyperbolic over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$.

This means that $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell})$ where $\sigma_{\mathrm{f}}^{\ell}(\mu)$ has either

- *a jump discontinuity in its first derivative* or
- *a jump discontinuity in its second derivative.*

Guided by these facts we now show how an ℓ-limited frontier can be approximated numerically with the Matlab command quadprog.

Computing Frontiers: Appproximating $\sigma_{\mathrm{f}}^{\ell}(\mu)$

First, partition the interval $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ as

$$\mu_{\mathrm{mn}}^{\ell} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathrm{mx}}^{\ell} .$$

For example, set $\mu_k = \mu_{\mathrm{mn}}^{\ell} + k(\mu_{\mathrm{mx}}^{\ell} - \mu_{\mathrm{mn}}^{\ell})/n$ for a uniform partition. Pick n large enough to resolve all the features of the ℓ-limited frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

Second, for every $k = 1, \cdots, n-1$ use quadprog to compute $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)\}_{k=0}^{n}$ should be saved.

Third, for every $k = 1, \cdots, n-1$ compute σ_k by

$$\sigma_k = \sigma_{\mathrm{f}}^{\ell}(\mu_k) = \sqrt{\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)} .$$

Computing Frontiers: Linear Interpolation in $\sigma\mu$-Plane

Fourth. There is typically a unique long-short pair allocation $\mathbf{e}_{ij}^{\ell}$ such that $\mu_{\mathrm{mn}}^{\ell} = (1+\ell)m_i - \ell m_j$ that is most efficient, in which case we have

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_0) = \mathbf{e}_{ij}^{\ell}\,, \qquad \sigma_0 = \sigma_{ij}^{\ell} = \sqrt{(\mathbf{e}_{ij}^{\ell})^{\mathrm{T}}\mathbf{V}\mathbf{e}_{ij}^{\ell}}\,.$$

Similarly, there is typically a unique long-short pair allocation $\mathbf{e}_{kl}^{\ell}$ such that $\mu_{\mathrm{mx}}^{\ell} = (1+\ell)m_k - \ell m_l$ that is most efficient, in which case we have

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_n) = \mathbf{e}_{kl}^{\ell}\,, \qquad \sigma_n = \sigma_{kl}^{\ell} = \sqrt{(\mathbf{e}_{kl}^{\ell})^{\mathrm{T}}\mathbf{V}\mathbf{e}_{kl}^{\ell}}\,.$$

Finally, we "connect the dots" between the points $\{(\sigma_k,\mu_k)\}_{k=0}^{n}$ to build an approximation to the ℓ-limited frontier in the $\sigma\mu$-plane. This can be done by linear interpolation. Specifically, for every $\mu \in (\mu_{k-1},\mu_k)$ we set

$$\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\,\sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\,\sigma_k\,.$$

Computing Frontiers: Linear Interpolation in Π^ℓ

A better way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)\,,$$

and then set

$$\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu)^{\mathrm{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu)}\,.$$

Remark. This will be a very good approximation if n is large enough. Over each interval (μ_{k-1}, μ_k) it approximates $\sigma_{\mathrm{f}}^{\ell}(\mu)$ with a hyperbola rather than with a line.

Computing Frontiers: Linear Interpolation in Π^ℓ

Remark. Because $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k) \in \Pi^\ell(\mu_k)$ and $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_{k-1}) \in \Pi^\ell(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu) \in \Pi^\ell(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k)\,.$$

Therefore $\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu)$ gives an approximation to the ℓ-limited frontier that lies on or to the right of the ℓ-limited frontier in the $\sigma\mu$-plane.

Remark. When there are no nodes in the interval (μ_{k-1}, μ_k) then we can use the two-fund property to show that $\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \sigma_{\mathrm{f}}^{\ell}(\mu)$.

General Portfolio with Two Risky Assets: m and V

Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\mathrm{mn}} = m_1$, $\mu_{\mathrm{mx}} = m_2$ and

$$\mu^{\ell}_{\mathrm{mn}} = m_1 - \ell(m_2 - m_1), \qquad \mu^{\ell}_{\mathrm{mn}} = m_2 + \ell(m_2 - m_1).$$

Recall that for every $\mu \in \mathbb{R}$ the unique portfolio allocation that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu) \in \Pi^{\ell}$ if and only if $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$.

General Portfolio with Two Risky Assets: $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ and $\sigma_{\mathrm{f}}^{\ell}(\mu)$

Therefore the set $\Pi^{\ell}(\mu)$ is given by

$$\Pi^{\ell} = \left\{ \mathbf{f}(\mu) \,:\, \mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}] \right\}.$$

In other words, the set Π^{ℓ} is the line segment in $\mathbb{R}^2$ that is the image of the interval $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$.

Because for every $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ the set $\Pi^{\ell}(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ over $\Pi^{\ell}(\mu)$ is $\mathbf{f}(\mu)$. Therefore the ℓ-limited frontier portfolios are

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu) = \mathbf{f}(\mu) \qquad \text{for } \mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}],$$

and the ℓ-limited frontier is given by

$$\sigma = \sigma_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\,\mathbf{f}(\mu)} \qquad \text{for } \mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}].$$

Hence, the ℓ-limited frontier is simply a segment of the frontier hyperbola. It has no nodes.

General Portfolio with Three Risky Assets: m and V

Recall the portfolio of three risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \leq m_2 \leq m_3 \,, \qquad m_1 < m_3 \,.$$

Then $\mu_{\mathrm{mn}} = m_1$, $\mu_{\mathrm{mx}} = m_3$ and

$$\mu_{\mathrm{mn}}^{\ell} = m_1 - \ell(m_3 - m_1)\,, \qquad \mu_{\mathrm{mn}}^{\ell} = m_3 + \ell(m_3 - m_1)\,.$$

General Portfolio with Three Risky Assets: $\mathbf{f}(\mu, \phi)$

Recall that for every $\mu \in \mathbb{R}$ the portfolios that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\,\mathbf{n}\,, \qquad \text{for some } \phi \in \mathbb{R}\,,$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1}\begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1}\begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

Here $\mathbf{f}_{13}(\mu)$ is the two-asset allocation for assets 1 and 3 that satisfies

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{13}(\mu) = 1\,, \qquad \mathbf{m}^{\mathrm{T}}\mathbf{f}_{13}(\mu) = \mu\,,$$

while $\mathbf{n}$ satisfies $\mathbf{1}^{\mathrm{T}}\mathbf{n} = 0$ and $\mathbf{m}^{\mathrm{T}}\mathbf{n} = 0$.

General Portfolio with Three Risky Assets: $\mathbf{f}(\mu,\phi) \in \Pi^\ell$

Because

$$\mathbf{f}(\mu,\phi) = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu - \phi\,(m_3 - m_2) \\ \phi\,(m_3 - m_1) \\ \mu - m_1 - \phi\,(m_2 - m_1) \end{pmatrix},$$

We see from (2.1) that $\mathbf{f}(\mu,\phi) \in \Pi^\ell$ if and only if $\mu \in [\mu^\ell_{\rm mn}, \mu^\ell_{\rm mx}]$ and

$$-\ell \leq \frac{m_3 - \mu}{m_3 - m_1} - \phi\,\frac{m_3 - m_2}{m_3 - m_1} \leq 1 + \ell\,,$$

$$-\ell \leq \phi \leq 1 + \ell\,,$$

$$-\ell \leq \frac{\mu - m_1}{m_3 - m_1} - \phi\,\frac{m_2 - m_1}{m_3 - m_1} \leq 1 + \ell\,.$$

General Portfolio with Three Risky Assets: $\phi^{\ell}_{\mathrm{mx}}(\mu)$, $\phi^{\ell}_{\mathrm{mn}}(\mu)$

For every $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$ these inequalities yield the bounds

$$
\begin{aligned}
-\frac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_3 - m_2} \leq \phi \leq \frac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_3 - m_2} \qquad & \text{if } m_2 < m_3 \,, \\
-\ell \leq \phi \leq 1 + \ell \,, \\
-\frac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_2 - m_1} \leq \phi \leq \frac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_2 - m_1} \qquad & \text{if } m_2 > m_1 \,.
\end{aligned}
$$

This region can be expressed as

$$
\phi^{\ell}_{\mathrm{mn}}(\mu) \leq \phi \leq \phi^{\ell}_{\mathrm{mx}}(\mu) \,,
$$

where $\phi^{\ell}_{\mathrm{mn}}(\mu)$ and $\phi^{\ell}_{\mathrm{mx}}(\mu)$ are defined on the next slide.

General Portfolio with Three Risky Assets: $\phi^{\ell}_{\mathrm{mx}}(\mu)$, $\phi^{\ell}_{\mathrm{mn}}(\mu)$

$$
\phi^{\ell}_{\mathrm{mx}}(\mu) = \begin{cases}
\min\left\{ 1+\ell \,,\; \dfrac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_3 - m_1} \right\} & \text{if } m_2 = m_1 \,, \\[2ex]
\min\left\{ \dfrac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_2 - m_1} \,,\; 1+\ell \,,\; \dfrac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_3 - m_2} \right\} & \text{if } m_2 \in (m_1, m_3) \,, \\[2ex]
\min\left\{ \dfrac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_3 - m_1} \,,\; 1+\ell \right\} & \text{if } m_2 = m_3 \,,
\end{cases}
$$

$$
\phi^{\ell}_{\mathrm{mn}}(\mu) = \begin{cases}
-\min\left\{ \dfrac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_3 - m_1} \,,\; \ell \right\} & \text{if } m_2 = m_1 \,, \\[2ex]
-\min\left\{ \dfrac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_3 - m_2} \,,\; \ell \,,\; \dfrac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_2 - m_1} \right\} & \text{if } m_2 \in (m_1, m_3) \,, \\[2ex]
-\min\left\{ \ell \,,\; \dfrac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_3 - m_1} \right\} & \text{if } m_2 = m_3 \,.
\end{cases}
$$

General Portfolio with Three Risky Assets: $\mathcal{H}^\ell$ and Π^ℓ

When $\ell > 0$ this region is the convex hexagon $\mathcal{H}^\ell$ in the $\mu\phi$-plane whose vertices are the six distinct points

$$\big(m_2 - \ell(m_3 - m_2)\,,\, 1 + \ell\big) \qquad \big(m_2 + \ell(m_2 - m_1)\,,\, 1 + \ell\big)$$

$$\big(m_1 - \ell(m_3 - m_1)\,,\, 0\,\big) \qquad \big(m_3 + \ell(m_3 - m_1)\,,\, 0\,\big)$$

$$\big(m_1 - \ell(m_2 - m_1), -\ell\big) \qquad \big(m_3 + \ell(m_3 - m_2), -\ell\big)$$

Therefore the set Π^ℓ is given by

$$\Pi^\ell = \Big\{ \mathbf{f}(\mu, \phi) \,:\, (\mu, \phi) \in \mathcal{H}^\ell \Big\}\,.$$

General Portfolio with Three Risky Assets: Π^ℓ and $\Pi^\ell(\mu)$

Therefore the sets Π^ℓ and $\Pi^\ell(\mu)$ can be visualized as follows.

- The set Π^ℓ is the hexagon in $\mathbb{R}^3$ that is the image of the hexagon $\mathcal{H}^\ell$ under the affine mapping $(\mu, \phi) \mapsto \mathbf{f}(\mu, \phi)$.
- For every $\mu \in [\mu^\ell_{\rm mn}, \mu^\ell_{\rm mx}]$ the set $\Pi^\ell(\mu)$ is the intersection of
  - the hexagon Π^ℓ in the plane $\{\mathbf{f} \in \mathbb{R}^3 \,:\, \mathbf{1}^{\rm T}\mathbf{f} = 1\}$ with
  - the transverse plane $\{\mathbf{f} \in \mathbb{R}^3 \,:\, \mathbf{m}^{\rm T}\mathbf{f} = \mu\}$.

  This is a line segment that might be a single point. It is given by

$$\Pi^\ell(\mu) = \left\{ \mathbf{f}(\mu, \phi) \,:\, \phi^\ell_{\rm mn}(\mu) \leq \phi \leq \phi^\ell_{\rm mx}(\mu) \right\}.$$

  Therefore the set $\Pi^\ell(\mu)$ is the line segment in $\mathbb{R}^3$ that is the image of the interval $[\phi^\ell_{\rm mn}(\mu), \phi^\ell_{\rm mx}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu, \phi)$.

General Portfolio with Three Risky Assets: $\phi_{\mathrm{mf}}(\mu)$

Hence, the point on the ℓ-limited frontier associated with $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$ is $(\sigma^{\ell}_{\mathrm{f}}(\mu), \mu)$ where $\sigma^{\ell}_{\mathrm{f}}(\mu)$ solves the constrained minimization problem

$$\begin{aligned} \sigma^{\ell}_{\mathrm{f}}(\mu)^2 &= \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \ : \ \mathbf{f} \in \Pi^{\ell}(\mu) \right\} \\ &= \min\left\{ \mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) \ : \ \phi^{\ell}_{\mathrm{mn}}(\mu) \leq \phi \leq \phi^{\ell}_{\mathrm{mx}}(\mu) \right\} . \end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\, \mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2 \mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}$$

is a quadratic in ϕ, we see that it has a unique global minimizer at

$$\phi = \phi_{\mathrm{mf}}(\mu) = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} .$$

The Markowitz frontier allocation is $\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}(\mu, \phi_{\mathrm{mf}}(\mu))$.

General Portfolio with Three Risky Assets: Minimizers

The global minimizer $\phi_{\mathrm{mf}}(\mu)$ will be the minimizer of our constrained minimization problem for the ℓ-limited frontier if and only if

$$\phi^{\ell}_{\mathrm{mn}}(\mu) \leq \phi_{\mathrm{mf}}(\mu) \leq \phi^{\ell}_{\mathrm{mx}}(\mu) .$$

Because the derivative of the objective function with respect to ϕ can be written as

$$\partial_\phi \mathbf{f}(\mu, \phi)^{\mathrm{T}} \mathbf{V} \mathbf{f}(\mu, \phi) = 2\, \mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{n} \left(\phi - \phi_{\mathrm{mf}}(\mu) \right) ,$$

we can read off the following.

- If $\phi_{\mathrm{mf}}(\mu) < \phi^{\ell}_{\mathrm{mn}}(\mu)$ then the objective function is increasing over $[\phi^{\ell}_{\mathrm{mn}}(\mu), \phi^{\ell}_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi^{\ell}_{\mathrm{mn}}(\mu)$.
- If $\phi^{\ell}_{\mathrm{mx}}(\mu) < \phi_{\mathrm{mf}}(\mu)$ then the objective function is decreasing over $[\phi^{\ell}_{\mathrm{mn}}(\mu), \phi^{\ell}_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi^{\ell}_{\mathrm{mx}}(\mu)$.

General Portfolio with Three Risky Assets: $\phi_{\mathrm{f}}^{\ell}(\mu)$, $\sigma_{\mathrm{f}}^{\ell}(\mu)$

Hence, the minimizer $\phi_{\mathrm{f}}^{\ell}(\mu)$ of our constrained minimization problem is

$$\begin{aligned}\phi_{\mathrm{f}}^{\ell}(\mu) &= \begin{cases} \phi_{\mathrm{mn}}^{\ell}(\mu) & \text{if } \phi_{\mathrm{mf}}(\mu) \leq \phi_{\mathrm{mn}}^{\ell}(\mu) \\ \phi_{\mathrm{mf}}(\mu) & \text{if } \phi_{\mathrm{mn}}^{\ell}(\mu) < \phi_{\mathrm{mf}}(\mu) < \phi_{\mathrm{mx}}^{\ell}(\mu) \\ \phi_{\mathrm{mx}}^{\ell}(\mu) & \text{if } \phi_{\mathrm{mx}}^{\ell}(\mu) < \phi_{\mathrm{mf}}(\mu) \end{cases} \\ &= \max\left\{\phi_{\mathrm{mn}}^{\ell}(\mu)\,,\ \min\left\{\phi_{\mathrm{mf}}(\mu)\,,\ \phi_{\mathrm{mx}}^{\ell}(\mu)\right\}\right\} \\ &= \min\left\{\max\left\{\phi_{\mathrm{mn}}^{\ell}(\mu)\,,\ \phi_{\mathrm{mf}}(\mu)\right\}\,,\ \phi_{\mathrm{mx}}^{\ell}(\mu)\right\}\,.\end{aligned}$$

Therefore the ℓ-limited frontier is given by

$$\sigma_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\mathbf{f}(\mu, \phi_{\mathrm{f}}^{\ell}(\mu))^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu, \phi_{\mathrm{f}}^{\ell}(\mu))}\,.$$