Portfolios that Contain Risky Assets
5.1. Limited-Leverage Portfolios

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

5. Models for Limited-Leverage Portfolios

Limited-Leverage Portfolios

Leveraged Markowitz Portfolios: Introduction

Leveraged portfolios are ones that take short positions. Short positions can offer the promise of great reward, but come with the potential for greater losses. They are favored by quantitative hedge funds, notable examples being the Medallion, RIEF, and RIDA funds run by Renaissance Technologies. They were also favored by investment banks during the first decade of the 21$^{\text{st}}$ century, and played a major role in bringing about the subsequent 2008 recession. They had a similar role in bringing about the great depression seventy eight years earlier. In fact, they have played a role in every major market crash, such as the dot-com crash of 2000.

Leveraged portfolios contribute to crashes because they have limits on their leverage. When their leverage exceeds their limit, their margins are called and they have to liquidate positions. This can lower asset values, which can stress other leveraged portfolios. Because leveraged portfolios can create systemic risk, every investor should know something about them.

Leveraged Markowitz Portfolios: Long-Short Decomp

The leverage of a general Markowitz portfolio can be quantified by decomposing its allocation $\mathbf{f}$ into its long and short positions as

$$\mathbf{f} = \mathbf{f}^+ - \mathbf{f}^- \,, \tag{1.1}$$

where $f_i^\pm$, the i^{th} entry of $\mathbf{f}^\pm$, is given by

$$f_i^+ = \max\{f_i \,, 0\} \,, \qquad f_i^- = \max\{-f_i \,, 0\} \,.$$

This is the so-called *long-short decomposition* of $\mathbf{f}$. The vectors $\mathbf{f}^+$ and $\mathbf{f}^-$ in this decomposition are characterized by

$$\mathbf{f}^+ \geq \mathbf{0} \,, \qquad \mathbf{f}^- \geq \mathbf{0} \,, \qquad (\mathbf{f}^+)^{\mathrm{T}}\mathbf{f}^- = 0 \,.$$

The multiples of the portfolio value that are held in long and short positions respectively are

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}^+ \,, \quad \text{and} \quad \mathbf{1}^{\mathrm{T}}\mathbf{f}^- \,. \tag{1.2}$$

Leveraged Markowitz Portfolios: Λ Characterization

Recall that the set of long Markowitz allocations is

$$\Lambda = \left\{ \mathbf{f} \in \mathcal{M} \; : \; \mathbf{f} \geq \mathbf{0} \right\}. \tag{1.3}$$

This can be characterized using the long-short decomposition.

Fact 1. We have

$$\Lambda = \left\{ \mathbf{f} \in \mathcal{M} \; : \; \mathbf{1}^{\mathrm{T}} \mathbf{f}^{-} = 0 \right\}. \tag{1.4}$$

Proof. Let $\mathbf{f} \in \Lambda$. Because $\mathbf{f} \geq \mathbf{0}$, we have $\mathbf{f}^{+} = \mathbf{f}$ and $\mathbf{f}^{-} = \mathbf{0}$. Therefore $\mathbf{1}^{\mathrm{T}} \mathbf{f}^{-} = 0$.

Conversely, let $\mathbf{f} \in \mathcal{M}$ such that $\mathbf{1}^{\mathrm{T}} \mathbf{f}^{-} = 0$. Let $\mathbf{f} = \mathbf{f}^{+} - \mathbf{f}^{-}$ be the long-short decomposition of $\mathbf{f}$ given by (1.1). Because $\mathbf{f}^{-} \geq \mathbf{0}$ while $\mathbf{1}^{\mathrm{T}} \mathbf{f}^{-} = 0$, we see that $\mathbf{f}^{-} = \mathbf{0}$. Therefore $\mathbf{f} = \mathbf{f}^{+} \geq \mathbf{0}$. Hence, $\mathbf{f} \in \Lambda$. □

Leveraged Markowitz Portfolios: Leverage Ratio $\lambda(\mathbf{f})$

Recall that leverged portfolios are ones that hold short positions – i.e. ones that are not long. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f}^{-}$ is the multiple of the value of a solvent portfolio that is held in short positions, we defined the *leverage ratio* of the portfolio by

$$\lambda(\mathbf{f}) = \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} . \tag{1.5}$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} \geq 0$, the leverage ratio is always nonnegative. From the decompostion (1.1) we see that $\mathbf{f}^{+} = \mathbf{f} + \mathbf{f}^{-}$, whereby the constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and definition (1.5) give

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}^{+} = \mathbf{1}^{\mathrm{T}}\mathbf{f} + \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} = 1 + \lambda(\mathbf{f}) . \tag{1.6}$$

We see from **Fact 1** that:

- long portfolios are those with $\lambda(\mathbf{f}) = 0$;
- leveraged portfolios are those with $\lambda(\mathbf{f}) > 0$.

Leveraged Markowitz Portfolios: $\|\mathbf{f}\|_1$ Formulation

The constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and decompostion (1.1) imply

$$1 = \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mathbf{1}^{\mathrm{T}}\mathbf{f}^{+} - \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} .$$

We also have

$$\|\mathbf{f}\|_1 = \mathbf{1}^{\mathrm{T}}\mathbf{f}^{+} + \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} ,$$

where $\|\mathbf{f}\|_1$ denotes the ℓ^1-norm of $\mathbf{f}$, which is defined by

$$\|\mathbf{f}\|_1 = \sum_{i=1}^{N} |f_i| .$$

Notice that $1 = |\mathbf{1}^{\mathrm{T}}\mathbf{f}| \leq \|\mathbf{f}\|_1$. By first adding and subtracting the top relation above from the second, and then multiplying by $\frac{1}{2}$, we obtain

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}^{+} = \tfrac{1}{2}\big(\|\mathbf{f}\|_1 + 1\big) , \qquad \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} = \tfrac{1}{2}\big(\|\mathbf{f}\|_1 - 1\big) . \tag{1.7}$$

Notice that $\mathbf{1}^{\mathrm{T}}\mathbf{f}^{+} \geq 1$ and that $\mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} \geq 0$.

Leveraged Markowitz Portfolios: $\lambda(\mathbf{f})$ and $\|\mathbf{f}\|_1$

The second formula of (1.7) gives a simple way to compute the leverage ratio (1.5) of a portfolio with allocation $\mathbf{f}$ — namely,

$$\lambda(\mathbf{f}) = \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} = \tfrac{1}{2}\big(\|\mathbf{f}\|_1 - 1\big)\,.$$

This formula does not require the long-short decomposition of $\mathbf{f}$. It is very easy to program.

Remark. By Fact 1 and the second formula of (1.7) we have

$$\Lambda = \Big\{\mathbf{f} \in \mathcal{M} \,:\, \|\mathbf{f}\|_1 = 1\Big\}\,. \tag{1.8}$$

This fact gives a simple way to determine if a portfolio with allocation $\mathbf{f}$ is long — namely, check if $\|\mathbf{f}\|_1 = 1$.

Limited-Leverage Portfolios: Introduction

The class $\mathcal{M}$ of Markowitz portfolios is unrealistic because it allows an investor to take short positions without any collateral. In practice short positions are restricted by *credit limits*.

If we assume that in each case the lender is the broker and the collateral is part of the portfolio then a simple model for credit limits is to constrain the total short position of the portfolio to be at most a positive multiple ℓ of the portfolio value. The value of ℓ is called the *leverage limit* of the portfolio and will depend upon market conditions, but brokers will often allow $\ell > 1$ and seldom allow $\ell > 5$.

Remark. Just because a broker allows a particular value of ℓ does not mean it is in the best interest of an investor to build a portfolio with that value of ℓ. We will use this model to understand what values of ℓ might not be prudent. This understanding will give us a measure of when markets are stressed.

Limited-Leverage Portfolios: Definition of Π^ℓ

The constraint that the multiple of the portfolio value held in short positions is bounded by a leverage limit ℓ can be expressed as

$$\lambda(\mathbf{f}) = \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} \leq \ell \,. \tag{2.9}$$

By (1.7) this is equivalent to

$$\tfrac{1}{2}\big(\|\mathbf{f}\|_1 - 1\big) = \mathbf{1}^{\mathrm{T}}\mathbf{f}^{-} \leq \ell \,.$$

Hence, the set of Markowitz allocations with a leverage limit $\ell \in [0,\infty)$ is

$$\Pi^\ell = \Big\{ \mathbf{f} \in \mathcal{M} \,:\, \|\mathbf{f}\|_1 \leq 1 + 2\,\ell \Big\} \,. \tag{2.10}$$

It is clear that if $\ell, \ell' \in [0,\infty)$ then

$$\ell \leq \ell' \qquad \Longrightarrow \qquad \Pi^\ell \subset \Pi^{\ell'} \,.$$

Limited-Leverage Portfolios: $\Pi^0 = \Lambda$, $\cup_\ell \Pi^\ell = \mathcal{M}$

By **Fact 1** and (1.8) we see that

$$\Pi^0 = \Big\{ \mathbf{f} \in \mathcal{M} \,:\, \|\mathbf{f}\|_1 \leq 1 \Big\} = \Lambda\,. \tag{2.11}$$

Hence, the limited leverage Markowitz allocations with leverage limit $\ell = 0$ are exactly the long Markowitz allocations.

It is clear from (2.10) that if $\ell \in [0, \infty)$ then $\Pi^\ell \subset \mathcal{M}$. Moreover, it is also clear that

$$\bigcup_{\ell \in [0,\infty)} \Pi^\ell = \mathcal{M}\,. \tag{2.12}$$

In words, the union of the sets Π^ℓ over $\ell \in [0, \infty)$ is $\mathcal{M}$.

Long-Short Pair Allocations: Definition of $\mathbf{e}_{ij}^{\lambda}$

Definition. The *long-short pair allocation* of asset i with asset j and leverage ratio $\lambda > 0$ is defined to be $\mathbf{e}_{ij}^{\lambda}$ given by

$$\mathbf{e}_{ij}^{\lambda} = (1+\lambda)\,\mathbf{e}_i - \lambda\,\mathbf{e}_j \,. \tag{3.13}$$

It is clear that

$$\mathbf{1}^{\mathrm{T}}\mathbf{e}_{ij}^{\lambda} = (1+\lambda)\,\mathbf{1}^{\mathrm{T}}\mathbf{e}_i - \lambda\,\mathbf{1}^{\mathrm{T}}\mathbf{e}_j = (1+\lambda) - \lambda = 1\,,$$

whereby $\mathbf{e}_{ij}^{\lambda} \in \mathcal{M}$. Moreover, the allocation $\mathbf{e}_{ij}^{\lambda}$ holds:

- a long position in asset i with allocation $1+\lambda$,
- a short position in asset j with allocation $-\lambda$,

so clearly $\lambda(\mathbf{e}_{ij}^{\lambda}) = \lambda$. Therefore $\mathbf{e}_{ij}^{\lambda} \in \Pi^{\ell}$ if and only if $\lambda \le \ell$.

Long-Short Pair Allocations: On the Line

For every i, j with $j \neq i$ and $\lambda > 0$ we have

$$\mathbf{e}_{ij}^{\lambda} = (1+\lambda)\,\mathbf{e}_i - \lambda\,\mathbf{e}_j\,, \qquad \mathbf{e}_{ji}^{\lambda} = (1+\lambda)\,\mathbf{e}_j - \lambda\,\mathbf{e}_i\,.$$

We see that both $\mathbf{e}_{ij}^{\lambda}$ and $\mathbf{e}_{ji}^{\lambda}$ lie on the line in $\mathbb{R}^N$ through $\mathbf{e}_i$ and $\mathbf{e}_j$. Moreover, they are distinct because

$$\mathbf{e}_{ij}^{\lambda} - \mathbf{e}_{ji}^{\lambda} = (1+2\,\lambda)\,(\mathbf{e}_i - \mathbf{e}_j) \neq \mathbf{0}\,.$$

Therefore $\mathbf{e}_{ij}^{\lambda}$ and $\mathbf{e}_{ji}^{\lambda}$ determine the same line and $\mathbf{e}_i$ and $\mathbf{e}_j$ can be expressed as

$$\begin{aligned} \mathbf{e}_i &= \frac{1+\lambda}{1+2\,\lambda}\,\mathbf{e}_{ij}^{\lambda} + \frac{\lambda}{1+2\,\lambda}\,\mathbf{e}_{ji}^{\lambda}\,, \\ \mathbf{e}_j &= \frac{\lambda}{1+2\,\lambda}\,\mathbf{e}_{ij}^{\lambda} + \frac{1+\lambda}{1+2\,\lambda}\,\mathbf{e}_{ji}^{\lambda}\,. \end{aligned} \tag{3.14}$$

Hence, $\mathbf{e}_i$ and $\mathbf{e}_j$ are convex combinations of $\mathbf{e}_{ij}^{\lambda}$ and $\mathbf{e}_{ji}^{\lambda}$, and thereby lie on the line segment connecting $\mathbf{e}_{ij}^{\lambda}$ and $\mathbf{e}_{ji}^{\lambda}$.

Long-Short Pair Allocations: On the Line

For every i, j with $j \neq i$ and $\ell > 0$ we have

$$\mathbf{e}_{ij}^{\ell} = (1+\ell)\,\mathbf{e}_i - \ell\,\mathbf{e}_j\,, \qquad \mathbf{e}_{ji}^{\ell} = (1+\ell)\,\mathbf{e}_j - \ell\,\mathbf{e}_i\,.$$

We see that both $\mathbf{e}_{ij}^{\ell}$ and $\mathbf{e}_{ji}^{\ell}$ also lie on the line in $\mathbb{R}^N$ through $\mathbf{e}_i$ and $\mathbf{e}_j$ and are distinct. Therefore $\mathbf{e}_{ij}^{\ell}$ and $\mathbf{e}_{ji}^{\ell}$ determine the same line and for any $\lambda > 0$ we can express $\mathbf{e}_{ij}^{\lambda}$ and $\mathbf{e}_{ji}^{\lambda}$ as

$$\begin{aligned} \mathbf{e}_{ij}^{\lambda} &= \frac{1+\ell+\lambda}{1+2\,\ell}\,\mathbf{e}_{ij}^{\ell} + \frac{\ell-\lambda}{1+2\,\ell}\,\mathbf{e}_{ji}^{\ell}\,, \\ \mathbf{e}_{ji}^{\lambda} &= \frac{\ell-\lambda}{1+2\,\ell}\,\mathbf{e}_{ij}^{\ell} + \frac{1+\ell+\lambda}{1+2\,\ell}\,\mathbf{e}_{ji}^{\ell}\,. \end{aligned} \tag{3.15}$$

Hence, $\mathbf{e}_{ij}^{\lambda}$ and $\mathbf{e}_{ji}^{\lambda}$ are convex combinations of $\mathbf{e}_{ij}^{\ell}$ and $\mathbf{e}_{ji}^{\ell}$ if and only if $\lambda \leq \ell$. In that case they lie on the line segment connecting $\mathbf{e}_{ij}^{\ell}$ and $\mathbf{e}_{ji}^{\ell}$.

Long-Short Pair Allocations: Convex Hulls

Recall the definition of a convex hull.

Definition. Let $\mathbb{X}$ be a linear space over $\mathbb{R}$. Let $\mathcal{S} \subset \mathbb{X}$. The *convex hull* of $\mathcal{S}$ is $\mathrm{Hull}(\mathcal{S}) \subset \mathbb{X}$ given by

$$\mathrm{Hull}(\mathcal{S}) = \Big\{\text{all convex combinations of vectors in } \mathcal{S}\Big\}\,.$$

Remark. $\mathrm{Hull}(\mathcal{S})$ is the smallest convex set that contains $\mathcal{S}$.

Example. Let $\mathcal{E} \subset \mathbb{R}^N$ be defined by

$$\mathcal{E} = \Big\{\mathbf{e}_i \;:\; i = 1, \cdots, N\Big\}\,. \qquad (3.16)$$

Then $\mathrm{Hull}(\mathcal{E}) = \Lambda$ because every convex combination of vectors in $\mathcal{E}$ has the form

$$\sum_i f_i\,\mathbf{e}_i\,,\,, \quad \text{where} \quad f_i \geq 0 \quad \text{and} \quad \sum_i f_i = 1\,,$$

which is the case if and only if $\mathbf{f} \in \Lambda$.

Long-Short Pair Allocations: **Fact 2**

An important fact about the long-short pair allocations is the following.

Fact 2. Let $\lambda > 0$ and $\mathcal{E}^\lambda \subset \mathbb{R}^N$ be defined by

$$\mathcal{E}^\lambda = \left\{ \mathbf{e}_{ij}^\lambda \,:\, i,j = 1, \cdots, N\,,\ j \neq i \right\}. \tag{3.17}$$

If $\mathbf{f} \in \mathcal{M}$ with $\lambda(\mathbf{f}) = \lambda$ then $\mathbf{f} \in \mathrm{Hull}(\mathcal{E}^\lambda)$.

Proof. Let $\mathbf{f} \in \mathcal{M}$ with $\lambda(\mathbf{f}) = \lambda$. Let $\mathbf{f} = \mathbf{f}^+ - \mathbf{f}^-$ be the long-short decomposition of $\mathbf{f}$. Then using the fact that

$$f_i^+ \geq 0\,, \qquad f_i^- \geq 0\,, \qquad f_i^+ f_i^- = 0 \qquad \text{for every } i\,,$$
$$\sum_i f_i^+ = \mathbf{1}^{\mathrm{T}}\mathbf{f}^+ = 1 + \lambda\,, \qquad \sum_i f_i^+ \mathbf{e}_i = \mathbf{f}^+\,,$$
$$\sum_j f_j^- = \mathbf{1}^{\mathrm{T}}\mathbf{f}^- = \lambda\,, \qquad \sum_j f_j^- \mathbf{e}_j = \mathbf{f}^-\,,$$

we have the following calculations.

Long-Short Pair Allocations: **Fact 2** Proof

Because

$$\sum_{i,j\neq i} \frac{f_i^+ f_j^-}{(1+\lambda)\lambda} = \sum_{i,j} \frac{f_i^+ f_j^-}{(1+\lambda)\lambda} = \left(\sum_i \frac{f_i^+}{1+\lambda}\right)\left(\sum_j \frac{f_j^-}{\lambda}\right) = 1\,,$$

and

$$\begin{aligned}\sum_{i,j\neq i} \frac{f_i^+ f_j^-}{(1+\lambda)\lambda}\,\mathbf{e}_{ij}^\lambda &= \sum_{i,j} \frac{f_i^+ f_j^-}{(1+\lambda)\lambda}\left((1+\lambda)\,\mathbf{e}_i - \lambda\,\mathbf{e}_j\right) \\ &= \left(\sum_j \frac{f_j^-}{\lambda}\right)\left(\sum_i f_i^+\,\mathbf{e}_i\right) - \left(\sum_i \frac{f_i^+}{1+\lambda}\right)\left(\sum_j f_j^-\,\mathbf{e}_j\right) \\ &= \mathbf{f}^+ - \mathbf{f}^- = \mathbf{f}\,.\end{aligned}$$

we see that $\mathbf{f} \in \mathrm{Hull}(\mathcal{E}^\lambda)$.

Long-Short Pair Allocations: **Fact 3**

Fact 2. leads to the following fact.

Fact 3. Let $\ell > 0$. Then $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$.

Proof. Because $\mathcal{E}^\ell \subset \Pi^\ell$ and Π^ℓ is a convex set, we have

$$\mathrm{Hull}(\mathcal{E}^\ell) \subset \mathrm{Hull}(\Pi^\ell) = \Pi^\ell .$$

Therefore $\mathrm{Hull}(\mathcal{E}^\ell) \subset \Pi^\ell$.

To prove the inclusion going the other way, let $\mathbf{f} \in \Pi^\ell$. Let $\lambda = \lambda(\mathbf{f})$. There are two cases to consider:

- $\lambda = 0$,
- $\lambda \in (0, \ell]$.

Long-Short Pair Allocations: **Fact 3** Proof

If $\lambda = 0$ then $\mathbf{f} \in \Lambda = \mathrm{Hull}(\mathcal{E})$, where $\mathcal{E}$ is given by (3.16). But (3.14) with $\lambda = \ell$ implies that $\mathcal{E} \subset \mathrm{Hull}(\mathcal{E}^\ell)$, whereby

$$\Lambda = \mathrm{Hull}(\mathcal{E}) \subset \mathrm{Hull}(\mathcal{E}^\ell)\,.$$

Hence, $\mathbf{f} \in \mathrm{Hull}(\mathcal{E}^\ell)$ for the first case.

If $\lambda \in (0, \ell]$ then **Fact 2** implies that $\mathbf{f} \in \mathrm{Hull}(\mathcal{E}^\lambda)$, where $\mathcal{E}^\lambda$ is given by (3.17). But (3.15) with $\lambda \in (0, \ell]$ implies that $\mathcal{E}^\lambda \subset \mathrm{Hull}(\mathcal{E}^\ell)$, whereby

$$\mathrm{Hull}(\mathcal{E}^\lambda) \subset \mathrm{Hull}(\mathcal{E}^\ell)\,.$$

Hence, $\mathbf{f} \in \mathrm{Hull}(\mathcal{E}^\ell)$ for the second casse.

Therefore $\mathbf{f} \in \mathrm{Hull}(\mathcal{E}^\ell)$ for both cases, whereby $\Pi^\ell \subset \mathrm{Hull}(\mathcal{E}^\ell)$. □

Return Means and Volatilities: Introduction

The return mean and volatility of any $\mathbf{f} \in \mathcal{M}$ are

$$\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,. \tag{4.18}$$

For every $\ell \geq 0$ the set $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$ is nonempty, convex, and bounded. Let $\mu(\Pi^\ell)$ and $\sigma(\Pi^\ell)$ be the images of Π^ℓ given by

$$\mu(\Pi^\ell) = \{\mu(\mathbf{f}) \,:\, \mathbf{f} \in \Pi^\ell\}\,, \qquad \sigma(\Pi^\ell) = \{\sigma(\mathbf{f}) \,:\, \mathbf{f} \in \Pi^\ell\}\,. \tag{4.19}$$

Observe the following.

- These functions are uniformly continuous over $\mathcal{M}$, and are thereby uniformly continuous over Π^ℓ.
- These functions are convex over $\mathcal{M}$, and are thereby convex over the convex set Π^ℓ.
- The images $\mu(\Pi^\ell)$ and $\sigma(\Pi^\ell)$ are pathwise connected, and are thereby nonempty intervals.

Here we study these intervals.

Return Means and Volatilities: Π^ℓ Closed

Fact 4. For every $\ell \geq 0$ the set Π^ℓ is a closed set.

Proof. For any $\mathbf{f}$ in the closure of Π^ℓ there exists a sequence $\{\mathbf{f}_n\}_{n\in\mathbb{N}} \subset \Pi^\ell$ such that

$$\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_n \,.$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{f}_n = 1$ and $\|\mathbf{f}_n\|_1 \leq 1 + 2\ell$ for every $n \in \mathbb{N}$, we see that

$$\mathbf{1}^{\mathrm{T}}\mathbf{f} = \lim_{n\to\infty} \mathbf{1}^{\mathrm{T}}\mathbf{f}_n = 1 \,, \qquad \|\mathbf{f}\|_1 = \lim_{n\to\infty} \|\mathbf{f}_n\|_1 \leq 1 + 2\ell \,.$$

Hence, $\mathbf{f} \in \Pi^\ell$. Therefore Π^ℓ is a closed set. □

Remark. Because Π^ℓ is convex, it is *pathwise connected*. Because Π^ℓ is closed and bounded, it is *compact*. Pathwise connectedness and compactness are properties of sets that are inherited by their images under continuous mappings.

Return Means and Volatilities: Endpoints

Because the mappings $\mathbf{f} \mapsto \mu(\mathbf{f})$ and $\mathbf{f} \mapsto \sigma(\mathbf{f})$ are continuous and the set Π^ℓ is compact, the sets $\mu(\Pi^\ell)$ and $\sigma(\Pi^\ell)$ are compact and are thereby closed intervals.

- Let $\mu(\Pi^\ell) = [\mu^\ell_{\rm mn}, \mu^\ell_{\rm mx}]$ where

$$\mu^\ell_{\rm mn} = \min\left\{\mathbf{m}^{\rm T}\mathbf{f} \,:\, \mathbf{f} \in \Pi^\ell\right\}, \qquad \mu^\ell_{\rm mx} = \max\left\{\mathbf{m}^{\rm T}\mathbf{f} \,:\, \mathbf{f} \in \Pi^\ell\right\}.$$

- Let $\sigma(\Pi^\ell) = [\sigma^\ell_{\rm mv}, \sigma^\ell_{\rm mx}]$ where

$$\sigma^\ell_{\rm mv} = \min\left\{\sqrt{\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}} \,:\, \mathbf{f} \in \Pi^\ell\right\}, \qquad \sigma^\ell_{\rm mx} = \max\left\{\sqrt{\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}} \,:\, \mathbf{f} \in \Pi^\ell\right\}.$$

Notice that $\sigma^\ell_{\rm mv} \geq \sigma_{\rm mv}$ with equality if and only if $\mathbf{f}_{\rm mv} \in \Pi^\ell$.

Return Means and Volatilities: Convexity and Maximums

The usefulness of **Fact 3** becomes evident in the next fact.

Fact 5. Let $\ell > 0$. Let $\psi : \Pi^\ell \to \mathbb{R}$ be a convex continuous function. Then

$$\max\Big\{\psi(\mathbf{f}) \;:\; \mathbf{f} \in \Pi^\ell\Big\} = \max\Big\{\psi(\mathbf{f}) \;:\; \mathbf{f} \in \mathcal{E}^\ell\Big\}. \tag{4.20}$$

Proof. Let $\mathbf{f}_*$ be a maximizer of the first problem. Because $\mathbf{f}_* \in \Pi^\ell$ and $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$ by **Fact 3**, there exists $\alpha_{ij} \geq 0$ such that

$$\mathbf{f}_* = \sum_{i,j\neq i} \alpha_{ij}\, \mathbf{e}^\lambda_{ij}, \qquad \sum_{i,j\neq i} \alpha_{ij} = 1.$$

By the convexity of ψ we then have

$$\psi(\mathbf{f}_*) \leq \sum_{i,j\neq i} \alpha_{ij}\, \psi(\mathbf{e}^\lambda_{ij}).$$

But this implies the maximum of the first problem is not greater than the maximum of the second. But because $\mathcal{E}^\ell \subset \Pi^\ell$, it cannot be less.

Return Means and Volatilities: Values on $\mathcal{E}^\ell$

To use **Fact 5** we need the return mean and variance of alloaction $\mathbf{e}_{ij}^\lambda$. These are

$$\begin{aligned}
\mathbf{m}^{\mathrm{T}}\mathbf{e}_{ij}^\lambda &= (1+\lambda)\,\mathbf{m}^{\mathrm{T}}\mathbf{e}_i - \lambda\,\mathbf{m}^{\mathrm{T}}\mathbf{e}_j \\
&= (1+\lambda)\,m_i - \lambda\,m_j\,, \\
(\mathbf{e}_{ij}^\lambda)^{\mathrm{T}}\mathbf{V}\mathbf{e}_{ij}^\lambda &= (1+\lambda)^2\mathbf{e}_i^{\mathrm{T}}\mathbf{V}\mathbf{e}_i - 2(1+\lambda)\lambda\,\mathbf{e}_i^{\mathrm{T}}\mathbf{V}\mathbf{e}_j + \lambda^2\mathbf{e}_j^{\mathrm{T}}\mathbf{V}\mathbf{e}_j \\
&= (1+\lambda)^2 v_{ii} - 2(1+\lambda)\lambda\,v_{ij} + \lambda^2 v_{jj}\,.
\end{aligned} \tag{4.21}$$

Remark. Investors who take such positions are betting that asset i will outperform asset j, no matter which direction the market moves.

Remark. If $m_i > m_j$ and $v_{ij} > v_{ii}$ then a classical long-short strategy is to minimize the volatility by setting

$$\lambda = \frac{v_{ij} - v_{ii}}{v_{ii} - 2v_{ij} + v_{jj}}\,.$$

Return Means and Volatilities: Return Mean Extrema

Upon applying **Fact 5** to $\mathbf{m}^{\rm T}\mathbf{f}$ and $-\mathbf{m}^{\rm T}\mathbf{f}$ and using (4.21) we find that

$$\begin{aligned}
\mu^{\ell}_{\rm mx} &= \max\left\{\mathbf{m}^{\rm T}\mathbf{f} \;:\; \mathbf{f} \in \Pi^{\ell}\right\} = \max\left\{\mathbf{m}^{\rm T}\mathbf{e}^{\ell}_{ij} \;:\; \mathbf{e}^{\ell}_{ij} \in \mathcal{E}^{\ell}\right\} \\
&= \max_{i,j\neq i}\left\{(1+\ell)\, m_i - \ell\, m_j\right\} = (1+\ell)\,\mu_{\rm mx} - \ell\,\mu_{\rm mn} \\
&= \mu_{\rm mx} + \ell\,(\mu_{\rm mx} - \mu_{\rm mn})\,. \\
\mu^{\ell}_{\rm mn}\, & \min\left\{\mathbf{m}^{\rm T}\mathbf{f} \;:\; \mathbf{f} \in \Pi^{\ell}\right\} = \min\left\{\mathbf{m}^{\rm T}\mathbf{e}^{\ell}_{ij} \;:\; \mathbf{e}^{\ell}_{ij} \in \mathcal{E}^{\ell}\right\} \\
&= \min_{i,j\neq i}\left\{(1+\ell)\, m_i - \ell\, m_j\right\} = (1+\ell)\,\mu_{\rm mn} - \ell\,\mu_{\rm mx} \\
&= \mu_{\rm mn} - \ell\,(\mu_{\rm mx} - \mu_{\rm mn})\,,
\end{aligned}$$

where $\mu_{\rm mx}$ and $\mu_{\rm mn}$ are given by

$$\mu_{\rm mx} = \max_{i}\,\{m_i\}\,, \qquad \mu_{\rm mn} = \min_{i}\,\{m_i\}\,.$$

Return Means and Volatilities: The Image $\mu(\Pi^\ell)$

We thereby see that the image $\mu(\Pi^\ell)$ is given by

$$\mu(\Pi^\ell) = [\mu^\ell_{\mathrm{mn}}\,,\,\mu^\ell_{\mathrm{mx}}]\,, \tag{4.22a}$$

with endpoints given by

$$\begin{aligned} \mu^\ell_{\mathrm{mn}} &= \mu_{\mathrm{mn}} - \ell\,(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,, \\ \mu^\ell_{\mathrm{mx}} &= \mu_{\mathrm{mx}} + \ell\,(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,, \end{aligned} \tag{4.22b}$$

where

$$\mu_{\mathrm{mn}} = \min_i \{m_i\}\,, \qquad \mu_{\mathrm{mx}} = \max_i \{m_i\}\,. \tag{4.22c}$$

- $\mu = \mu^\ell_{\mathrm{mn}}$ for any portfolio with allocation $-\ell$ in an asset with return mean μ_{mx} and allocation $1+\ell$ in an asset with return mean μ_{mn}.
- $\mu = \mu^\ell_{\mathrm{mx}}$ for any portfolio with allocation $-\ell$ in an asset with return mean μ_{mn} and allocation $1+\ell$ in an asset with return mean μ_{mx}.

Return Means and Volatilities: Variance Maximum

Upon applying **Fact 5** to $\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ and using (4.21) we find that

$$\begin{aligned} v_{\mathrm{mx}}^{\ell} &\equiv \max\left\{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Pi^{\ell}\right\} = \max\left\{\left(\mathbf{e}_{ij}^{\ell}\right)^{\mathrm{T}}\mathbf{V}\mathbf{e}_{ij}^{\ell} \;:\; \mathbf{e}_{ij}^{\ell} \in \mathcal{E}^{\ell}\right\} \\ &= \max_{i,j\neq i}\left\{(1+\ell)^2 v_{ii} - 2\,(1+\ell)\,\ell\, v_{ij} + \ell^2 v_{jj}\right\}. \end{aligned} \tag{4.23}$$

This cannot be expressed in terms of v_{mx}, where

$$v_{\mathrm{mx}} = \max_{i}\left\{v_{ii}\right\}.$$

However, we can bound v_{mx}^{ℓ} above by using the rough bounds

$$v_{ii} \leq v_{\mathrm{mx}}\,, \qquad -v_{ij} \leq v_{\mathrm{mx}}\,, \qquad v_{jj} \leq v_{\mathrm{mx}}\,,$$

to get the upper bound

$$v_{\mathrm{mx}}^{\ell} \leq (1+\ell)^2 v_{\mathrm{mx}} + 2(1+\ell)\ell\, v_{\mathrm{mx}} + \ell^2 v_{\mathrm{mx}} = (1+2\,\ell)^2 v_{\mathrm{mx}}\,. \tag{4.24}$$

Return Means and Volatilities: Maximum Volatility

The maximum volatility $\sigma^\ell_{\mathrm{mx}}$ is defined by $\sigma^\ell_{\mathrm{mx}} = \sqrt{v^\ell_{\mathrm{mx}}}$, where v^ℓ_{mx} is given by (4.23). We have the lower bound

$$\begin{aligned}(1+\ell)^2 v_{ii} - 2\,(1+\ell)\,\ell\, v_{ij} + \ell^2 v_{jj} &\geq (1+\ell)^2\sigma_i^2 - 2\,(1+\ell)\,\ell\,\sigma_i\,\sigma_j + \ell^2\sigma_j^2 \\ &= \left((1+\ell)\,\sigma_i - \ell\,\sigma_j\right)^2 ,\end{aligned}$$

where $\sigma_i = \sqrt{v_{ii}}$ and $\sigma_j = \sqrt{v_{jj}}$. We then see from (4.23) that

$$v^\ell_{\mathrm{mx}} \geq \max_{i,j\neq i} \left\{ \left((1+\ell)\,\sigma_i - \ell\,\sigma_j\right)^2 \right\} = \left((1+\ell)\,\sigma_{\mathrm{mx}} - \ell\,\sigma_{\mathrm{mn}}\right)^2 ,$$

where

$$\sigma_{\mathrm{mx}} = \max_i \{\sigma_i\} , \qquad \sigma_{\mathrm{mn}} = \min_i \{\sigma_i\} .$$

Upon combining this lower bound with upper bound (4.24) we get

$$\sigma_{\mathrm{mx}} + \ell\,(\sigma_{\mathrm{mx}} - \sigma_{\mathrm{mn}}) \leq \sigma^\ell_{\mathrm{mx}} \leq (1 + 2\,\ell)\,\sigma_{\mathrm{mx}} .$$

Return Means and Volatilities: The Image $\sigma(\Pi^\ell)$

We thereby see that the image $\sigma(\Pi^\ell)$ is given by

$$\sigma(\Pi^\ell) = \left[\sigma^\ell_{\rm mv}\,,\, \sigma^\ell_{\rm mx}\right]\,, \tag{4.25a}$$

with endpoints given by

$$\begin{aligned} \sigma^\ell_{\rm mv} &= \min\left\{\sqrt{\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}}\ :\ \mathbf{f} \in \Pi^\ell\right\}\,, \\ \sigma^\ell_{\rm mx} &= \max\left\{\sqrt{(\mathbf{e}^\ell_{ij})^{\rm T}\mathbf{V}\mathbf{e}^\ell_{ij}}\ :\ \mathbf{e}^\ell_{ij} \in \mathcal{E}^\ell\right\}\,, \end{aligned} \tag{4.25b}$$

which satisfy the rough bounds

$$\begin{aligned} \sigma_{\rm mv} \leq \sigma^\ell_{\rm mv} &\leq \sigma_{\rm mn}\,, \\ \sigma_{\rm mx} + \ell\,(\sigma_{\rm mx} - \sigma_{\rm mn}) \leq \sigma^\ell_{\rm mx} &\leq (1 + 2\,\ell)\,\sigma_{\rm mx}\,. \end{aligned} \tag{4.25c}$$

where

$$\sigma_{\rm mn} = \min_i \{\sigma_i\}\,, \qquad \sigma_{\rm mx} = \max_i \{\sigma_i\}\,. \tag{4.25d}$$

The Slice $\Pi^\ell(\mu)$

Recall that Π^ℓ is the set of all limited-leverage portfolio allocations with leverage limit $\ell \geq 0$ and that $\Pi^\ell(\mu)$ is the set of all such allocations with return mean μ. These sets are given by

$$\begin{aligned} \Pi^\ell &= \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \; \|\mathbf{f}\|_1 \leq 1 + 2\,\ell\,, \right\}, \\ \Pi^\ell(\mu) &= \left\{ \mathbf{f} \in \Pi^\ell \;:\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}. \end{aligned} \tag{5.26}$$

Clearly $\Pi^\ell(\mu) \subset \Pi^\ell$ for every $\mu \in \mathbb{R}$.

- The set $\Pi^\ell = \mathrm{Hull}(\mathcal{E}^\ell)$ is a convex polytope of dimension $N - 1$ that is contained in the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\}$.
- The set $\Pi^\ell(\mu)$ is the intersection of Π^ℓ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. Because we have assumed that $\mathbf{m}$ and $\mathbf{1}$ are not proportional, the intersection of these hyperplanes is a set of dimension $N - 2$.
- The set $\Pi^\ell(\mu)$ is nonempty if and only if $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$.

The Slice $\Pi^\ell(\mu)$

Therefore for every $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ the set $\Pi^\ell(\mu)$ is a nonempty, closed, bounded, convex polytope of dimension at most $N - 2$.

- When $N = 2$ it is a point.
- When $N = 3$ it is either a point or a line segment.
- When $N = 4$ it is either a point, a line segment, or a convex polygon.