Portfolios that Contain Risky Assets
4.3. Metrics for Long Portfolios

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Part I: Portfolio Models

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4. Models for Long Portfolios

Metrics for Long Portfolios

Introduction

Now we again enlarge the set of metrics that we will explore to include information about long portfolios. Specifically, we consider applying the five portfolio metric functions developed previously

$$\omega^{\lambda}(\mathbf{f}), \quad \omega^{\mu}(\mathbf{f}), \quad \omega^{\sigma}(\mathbf{f}), \quad \omega^{\delta}(\mathbf{f}), \quad \omega^{\rho}(\mathbf{f}), \tag{1.1}$$

to portfolio allocations $\mathbf{f}$ on the efficient long frontier. The natural candidate for this role is the efficient long tangent allocation, $\mathbf{f}_{\mathrm{elt}}$.

We will also finalize the set of metrics that will be used for **Project One**. The metrics will be organized into six classes: Stability, Leverage, Liquidity, Efficiency, Proximity and Sharpe. Each team has three of these classes assigned to them. The goal is too explore whether any of the assigned metrics is a good leading indicator of a potential systemic downturn in the market.

Introduction: Stability Metrics

The stability metrics are the only ones not derived from one of the five portfolio metric functions (1.1). Rather, they derive from the Tobin frontier parameter $\nu_{\mathrm{rf}} > 0$, which for a given risk-free rate μ_{rf} is determined by

$$\nu_{\mathrm{rf}}^2 = \nu_{\mathrm{mv}}^2 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}} \right)^2 .$$

The associated tangency point and asymptote intersection metrics are

$$\omega_{\mathrm{rf}}^{\mathrm{tg}} = \frac{\nu_{\mathrm{mv}}^2}{\nu_{\mathrm{rf}}^2} , \qquad \omega_{\mathrm{rf}}^{\mathrm{as}} = \frac{\nu_{\mathrm{mv}}}{\nu_{\mathrm{rf}}} .$$

Applying these to $\mu_{\mathrm{rf}} = \mu_{\mathrm{si}}$ and $\mu_{\mathrm{rf}} = \mu_{\mathrm{cl}}$ gives the four stability metrics:

$$\omega_{\mathrm{si}}^{\mathrm{tg}} , \qquad \omega_{\mathrm{si}}^{\mathrm{as}} , \qquad \omega_{\mathrm{cl}}^{\mathrm{tg}} , \qquad \omega_{\mathrm{cl}}^{\mathrm{as}} . \tag{1.2}$$

Portfolio Metrics: Leverage

The five portfolio metric functions (1.1) are built from five functions of $\mathbf{f} \in \mathcal{M}$: the leverage ratio $\lambda(\mathbf{f})$, the downside potential $\delta(\mathbf{f})$, the return mean $\mu(\mathbf{f})$, the volatility $\sigma(\mathbf{f})$, and the Sharpe ratio $\rho(\mathbf{f})$.

The leverage ratio is given by

$$\lambda(\mathbf{f}) = \tfrac{1}{2}\left(\|\mathbf{f}\|_1 - 1\right).$$

It was used to define the leverge metric function by

$$\omega^{\lambda}(\mathbf{f}) = \frac{\lambda(\mathbf{f})}{1 + \lambda(\mathbf{f})}. \tag{2.3}$$

The leverage ratio and metric do not have an explicit dependence upon the return history. They can gain statictical meaning by applying them to allocations $\mathbf{f}$ that do depend on the return history.

Portfolio Metrics: Leverage Metrics

The leverage metric function $\omega^\lambda(\mathbf{f})$ given by (2.3) had been applied to

- the minimum volatility allocation $\mathbf{f}_{\mathrm{mv}}$,
- the safe tangent allocation $\mathbf{f}_{\mathrm{st}}$,
- the credit tangent allocation $\mathbf{f}_{\mathrm{ct}}$.

Because $\lambda(\mathbf{f}) = 0$ for every $\mathbf{f} \in \Lambda$, no knowledge is gained by applying it to a long portfolio. In particular, it makes no sense to apply it to individual assets. We are thereby left with three leverage metrics:

$$\omega^\lambda_{\mathrm{mv}} = \omega^\lambda(\mathbf{f}_{\mathrm{mv}}) \,, \qquad \omega^\lambda_{\mathrm{st}} = \omega^\lambda(\mathbf{f}_{\mathrm{st}}) \,, \qquad \omega^\lambda_{\mathrm{ct}} = \omega^\lambda(\mathbf{f}_{\mathrm{ct}}) \,. \tag{2.4}$$

Remark. Markowitz frontier allocations lie on the line $\mathbf{f} = \mathbf{f}_{\mathrm{mf}}(\mu)$ in $\mathcal{M}$. Hence, because $\lambda(\mathbf{f})$ is a convex function over $\mathcal{M}$, we can bound $\lambda(\mathbf{f})$ above by convex combinations of its value at $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$ and $\mathbf{f}_{\mathrm{ct}}$.

Portfolio Metrics: Liquidity

The return mean, $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, and downside potential,

$$\delta(\mathbf{f}) = \max\left\{ -\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \;:\; d = 1, \cdots, D \right\},$$

were used to define the liquidity metric function by

$$\omega^{\delta}(\mathbf{f}) = \begin{cases} \dfrac{\delta(\mathbf{f}) + \mu(\mathbf{f})}{1 + \mu(\mathbf{f})} & \text{if } \delta(\mathbf{f}) < 1\,, \\ 1 & \text{if } \delta(\mathbf{f}) \geq 1\,. \end{cases} \tag{2.5}$$

The downside potential (and through it, the liquidity metric function) has an explicit dependence on the return history. Moreover, this dependence goes beyond the mean-variance statistics of $\mathbf{m}$ and $\mathbf{V}$.

Portfolio Metrics: Liquidity Metrics

The liquidity metric function $\omega^\delta(\mathbf{f})$ given by (2.5) can be applied to any $\mathbf{f} \in \mathcal{M}$. However, it is most useful when applied to leveraged portfolio allocations that have the potential of not being solvent. It had been applied to

- the minimum volatility allocation $\mathbf{f}_{\mathrm{mv}}$,
- the safe tangent allocation $\mathbf{f}_{\mathrm{st}}$,
- the credit tangent allocation $\mathbf{f}_{\mathrm{ct}}$.

Because long portfolios are not leveraged, we will not add to this list. This leaves us with a total of three liquidity metrics:

$$\omega^\delta_{\mathrm{mv}} = \omega^\delta(\mathbf{f}_{\mathrm{mv}}) \,, \qquad \omega^\delta_{\mathrm{st}} = \omega^\delta(\mathbf{f}_{\mathrm{st}}) \,, \qquad \omega^\delta_{\mathrm{ct}} = \omega^\delta(\mathbf{f}_{\mathrm{ct}}) \,. \tag{2.6}$$

Remark. Markowitz frontier allocations lie on the line $\mathbf{f} = \mathbf{f}_{\mathrm{mf}}(\mu)$ in $\mathcal{M}$. Hence, because $\delta(\mathbf{f})$ is a convex function over $\mathcal{M}$, we can bound $\delta(\mathbf{f})$ above by convex combinations of its value at $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$ and $\mathbf{f}_{\mathrm{ct}}$.

Portfolio Metrics: Efficiency and Proximity

The volatility, $\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$, and return mean, $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, were used to define the efficiency and proximity metric functions by

$$\omega^{\mu}(\mathbf{f}) = \frac{\mu_{\mathrm{emf}}(\sigma(\mathbf{f})) - \mu(\mathbf{f})}{\mu_{\mathrm{emf}}(\sigma(\mathbf{f})) - \mu_{\mathrm{imf}}(\sigma(\mathbf{f}))}\,, \qquad \omega^{\sigma}(\mathbf{f}) = \sqrt{1 - \frac{\sigma_{\mathrm{mf}}(\mu(\mathbf{f}))^2}{\sigma(\mathbf{f})^2}}\,. \tag{2.7}$$

The efficiency and proximity metric functions have explicit dependence upon the return history through the Markowitz frontier parameters μ_{mv}, σ_{mv} and ν_{mv}, which determine $\mu_{\mathrm{emf}}(\sigma)$, $\mu_{\mathrm{imf}}(\sigma)$ and $\sigma_{\mathrm{mf}}(\mu)$.

Portfolio Metrics: Efficiency and Proximity Metrics

The efficiency and proximity metric functions, $\omega^\mu(\mathbf{f})$ and $\omega^\sigma(\mathbf{f})$ given by (2.7), had been applied to

- the equity index fund $\mathbf{e}_{\mathrm{EI}}$,
- the total bond index fund $\mathbf{e}_{\mathrm{BI}}$.

They were not applied to portfolios on the Markowitz frontier because

- $\omega^\mu(\mathbf{f}) = \omega^\sigma(\mathbf{f}) = 0$ for all portfolios on the efficient frontier,
- $\omega^\mu(\mathbf{f}) = 1$ and $\omega^\sigma(\mathbf{f}) = 0$ for all portfolios on the inefficient frontier.

In particular, it made no sense to apply them to $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$, or $\mathbf{f}_{\mathrm{ct}}$.

However, long portfolios generally do not lie on the Markowitz, so we add the efficient long tangent allocation $\mathbf{f}_{\mathrm{elt}}$ to the list. This gives three efficiency metrics and three proximity metrics:

$$\begin{aligned} \omega^\mu_{\mathrm{EI}} &= \omega^\mu(\mathbf{e}_{\mathrm{EI}})\,, & \omega^\mu_{\mathrm{BI}} &= \omega^\mu(\mathbf{e}_{\mathrm{BI}})\,, & \omega^\mu_{\mathrm{elt}} &= \omega^\mu(\mathbf{f}_{\mathrm{elt}})\,, \\ \omega^\sigma_{\mathrm{EI}} &= \omega^\sigma(\mathbf{e}_{\mathrm{EI}})\,, & \omega^\sigma_{\mathrm{BI}} &= \omega^\sigma(\mathbf{e}_{\mathrm{BI}})\,, & \omega^\sigma_{\mathrm{elt}} &= \omega^\sigma(\mathbf{f}_{\mathrm{elt}})\,. \end{aligned} \tag{2.8}$$

Portfolio Metrics: Sharpe Ratio

The volatility, $\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$, return mean, $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, and safe investment rate μ_{si} were used to define the Shape ratio as

$$\rho(\mathbf{f}) = \frac{\mu(\mathbf{f}) - \mu_{\mathrm{si}}}{\sigma(\mathbf{f})} \,,$$

from which we defined the Sharpe metric function by

$$\omega^{\rho}(\mathbf{f}) = 1 - \max\left\{0 \,,\, \frac{\rho(\mathbf{f})}{\nu_{\mathrm{si}}}\right\} \,. \tag{2.9}$$

The Sharpe ratio and metric function have an explicit dependence upon the return history and the risk-free rate μ_{si}. Recall that $\rho(\mathbf{f})/\nu_{\mathrm{si}}$ is the correlation of the portfolio with allocation $\mathbf{f}$ with any portfolio on the efficient Tobin frontier for the rate μ_{si}.

Portfolio Metrics: Sharpe Metrics

The Sharpe metric function $\omega^\rho(\mathbf{f})$ can be applied to any $\mathbf{f} \in \mathcal{M}$. It vanishes only for $\mathbf{f} = \mathbf{f}_{\mathrm{st}}$ when $\mathbf{f}_{\mathrm{st}}$ lies on the efficient Markowitz frontier. Recall that $\mathbf{f}_{\mathrm{st}}$ lies on the efficient Markowitz frontier if and only if $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}}$. When $\mu_{\mathrm{si}} > \mu_{\mathrm{mv}}$ we have $\omega^\rho(\mathbf{f}_{\mathrm{st}}) = 1$. Hence, $\omega^\rho(\mathbf{f}_{\mathrm{st}})$ can take just two values, 0 or 1. It is more informative for other portfolios. We had applied it to

- the credit tangent allocation $\mathbf{f}_{\mathrm{ct}}$,
- the equity index fund $\mathbf{e}_{\mathrm{EI}}$.

We now add the efficient long tangent allocation $\mathbf{f}_{\mathrm{elt}}$ to this list. This gives the three Sharpe metrics:

$$\omega^\rho_{\mathrm{EI}} = \omega^\rho(\mathbf{e}_{\mathrm{EI}}) \,, \qquad \omega^\rho_{\mathrm{ct}} = \omega^\rho(\mathbf{f}_{\mathrm{ct}}) \,, \qquad \omega^\rho_{\mathrm{elt}} = \omega^\rho(\mathbf{f}_{\mathrm{elt}}) \,. \tag{2.10}$$

Summary of the Metric Classes

The six metric classes for **Project One** are comprised as follows.

- **Stability Metrics:** $\omega_{\mathrm{si}}^{\mathrm{tg}}\,, \quad \omega_{\mathrm{cl}}^{\mathrm{tg}}\,, \quad \omega_{\mathrm{si}}^{\mathrm{as}}\,, \quad \omega_{\mathrm{cl}}^{\mathrm{as}}\,,$ given by (1.2).
- **Leverage Metrics:** $\omega_{\mathrm{mv}}^{\lambda}\,, \quad \omega_{\mathrm{st}}^{\lambda}\,, \quad \omega_{\mathrm{ct}}^{\lambda}\,,$ given by (2.4).
- **Liquidity Metrics:** $\omega_{\mathrm{mv}}^{\delta}\,, \quad \omega_{\mathrm{st}}^{\delta}\,, \quad \omega_{\mathrm{ct}}^{\delta}\,,$ given by (2.6).
- **Efficiency Metrics:** $\omega_{\mathrm{EI}}^{\mu}\,, \quad \omega_{\mathrm{BI}}^{\mu}\,, \quad \omega_{\mathrm{elt}}^{\mu}\,,$ given by (2.8).
- **Proximity Metrics:** $\omega_{\mathrm{EI}}^{\sigma}\,, \quad \omega_{\mathrm{BI}}^{\sigma}\,, \quad \omega_{\mathrm{elt}}^{\sigma}\,,$ given by (2.8).
- **Sharpe Metrics:** $\omega_{\mathrm{EI}}^{\rho}\,, \quad \omega_{\mathrm{ct}}^{\rho}\,, \quad \omega_{\mathrm{elt}}^{\rho}\,,$ given by (2.10).

Each team has been assigned a unique combination of three classes.