Portfolios that Contain Risky Assets
4.2. Long Portfolios with a Safe Investment

C. David Levermore

University of Maryland, College Park, MD

Math 420: *Mathematical Modeling*
March 15, 2022 version

Portfolios that Contain Risky Assets
Part I: Portfolio Models

1. Preliminary Topics
2. Markowitz Portfolio Model
3. Models for Portfolios with Risk-Free Assets
4. Models for Long Portfolios
5. Models for Limited-Leverage Portfolios

Portfolios that Contain Risky Assets
Part I: Portfolio Models

4. Models for Long Portfolios

Long Portfolios with a Safe Investment

Efficient Long Frontier

We now consider investors who will not hold a short position in *any asset*. Such an investor should not borrow to invest in risky assets, so the safe investment is the only risky-free asset that we will consider.

We will use the capital allocation line construction to obtain the *efficient long frontier* for long portfolios that might include the safe investment. We assume that the long frontier for the risky assets has already been constructed, and is given by $\sigma = \sigma_{\mathrm{lf}}(\mu)$ for $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$, where

$$\mu_{\mathrm{mn}} = \min\{m_i \,:\, i = 1, \cdots, N\},$$
$$\mu_{\mathrm{mx}} = \max\{m_i \,:\, i = 1, \cdots, N\}.$$

We will assume that $\mu_{\mathrm{si}} < \mu_{\mathrm{mx}}$, because otherwise the safe investment is more efficient than any long portfolio of risky asset.

Efficient Long Frontier

The capital allocation line between the safe investment and the portfolio on the long frontier with return μ is the line segment in the $\sigma\mu$-plane between the points $(0, \mu_{\mathrm{si}})$ and $(\sigma_{\mathrm{lf}}(\mu), \mu)$, the slope of which is

$$\nu_{\mathrm{ca}}(\mu) = \frac{\mu - \mu_{\mathrm{si}}}{\sigma_{\mathrm{lf}}(\mu)} .$$

The efficient long frontier is obtained by first finding the capital allocation line with the greatest slope. In other words, we solve

$$\mu_{\mathrm{lt}} = \arg\max\{\nu_{\mathrm{ca}}(\mu) \, : \, \mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]\} . \tag{1.1}$$

We set $\nu_{\mathrm{lt}} = \nu_{\mathrm{ca}}(\mu_{\mathrm{lt}})$ and $\sigma_{\mathrm{lt}} = \sigma_{\mathrm{lf}}(\mu_{\mathrm{lt}})$.

Efficient Long Frontier

Let us consider the maximization problem given in (1.1):

$$\mu_{\rm lt} = \arg\max\{\nu_{\rm ca}(\mu) \,:\, \mu \in [\mu_{\rm mn}, \mu_{\rm mx}]\}\,,$$

where

$$\nu_{\rm ca}(\mu) = \frac{\mu - \mu_{\rm si}}{\sigma_{\rm lf}(\mu)}\,.$$

Recall that the function $\mu \mapsto \sigma_{\rm lf}(\mu)$ is positive and continuous over $[\mu_{\rm mn}, \mu_{\rm mx}]$. This implies that the function $\mu \mapsto \nu_{\rm ca}(\mu)$ is continuous over $[\mu_{\rm mn}, \mu_{\rm mx}]$, which implies that it has a maximum over $[\mu_{\rm mn}, \mu_{\rm mx}]$. Because $\mu_{\rm si} < \mu_{\rm mx}$ we see that

$$\nu_{\rm ca}(\mu_{\rm mx}) = \frac{\mu_{\rm mx} - \mu_{\rm si}}{\sigma_{\rm lf}(\mu_{\rm mx})} > 0\,,$$

which implies that the maximum must be positive.

Efficient Long Frontier

Because the function $\mu \mapsto \sigma_{\rm lf}(\mu)$ is strictly convex over $[\mu_{\rm mn}, \mu_{\rm mx}]$, the maximizer $\mu_{\rm lt}$ must be unique.

If $\mu_{\rm lt} < \mu_{\rm mx}$ then the function $\mu \mapsto \sigma_{\rm lf}(\mu)$ is increasing over $[\mu_{\rm lt}, \mu_{\rm mx}]$ that maps onto $[\sigma_{\rm lt}, \sigma_{\rm mx}]$, where $\sigma_{\rm mx} = \sigma_{\rm lf}(\mu_{\rm mx})$.

Let $\sigma \mapsto \sigma_{\rm lf}^{-1}(\sigma)$ denote the inverse function of $\mu \mapsto \sigma_{\rm lf}(\mu)$ over $[\sigma_{\rm lt}, \sigma_{\rm mx}]$. Then the efficient long frontier is then given by $\mu = \mu_{\rm elf}(\sigma)$ where

$$\mu_{\rm elf}(\sigma) = \begin{cases} \mu_{\rm si} + \nu_{\rm lt}\sigma & \text{for } \sigma \in [0, \sigma_{\rm lt}]\,, \\ \sigma_{\rm lf}^{-1}(\sigma) & \text{for } \sigma \in [\sigma_{\rm lt}, \sigma_{\rm lf}(\mu_{\rm mx})]\,. \end{cases} \tag{1.2}$$

Efficient Long Frontier

Remark. If we had also added a credit line to the portolio then we would have had to find the credit tangency portfolio and added the appropriate capital allocation line to the efficient long frontier. Typically there are two kinds of credit lines an investor might consider.

- A cedit line available from a broker usually requires that some of our risky assets be held as collateral.
- A credit line available from a bank might use real estate as collateral.

The downside of using a credit line from a broker is that when the market goes down then the broker can force us either to add assets to our collateral or to sell assets in a low market to pay off the loan.

Shape of the Efficient Long Frontier

The downside of using a credit line from a bank with real estate as collateral is that if the price of real estate falls then we again might be forced to sell assets in a low market to pay off our loan.

For investors who hold short positions in risky assets, these risks are hedged because they will also make money when markets go down. *Investors who hold long portfolios in risky assets and use a credit line can find themselves highly exposed to large losses in a market downturn.* It is an unwise position to take — yet many do in a bubble.

Computing the Tangent Portfolio

We now address how to solve the maximization problem given in (1.1):

$$\mu_{\mathrm{lt}} = \arg\max\left\{ \frac{\mu - \mu_{\mathrm{si}}}{\sigma_{\mathrm{lf}}(\mu)} \,:\, \mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}] \right\} . \tag{2.3}$$

Recall that $\sigma_{\mathrm{lf}}(\mu)$ was approximated by partitioning the interval $[\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ as

$$\mu_{\mathrm{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathrm{mx}} , \tag{2.4}$$

where n was large enough to resolve all the features of the long frontier. At each μ_k we then used quadprog to compute $\mathbf{f}_{\mathrm{lf}}(\mu_k)$ and set

$$\sigma_k = \sigma_{\mathrm{lf}}(\mu_k) = \sqrt{\mathbf{f}_{\mathrm{lf}}(\mu_k)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{lf}}(\mu_k)} . \tag{2.5}$$

Finally, we "connected the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the long frontier in the $\sigma\mu$-plane.

Computing the Tangent Portfolio

The way we "connected the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ was motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we approximated the long frontier portfolios by

$$\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{lf}}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{lf}}(\mu_k)\,, \tag{2.6}$$

and approximated the long frontier by

$$\tilde{\sigma}_{\mathrm{lf}}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu)^{\mathrm{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathrm{lf}}(\mu)}\,. \tag{2.7}$$

Computing the Tangent Portfolio

Now consider the numbers

$$\left\{ \frac{\mu_k - \mu_{\mathrm{si}}}{\sigma_k} \;:\; k = 0, 1, \cdots, n \right\} .$$

These will be an increasing function of k until some $k = k_{\mathrm{mx}}$, after which they will decrease. We can then make the rough approximations

$$\tilde{\mu}_{\mathrm{lt}} = \mu_{k_{\mathrm{mx}}} , \qquad \tilde{\sigma}_{\mathrm{lt}} = \sigma_{k_{\mathrm{mx}}} , \qquad \tilde{\mathbf{f}}_{\mathrm{lt}} = \sigma_{\mathrm{lf}}(\mu_{k_{\mathrm{mx}}}) . \tag{2.8}$$

These approximations can be made better by taking n larger. More specifically, if the $\{\mu_k\}$ are uniformly spaced then these approximations will be good enough whenever

$$\frac{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}{n} \quad \text{is smaller than our desired resolution} .$$

Computing the Tangent Portfolio

Better approximations can be found by using approximation (2.7) to solve

$$\tilde{\mu}_{\mathrm{lt}} = \arg\max\left\{ \frac{\mu - \mu_{\mathrm{si}}}{\tilde{\sigma}_{\mathrm{lf}}(\mu)} \;:\; \mu \in [\mu_{k_{\mathrm{mx}}-1}, \mu_{k_{\mathrm{mx}}+1}] \right\}, \tag{2.9a}$$

then using approximations (2.7) and (2.6) to obtain

$$\tilde{\sigma}_{\mathrm{lt}} = \tilde{\sigma}_{\mathrm{lf}}(\tilde{\mu}_{\mathrm{lt}}), \qquad \tilde{\mathbf{f}}_{\mathrm{lt}} = \tilde{\mathbf{f}}_{\mathrm{lf}}(\tilde{\mu}_{\mathrm{lt}}). \tag{2.9b}$$

Here we set $\mu_{k_{\mathrm{mx}}+1} = \mu_{\mathrm{mx}}$ in (2.9a) when $k_{\mathrm{mx}} = n$. The solution of (2.9a) can be found analytically, but we skip the details here.

Shape of the Efficient Long Frontier

We will now examine the shape of the efficient long frontier for the case when $\sigma'_{\mathrm{lf}}(\mu_{\mathrm{mx}}) > 0$ and $\sigma'_{\mathrm{lf}}(\mu_{\mathrm{mn}}) \leq 0$, which is a common situation.

The tangent line to the curve $\sigma = \sigma_{\mathrm{lf}}(\mu)$ at the point $(\sigma_{\mathrm{mx}}, \mu_{\mathrm{mx}})$ will intersect the μ-axis at $\mu = \eta_{\mathrm{mx}}$ where

$$\eta_{\mathrm{mx}} = \mu_{\mathrm{mx}} - \frac{\sigma_{\mathrm{lf}}(\mu_{\mathrm{mx}})}{\sigma'_{\mathrm{lf}}(\mu_{\mathrm{mx}})}\,.$$

We will consider the cases $\mu_{\mathrm{si}} \geq \eta_{\mathrm{mx}}$ and $\mu_{\mathrm{si}} < \eta_{\mathrm{mx}}$ separately.

Shape of the Efficient Long Frontier

For the case when $\mu_{\mathrm{si}} \geq \eta_{\mathrm{mx}}$ we will make the additional assumption that $\mu_{\mathrm{si}} < \mu_{\mathrm{mx}}$. Then the efficient long frontier is simply given by

$$\mu_{\mathrm{elf}}(\sigma) = \mu_{\mathrm{si}} + \frac{\mu_{\mathrm{mx}} - \mu_{\mathrm{si}}}{\sigma_{\mathrm{mx}}}\,\sigma \qquad \text{for } \sigma \in [0, \sigma_{\mathrm{mx}}]\,.$$

Our additional assumption states that there is at least one risky asset that has a return mean greater than the return for the safe investment. This is usually the case. If it is not, the formula for $\mu_{\mathrm{elf}}(\sigma)$ can be modified by appealing to the capital allocation line construction.

Remark. Notice that $\mu_{\mathrm{elf}}(\sigma)$ given above is increasing over $\sigma \in [0, \sigma_{\mathrm{mx}}]$. When $\mu_{\mathrm{si}} = \mu_{\mathrm{mx}}$ the capital allocation line construction would produce an expression for $\mu_{\mathrm{elf}}(\sigma)$ that is constant, but might be defined over an interval larger than $[0, \sigma_{\mathrm{mx}}]$. When $\mu_{\mathrm{si}} > \mu_{\mathrm{mx}}$ the capital allocation line construction would produce an expression for $\mu_{\mathrm{elf}}(\sigma)$ that is decreasing over an interval larger than $[0, \sigma_{\mathrm{mx}}]$.

Shape of the Efficient Long Frontier

For the case when $\mu_{\rm si} < \eta_{\rm mx}$ there is a frontier portfolio $(\sigma_{\rm lt}, \mu_{\rm lt})$ such that the capital allocation between it and $(0, \mu_{\rm si})$ lies above the efficient long frontier. This means that $\mu_{\rm lt} > \mu_{\rm si}$ and

$$\frac{\mu - \mu_{\rm si}}{\mu_{\rm lt} - \mu_{\rm si}}\,\sigma_{\rm lt} \leq \sigma_{\rm lf}(\mu) \quad \text{for every } \mu \in [\mu_{\rm mn}, \mu_{\rm mx}]\,.$$

Because $\sigma_{\rm lf}(\mu)$ is an increasing, continuous function over $[\mu_{\rm lt}, \mu_{\rm mx}]$ with image $[\sigma_{\rm lt}, \sigma_{\rm mx}]$, it has an increasing, continuous inverse function $\sigma_{\rm lf}^{-1}(\sigma)$ over $[\sigma_{\rm lt}, \sigma_{\rm mx}]$ with image $[\mu_{\rm lt}, \mu_{\rm mx}]$. The efficient long frontier is then given by

$$\mu_{\rm elf}(\sigma) = \begin{cases} \mu_{\rm si} + \dfrac{\mu_{\rm lt} - \mu_{\rm si}}{\sigma_{\rm lt}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\rm lt}]\,, \\ \sigma_{\rm lf}^{-1}(\sigma) & \text{for } \sigma \in [\sigma_{\rm lt}, \sigma_{\rm mx}]\,. \end{cases}$$

General Portfolio with Two Risky Assets

Recall the portfolio of two risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Here we will assume that $m_1 < m_2$, so that $\mu_{\text{mn}} = m_1$ and $\mu_{\text{mx}} = m_2$.

The minimum volatility portfolio is

$$\mathbf{f}_{\text{mv}} = \frac{1}{v_{11} + v_{22} - 2v_{12}} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix}.$$

This will be a long portfolio if and only if $v_{12} \le v_{11}$ and $v_{12} \le v_{22}$.

General Portfolio with Two Risky Assets

The long frontier is given by

$$\sigma_{\mathrm{lf}}(\mu) = \sqrt{\sigma_{\mathrm{mv}}^2 + \left(\frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}}\right)^2} \qquad \text{for } \mu \in [m_1, m_2]\,,$$

where the frontier parameters are

$$\sigma_{\mathrm{mv}} = \sqrt{\frac{v_{11}v_{22} - v_{12}^2}{v_{11} + v_{22} - 2v_{12}}}\,, \qquad \nu_{\mathrm{mv}} = \sqrt{\frac{(m_2 - m_1)^2}{v_{11} + v_{22} - 2v_{12}}}\,,$$

$$\mu_{\mathrm{mv}} = \frac{(v_{22} - v_{12})m_1 + (v_{11} - v_{12})m_2}{v_{11} + v_{22} - 2v_{12}}\,.$$

Recall that $v_{11} + v_{22} - 2v_{12} > 0$ because $\mathbf{V}$ is positive definite.

General Portfolio with Two Risky Assets

It follows that

$$\mu_{\rm mv} - m_1 = \frac{(v_{11} - v_{12})(m_2 - m_1)}{v_{11} + v_{22} - 2v_{12}}\,,$$
$$m_2 - \mu_{\rm mv} = \frac{(v_{22} - v_{12})(m_2 - m_1)}{v_{11} + v_{22} - 2v_{12}}\,.$$

There are three cases to consider.

Case 1. If $v_{12} > v_{11}$ then $v_{12} < v_{22}$ and

$$\mu_{\rm mv} < m_1 < m_2\,.$$

In this case the efficient long frontier is given by

$$(\sigma_{\rm lf}(\mu), \mu) \quad \text{where} \quad \mu \in [m_1, m_2]\,.$$

In other words, the entire long frontier is efficient.

General Portfolio with Two Risky Assets

Case 2. If $v_{12} \leq v_{11}$ and $v_{12} \leq v_{22}$ then $\mathbf{f}_{\mathrm{mv}} \geq \mathbf{0}$ and

$$m_1 \leq \mu_{\mathrm{mv}} \leq m_2 \,.$$

This will be the case when $v_{12} \leq 0$, which is how two-asset portfolios are often built. In this case the efficient long frontier is given by

$$(\sigma_{\mathrm{lf}}(\mu), \mu) \quad \text{where} \quad \mu \in [\mu_{\mathrm{mv}}, m_2] \,.$$

In other words, only part of the long frontier is efficient.

Case 3. If $v_{12} > v_{22}$ then $v_{12} < v_{11}$ and

$$m_1 < m_2 < \mu_{\mathrm{mv}} \,.$$

In this case the efficient long frontier is the single point (σ_2, m_2).

General Portfolio with Two Risky Assets

Now we show how Case 2 is modified by the inclusion of a safe investment. The μ-intercept of the tangent line through $(\sigma_{\rm mx}, \mu_{\rm mx}) = (\sigma_2, m_2)$ is

$$\eta_{\rm mx} = \mu_{\rm mx} - \frac{\sigma_{\rm lf}(\mu_{\rm mx})}{\sigma_{\rm lf}'(\mu_{\rm mx})}$$
$$= m_2 - \frac{\nu_{\rm mv}^2\,\sigma_2^2}{m_2 - \mu_{\rm mv}} = \frac{v_{22}m_1 - v_{12}m_2}{v_{22} - v_{12}}\,.$$

We will present the two cases that arise in order of increasing complexity: $\eta_{\rm mx} \leq \mu_{\rm si}$ and $\mu_{\rm si} < \eta_{\rm mx}$.

When $\eta_{\rm mx} \leq \mu_{\rm si}$ the efficient long frontier is determined by

$$\mu_{\rm elf}(\sigma) = \mu_{\rm si} + \frac{m_2 - \mu_{\rm si}}{\sigma_2}\,\sigma \quad \text{for } \sigma \in [0, \sigma_2]\,.$$

General Portfolio with Two Risky Assets

When $\mu_{\mathrm{si}} < \eta_{\mathrm{mx}}$ the tangency portfolio parameters are given by

$$\nu_{\mathrm{lt}} = \nu_{\mathrm{mv}} \sqrt{1 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}{\nu_{\mathrm{mv}}\, \sigma_{\mathrm{mv}}} \right)^2}\,, \qquad \sigma_{\mathrm{lt}} = \sigma_{\mathrm{mv}} \sqrt{1 + \left(\frac{\nu_{\mathrm{mv}}\, \sigma_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}} \right)^2}\,,$$

and the efficient long frontier is determined by

$$\mu_{\mathrm{elf}}(\sigma) = \begin{cases} \mu_{\mathrm{si}} + \nu_{\mathrm{lt}}\, \sigma & \text{for } \sigma \in [0, \sigma_{\mathrm{lt}}]\,, \\ \mu_{\mathrm{mv}} + \nu_{\mathrm{mv}} \sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2} & \text{for } \sigma \in [\sigma_{\mathrm{lt}}, \sigma_2]\,. \end{cases}$$