# Portfolios that Contain Risky Assets
# 4.1. Long Portfolios and Their Frontiers

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# Portfolios that Contain Risky Assets
# Part I: Portfolio Models


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# Portfolios that Contain Risky Assets
# Part I: Portfolio Models

## 4. Models for Long Portfolios

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# Long Portfolios and Their Frontiers


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# The Set $\Lambda$: Definition

Because the value of any portfolio with short positions can become negative, many investors will not hold a short position in any risky asset. Portfolios that hold no short positions are called *long portfolios*.

A Markowitz portfolio with allocation $\mathbf{f}$ is long if and only if $f_i \geq 0$ for every $i$. This can be expressed compactly as

$$\mathbf{f} \geq \mathbf{0}\,, \tag{1.1}$$

where $\mathbf{0}$ denotes the $N$-vector with each entry equal to 0 and the inequality is understood entrywise. Therefore the set of all *long Markowitz allocations* $\Lambda$ is given by

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\, \mathbf{f} \geq \mathbf{0} \right\}. \tag{1.2}$$

# The Set Λ: Convex Combinations

Let $\mathbf{e}_i$ denote the vector whose $i^{\text{th}}$ entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where $f_i \geq 0$ for every $i = 1, \cdots, N$ and

$$\sum_{i=1}^{N} f_i = \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \,.$$

This shows that Λ is just all convex combinations of the vectors $\{\mathbf{e}_i\}_{i=1}^{N}$. We can visualize Λ when $N$ is small.

When $N = 2$ it is the line segment with vertices

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

# The Set $\Lambda$: Visualization for $N = 3$ and $N = 4$

When $N = 3$ it is the triangle with vertices

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron with vertices

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general $N$ it is the simplex with vertices at the vectors $\{\mathbf{e}_i\}_{i=1}^{N}$.

# The Set $\Lambda$: Visualization for $N = 4$ in $\mathbb{R}^3$

**Remark.** When $N = 4$ it is easy to check that the tetrahedron $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ given by

$$\mathcal{T} = \left\{ \mathbf{z} \in \mathbb{R}^3 \;:\; \mathbf{w}_k \cdot \mathbf{z} \leq 1 \;\text{ for } k = 1, 2, 3, 4 \right\},$$

where

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

under the one-to-one affine mapping $\mathbf{\Phi} : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\mathbf{\Phi}(\mathbf{z}) = \frac{1}{4} \begin{pmatrix} 1 - \mathbf{w}_1 \cdot \mathbf{z} \\ 1 - \mathbf{w}_2 \cdot \mathbf{z} \\ 1 - \mathbf{w}_3 \cdot \mathbf{z} \\ 1 - \mathbf{w}_4 \cdot \mathbf{z} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -z_1 \\ -z_2 \\ -z_3 \end{pmatrix}.$$

# The Set Λ: Closed, Bounded and Convex

Because $\Lambda$ is the simplex with vertices at the vectors $\{\mathbf{e}_i\}_{i=1}^N$, it is a nonempty, convex, and bounded set. In addition, $\Lambda$ is a closed set.

**Proof.** For any $\mathbf{f}$ in the closure of $\Lambda$ there exists a sequence $\{\mathbf{f}_n\}_{n\in\mathbb{N}} \subset \Lambda$ such that

$$\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_n \,.$$

Because $\mathbf{f}_n \geq \mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}\mathbf{f}_n = 1$ for every $n \in \mathbb{N}$, we see that

$$\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_n \geq \mathbf{0} \,, \qquad \mathbf{1}^{\mathrm{T}}\mathbf{f} = \lim_{n\to\infty} \mathbf{1}^{\mathrm{T}}\mathbf{f}_n = 1 \,.$$

Hence, $\mathbf{f} \in \Lambda$. Therefore $\Lambda$ is a closed set. □

Therefore $\Lambda$ is a nonempty, closed, bounded, convex set.

# The Set Λ: Return Means Bounded

Because the set Λ is a bounded, its return means are bounded. Let

$$\mu_{\mathrm{mn}} = \min_i \{m_i\} \,, \qquad \mu_{\mathrm{mx}} = \max_i \{m_i\} \,.$$

Then because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda$ we have

$$\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \sum_{i=1}^{N} m_i f_i \geq \mu_{\mathrm{mn}} \sum_{i=1}^{N} f_i = \mu_{\mathrm{mn}} \, \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mu_{\mathrm{mn}} \,,$$

$$\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \sum_{i=1}^{N} m_i f_i \leq \mu_{\mathrm{mn}} \sum_{i=1}^{N} f_i = \mu_{\mathrm{mx}} \, \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mu_{\mathrm{mx}} \,.$$

Therefore the return mean $\mu$ of any $\mathbf{f} \in \Lambda$ satisfies

$$\mu_{\mathrm{mn}} \leq \mu \leq \mu_{\mathrm{mx}} \,.$$

These bounds are sharp.

# The Set Λ: Bounded Volatilities

Because Λ is bounded, its return variances and volatilities are bounded. Let

$$v_{\mathrm{mx}} = \max_i \{v_{ii}\}\,, \qquad \sigma_{\mathrm{mx}} = \sqrt{v_{\mathrm{mx}}}\,.$$

Because $v_{ij} = c_{ij}\sqrt{v_{ii}v_{jj}}$ and $|c_{ij}| \leq 1$ we see that

$$|v_{ij}| = |c_{ij}|\sqrt{v_{ii}v_{jj}} \leq \sqrt{v_{ii}v_{jj}} \leq v_{\mathrm{mx}}\,.$$

Then because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda$ we have

$$v = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} = \sum_{i,j=1}^{N} f_i v_{ij} f_j \leq \sum_{i,j=1}^{N} f_i |v_{ij}| f_j \leq v_{\mathrm{mx}} \sum_{i,j=1}^{N} f_i f_j = v_{\mathrm{mx}}(\mathbf{1}^{\mathrm{T}}\mathbf{f})^2 = v_{\mathrm{mx}}\,.$$

Therefore the return variance $v$ and volatility $\sigma$ of any $\mathbf{f} \in \Lambda$ satisfy

$$v_{\mathrm{mv}} \leq v \leq v_{\mathrm{mx}}\,, \qquad \sigma_{\mathrm{mv}} \leq \sigma \leq \sigma_{\mathrm{mx}}\,.$$

These upper bounds are sharp. The lower bounds will be sharp only when $\mathbf{f}_{\mathrm{mv}} \in \Lambda$, which is generally not the case. They will be improved soon.

# Slices of Λ: The Slice $\Lambda(\mu)$

Let $\Lambda(\mu)$ be the set of all *long portfolio allocations with return mean $\mu$.* This set is given by

$$\Lambda(\mu) = \left\{ \mathbf{f} \in \Lambda \; : \; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\} . \tag{2.3}$$

Clearly $\Lambda(\mu) \subset \Lambda$ for every $\mu \in \mathbb{R}$. It is a *slice* of Λ. It is the intersection of the simplex Λ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \; : \; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$.

We now characterize those $\mu$ for which $\Lambda(\mu)$ is nonempty.

**Fact.** The set $\Lambda(\mu)$ is nonempty if and only if $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$.

**Remark.** Because we have assumed that $\mathbf{m}$ is not proportional to $\mathbf{1}$, the return means $\{m_i\}_{i=1}^N$ are not identical. This implies that $\mu_{\mathrm{mn}} < \mu_{\mathrm{mx}}$, which implies that the interval $[\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ does not reduce to a point.

# Slices of Λ: Nonempty Characterization Proof

**Proof.** Because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda(\mu)$ we have the inequalities

$$\mu_{\mathrm{mn}} = \mu_{\mathrm{mn}} \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mu_{\mathrm{mn}} \sum_{i=1}^{N} f_i \leq \sum_{i=1}^{N} m_i f_i = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \,,$$

$$\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \sum_{i=1}^{N} m_i f_i \leq \mu_{\mathrm{mx}} \sum_{i=1}^{N} f_i = \mu_{\mathrm{mx}} \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mu_{\mathrm{mx}} \,.$$

Therefore if $\Lambda(\mu)$ is nonempty then $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$.

Conversely, first choose $\mathbf{e}_{\mathrm{mn}}$ and $\mathbf{e}_{\mathrm{mx}}$ so that

$$\begin{aligned} \mathbf{e}_{\mathrm{mn}} &= \mathbf{e}_i \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\mathrm{mn}} \,, \\ \mathbf{e}_{\mathrm{mx}} &= \mathbf{e}_j \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\mathrm{mx}} \,. \end{aligned}$$

# Slices of Λ: Nonempty Characterization Proof

Now let $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ and set

$$\mathbf{f} = \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mathbf{e}_{\mathrm{mx}}\,.$$

Clearly $\mathbf{f} \geq \mathbf{0}$. Because $\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} = \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} = 1$, $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} = \mu_{\mathrm{mn}}$, and $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} = \mu_{\mathrm{mx}}$, we see that

$$\begin{aligned}
\mathbf{1}^{\mathrm{T}}\mathbf{f} &= \tfrac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} + \tfrac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} \\
&= \tfrac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} + \tfrac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} = 1\,, \\
\mathbf{m}^{\mathrm{T}}\mathbf{f} &= \tfrac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} + \tfrac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} \\
&= \tfrac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mu_{\mathrm{mn}} + \tfrac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\,\mu_{\mathrm{mx}} = \mu\,.
\end{aligned}$$

Hence, $\mathbf{f} \in \Lambda(\mu)$. Therefore if $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ then $\Lambda(\mu)$ is nonempty. □

# Slices of $\Lambda$: $\Lambda(\mu)$ as a Polytope

For every $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ the set $\Lambda(\mu)$ is the nonempty intersection in $\mathbb{R}^N$ of the $N-1$ dimensional simplex $\Lambda$ with the $N-1$ dimensional hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. Therefore $\Lambda(\mu)$ will be a nonempty, closed, bounded, convex polytope of dimension at most $N-2$.

**Remark.** If there are

- $n$ assets with $m_i > \mu$ and
- $N-n$ assets with $m_i < \mu$

then there are $n(N-n)$ edges of $\Lambda$ that cross the $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ hyperplane, whereby $\Lambda(\mu)$ will have $n(N-n)$ vertices. This means that $\Lambda(\mu)$ can have

- at most $\frac{1}{4}N^2$ vertices when $N$ is even and
- at most $\frac{1}{4}(N^2-1)$ vertices when $N$ is odd.

# Slices of $\Lambda$: Visualization for Small $N$

We can visualize the polytope $\Lambda(\mu)$ when $N$ is small.

- When $N = 2$ it is a point because it is the intersection of the line segment $\Lambda$ with a transverse line.
- When $N = 3$ it is either a point or line segment because it is the intersection of the triangle $\Lambda$ with a transverse plane.
- When $N = 4$ it is either a point, line segment, triangle, or convex quadralateral because it is the intersection of the tetrahedron $\Lambda$ with a transverse hyperplane.

# Slices of $\Lambda$: Visualization for $N = 4$ in $\mathbb{R}^3$

**Remark.** Recall from an earlier remark that when $N = 4$ the set $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ under the one-to-one affine mapping $\mathbf{\Phi} : \mathbb{R}^3 \to \mathbb{R}^4$ given there.

The set $\Lambda(\mu) \subset \mathbb{R}^4$ is thereby the image under $\mathbf{\Phi}$ of the intersection of $\mathcal{T}$ with the hyperplane $H_\mu$ given by

$$H_\mu = \left\{ \mathbf{z} \in \mathbb{R}^3 \,;\, \mathbf{m}^{\mathrm{T}} \mathbf{\Phi}(\mathbf{z}) = \mu \right\}.$$

Hence, the set $\Lambda(\mu)$ in $\mathbb{R}^4$ can be visualized in $\mathbb{R}^3$ as the set $\mathcal{T}_\mu = \mathcal{T} \cap H_\mu$.

As $\mathbf{\Phi}$ is one-to-one and $\mathbf{m}$ is arbitrary, $H_\mu$ can be any hyperplane in $\mathbb{R}^3$. Therefore $\mathcal{T}_\mu$ can be the intersection of the tetrahedron $\mathcal{T}$ with any hyperplane in $\mathbb{R}^3$.

## Slices of Λ: Visualization for $N = 4$

When such an intersection is nonempty it can be either

1. a point that is a vertex of $\mathcal{T}$,
2. a line segment that is an edge of $\mathcal{T}$,
3. a triangle with vertices on edges of $\mathcal{T}$,
4. a convex quadrilateral with vertices on edges of $\mathcal{T}$.

These are each convex polytopes of dimension at most 2.

- The first and second cases arise only when $\mu = \mu_{\text{mn}}$ or $\mu = \mu_{\text{mx}}$. The second case is extremely rare.
- Either the third or fourth case arises for every $\mu \in (\mu_{\text{mn}}, \mu_{\text{mx}})$.

# Long Frontiers: $\Sigma(\Lambda)$ in the $\sigma\mu$-Plane

The set $\Lambda$ in $\mathbb{R}^N$ of all long portfolios is associated with the set $\Sigma(\Lambda)$ in the $\sigma\mu$-plane of volatilities and return means given by

$$\Sigma(\Lambda) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \ : \ \sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \ \mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \ \mathbf{f} \in \Lambda \right\} . \qquad (3.4)$$

The set $\Sigma(\Lambda)$ is the image in $\mathbb{R}^2$ of the simplex $\Lambda$ in $\mathbb{R}^N$ under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set $\Lambda$ is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set $\Sigma(\Lambda)$ is compact.

We have seen that the set $\Lambda(\mu)$ of all long portfolios with return mean $\mu$ is nonempty if and only if $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$. Hence, $\Sigma(\Lambda)$ can be expressed as

$$\Sigma(\Lambda) = \left\{ \left( \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \ \mu \right) \ : \ \mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]\,, \ \mathbf{f} \in \Lambda(\mu) \right\} .$$

The points on the boundary of $\Sigma(\Lambda)$ that correspond to those long portfolios that have less volatility than every other long portfolio with the same return mean is called the *long frontier.*

# Long Frontiers: Definition of $\sigma_{\rm lf}(\mu)$

The *long frontier* is the curve in the $\sigma\mu$-plane given by the equation

$$\sigma = \sigma_{\rm lf}(\mu) \quad \text{over} \quad \mu \in [\mu_{\rm mn}, \mu_{\rm mx}] , \tag{3.5}$$

where the value of $\sigma_{\rm lf}(\mu)$ is obtained for each $\mu \in [\mu_{\rm mn}, \mu_{\rm mx}]$ by solving the constrained minimization problem

$$\sigma_{\rm lf}(\mu)^2 = \min\left\{ \sigma^2 \;:\; (\sigma, \mu) \in \Sigma(\Lambda) \right\} = \min\left\{ \mathbf{f}^{\rm T}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Lambda(\mu) \right\} .$$

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}$ is continuous over the compact set $\Lambda(\mu)$, a minimizer exists.

Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Lambda(\mu)$, whereby the minimizer is unique.

# Long Frontiers: Definition of $\mathbf{f}_{\mathrm{lf}}(\mu)$

If we denote this unique minimizer by $\mathbf{f}_{\mathrm{lf}}(\mu)$ then for every $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ the function $\sigma_{\mathrm{lf}}(\mu)$ is given by

$$\sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}_{\mathrm{lf}}(\mu)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{lf}}(\mu)}\,, \tag{3.6}$$

where $\mathbf{f}_{\mathrm{lf}}(\mu)$ can be expressed as

$$\mathbf{f}_{\mathrm{lf}}(\mu) = \arg\min \left\{ \tfrac{1}{2} \mathbf{f}^{\mathrm{T}} \mathbf{V} \mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N ,\; \mathbf{f} \geq \mathbf{0} ,\; \mathbf{1}^{\mathrm{T}} \mathbf{f} = 1 ,\; \mathbf{m}^{\mathrm{T}} \mathbf{f} = \mu \right\} .$$

Here $\arg\min$ is read "the argument that minimizes". It means that $\mathbf{f}_{\mathrm{lf}}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2} \mathbf{f}^{\mathrm{T}} \mathbf{V} \mathbf{f}$ subject to the constraints.

**Remark.** This problem can not be solved by Lagrange multipliers because of the inequality constraints $\mathbf{f} \geq \mathbf{0}$ associated with the set $\Lambda(\mu)$. It is harder to solve analytically than the analogous minimization problem for portfolios with unlimited leverage. Therefore we will first present a numerical approach that can generally be applied.

# Long Frontiers: Quadratic Programming

Because the function being minimized is quadratic in $\mathbf{f}$ while the constraints are linear in $\mathbf{f}$, this is called a *quadratic programming problem*. It can be solved for a particular $\mathbf{V}$, $\mathbf{m}$, and $\mu$ by using either the Matlab command "quadprog" or an equivalent command in some other language.

The Matlab command quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\mathrm{eq}}, \mathbf{d}_{\mathrm{eq}})$ returns the solution of a quadratic programming problem in the *standard form*

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} \;:\; \mathbf{x} \in \mathbb{R}^{M},\; \mathbf{C}\mathbf{x} \leq \mathbf{d},\; \mathbf{C}_{\mathrm{eq}}\mathbf{x} = \mathbf{d}_{\mathrm{eq}} \right\},$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^{M}$, $\mathbf{C} \in \mathbb{R}^{K\times M}$, $\mathbf{d} \in \mathbb{R}^{K}$, $\mathbf{C}_{\mathrm{eq}} \in \mathbb{R}^{K_{\mathrm{eq}}\times M}$, and $\mathbf{d}_{\mathrm{eq}} \in \mathbb{R}^{K_{\mathrm{eq}}}$. Here $K$ and $K_{\mathrm{eq}}$ are the number of inequality and equality constraints respectively.

# Long Frontiers: Converting to the Standard Form

Given $\mathbf{V}$, $\mathbf{m}$, and $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$, the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{lf}}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N ,\; \mathbf{f} \geq \mathbf{0} ,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 ,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\} .$$

We can put this into the standard form given on the previous slide by setting $\mathbf{x} = \mathbf{f}$ then $M = N$, $K = N$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \mathbf{V} , \quad \mathbf{b} = \mathbf{0} , \quad \mathbf{C} = -\mathbf{I} , \quad \mathbf{d} = \mathbf{0} , \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix} , \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} ,$$

where $\mathbf{I}$ is the $N \times N$ identity. Notice that

- $M = N$ because $\mathbf{x} = \mathbf{f} \in \mathbb{R}^N$,
- $K = N$ because $\mathbf{f} \geq \mathbf{0}$ gives $N$ inequality constraints,
- $K_{\mathrm{eq}} = 2$ because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are two equality constraints.

# Long Frontiers: Matlab "quadprog" Command

Therefore $\mathbf{f}_{\mathrm{lf}}(\mu)$ can be obtained as the output $\mathrm{f}$ of a quadprog command that is formated as

$$\mathrm{f} = \mathsf{quadprog}(\mathrm{V}, \mathrm{z}, -\mathrm{I}, \mathrm{z}, \mathrm{Ceq}, \mathrm{deq})\,,$$

where the matrices $\mathrm{V}$, $\mathrm{I}$, and $\mathrm{Ceq}$, and vectors $\mathrm{z}$ and $\mathrm{deq}$ are given by

$$\mathrm{V} = \mathbf{V}\,, \quad \mathrm{z} = \mathbf{0}\,, \quad \mathrm{I} = \mathbf{I}\,, \quad \mathrm{Ceq} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix}\,, \quad \mathrm{deq} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}\,.$$

**Remark.** There are other ways to use quadprog to obtain $\mathbf{f}_{\mathrm{lf}}(\mu)$. Documentation for this command is easy to find on the web. The similar command in R is also called "quadprog".

# Long Frontiers: Properties of $\sigma_{\rm lf}(\mu)$

When computing a long frontier, it helps to know some general properties of the function $\sigma_{\rm lf}(\mu)$. These include:

- $\sigma_{\rm lf}(\mu)$ is continuous over $[\mu_{\rm mn}, \mu_{\rm mx}]$;
- $\sigma_{\rm lf}(\mu)$ is strictly convex over $[\mu_{\rm mn}, \mu_{\rm mx}]$;
- $\sigma_{\rm lf}(\mu)$ is piecewise hyperbolic over $[\mu_{\rm mn}, \mu_{\rm mx}]$.

This means that $\sigma_{\rm lf}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\rm mn}, \mu_{\rm mx})$ where $\sigma_{\rm lf}(\mu)$ has either

- a jump discontinuity in its first derivative, or
- a jump discontinuity in its second derivative.

Guided by these facts we now show how a long frontier can be approximated numerically with the Matlab command quadprog.

# Long Frontiers: Approximating $\sigma_{\mathrm{lf}}(\mu)$

**First**, partition the interval $[\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ as

$$\mu_{\mathrm{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathrm{mx}} \,.$$

For example, set $\mu_k = \mu_{\mathrm{mn}} + k(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})/n$ for a uniform partition. Pick $n$ large enough to resolve all the features of the long frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

**Second**, for every $k = 1, \cdots, n-1$ use quadprog to compute $\mathbf{f}_{\mathrm{lf}}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}_{\mathrm{lf}}(\mu_k)\}_{k=0}^{n}$ should be saved.

**Third**, for every $k = 1, \cdots, n-1$ compute $\sigma_k$ by

$$\sigma_k = \sigma_{\mathrm{lf}}(\mu_k) = \sqrt{\mathbf{f}_{\mathrm{lf}}(\mu_k)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{lf}}(\mu_k)} \,.$$

# Long Frontiers: Linear Interpolation in the $\sigma\mu$-Plane

**Fourth**, there is typically a unique $m_i$ such that $\mu_{\mathrm{mn}} = m_i$, in which case we set

$$\mathbf{f}_{\mathrm{lf}}(\mu_0) = \mathbf{e}_i \,, \qquad \sigma_0 = \sqrt{v_{ii}} \,.$$

Similarly, there is typically a unique $m_j$ such that $\mu_{\mathrm{mx}} = m_j$, in which case we set

$$\mathbf{f}_{\mathrm{lf}}(\mu_n) = \mathbf{e}_j \,, \qquad \sigma_n = \sqrt{v_{jj}} \,.$$

**Finally**, we "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the long frontier. This can be done by linear interpolation in the $\sigma\mu$-plane. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\sigma}_{\mathrm{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\,\sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\,\sigma_k \,.$$

# Long Frontiers: Linear Interpolation in Λ

A better way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ that is motivated by the two-fund property is to use linear interpolation in Λ. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{lf}}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{lf}}(\mu_k)\,,$$

and then set

$$\tilde{\sigma}_{\mathrm{lf}}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu)^{\mathrm{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathrm{lf}}(\mu)}\,.$$

**Remark.** This will be a very good approximation if $n$ is large enough. Over each interval $(\mu_{k-1}, \mu_k)$ it approximates $\sigma_{\mathrm{lf}}(\mu)$ with a hyperbola rather than with a line.

# Long Frontiers: Linear Interpolation in Λ

**Remark.** Because $\mathbf{f}_{\rm lf}(\mu_k) \in \Lambda(\mu_k)$ and $\mathbf{f}_{\rm lf}(\mu_{k-1}) \in \Lambda(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_{\rm lf}(\mu) \in \Lambda(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k)\,.$$

Therefore $\tilde{\sigma}_{\rm lf}(\mu)$ gives an approximation to the long frontier that lies on or to the right of the long frontier in the $\sigma\mu$-plane.

**Remark.** When there are no nodes in the interval $(\mu_{k-1}, \mu_k)$ then we can use the two-fund property to show that $\tilde{\sigma}_{\rm lf}(\mu) = \sigma_{\rm lf}(\mu)$.

# General Portfolio with Two Risky Assets: Λ

Recall the portfolio of two risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\mathrm{mn}} = m_1$ and $\mu_{\mathrm{mx}} = m_2$. Recall that for every $\mu \in \mathbb{R}$ the unique portfolio that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_2] = [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$. Therefore the set Λ of long portfolios is given by

$$\Lambda = \left\{ \mathbf{f}(\mu) \,:\, \mu \in [m_1, m_2] \right\}.$$

# General Portfolio with Two Risky Assets: $\sigma_{\mathrm{lf}}(\mu)$

In other words, the line segment $\Lambda$ in $\mathbb{R}^2$ is the image of the interval $[m_1, m_2]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$.

Because for every $\mu \in [m_1, m_2]$ the set $\Lambda(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ over $\Lambda(\mu)$ is $\mathbf{f}(\mu)$. Therefore the long frontier portfolios are

$$\mathbf{f}_{\mathrm{lf}}(\mu) = \mathbf{f}(\mu) \qquad \text{for } \mu \in [m_1, m_2]\,,$$

and the long frontier is given by

$$\sigma = \sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\,\mathbf{f}(\mu)} \qquad \text{for } \mu \in [m_1, m_2]\,.$$

Hence, the long frontier is simply a segment of the frontier hyperbola. It has no nodes.

# General Portfolio with Three Risky Assets: m and V

Recall the portfolio of three risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \leq m_2 \leq m_3, \qquad m_1 < m_3.$$

Then $\mu_{\text{mn}} = m_1$ and $\mu_{\text{mx}} = m_3$.

# General Portfolio with Three Risky Assets: $\mathbf{f}(\mu, \phi)$

Recall that for every $\mu \in \mathbb{R}$ the portfolio allocations that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\, \mathbf{n}\,, \qquad \text{for some } \phi \in \mathbb{R}\,, \tag{5.7a}$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1}\begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}\,, \qquad \mathbf{n} = \frac{1}{m_3 - m_1}\begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}\,. \tag{5.7b}$$

Here $\mathbf{f}_{13}(\mu)$ is the two-asset allocation for assets 1 and 3 that satisfies

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{13}(\mu) = 1\,, \qquad \mathbf{m}^{\mathrm{T}}\mathbf{f}_{13}(\mu) = \mu\,,$$

while $\mathbf{n}$ satisfies $\mathbf{1}^{\mathrm{T}}\mathbf{n} = 0$ and $\mathbf{m}^{\mathrm{T}}\mathbf{n} = 0$.

# General Portfolio with Three Risky Assets: $\mathbf{f}(\mu, \phi) \geq \mathbf{0}$

Because

$$\mathbf{f}(\mu, \phi) = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu - \phi\,(m_3 - m_2) \\ \phi\,(m_3 - m_1) \\ \mu - m_1 - \phi\,(m_2 - m_1) \end{pmatrix},$$

we see that $\mathbf{f}(\mu, \phi) \geq \mathbf{0}$ if and only if

- $\mu \in [m_1, m_3] = [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$, and
- $\phi \in [0, \phi_{\mathrm{mx}}(\mu)]$, where

$$\phi_{\mathrm{mx}}(\mu) = \begin{cases} \dfrac{m_3 - \mu}{m_3 - m_1} & \text{if } m_2 = m_1\,, \\ \dfrac{\mu - m_1}{m_3 - m_1} & \text{if } m_2 = m_3\,, \\ \min\left\{ \dfrac{m_3 - \mu}{m_3 - m_2}\,, \dfrac{\mu - m_1}{m_2 - m_1} \right\} & \text{if } m_2 \in (m_1, m_3)\,. \end{cases} \tag{5.8}$$

# General Portfolio with Three Risky Assets: $\Lambda$ and $\mathcal{T}_\Lambda$

Then the set $\Lambda$ of long portfolios is given by

$$\Lambda = \left\{ \mathbf{f}(\mu, \phi) \,:\, (\mu, \phi) \in \mathcal{T}_\Lambda \right\}, \tag{5.9}$$

where $\mathcal{T}_\Lambda$ is the triangle in the $\mu\phi$-plane given by

$$\mathcal{T}_\Lambda = \left\{ (\mu, \phi) \in \mathbb{R}^2 \,:\, \mu \in [m_1, m_3]\,,\, 0 \leq \phi \leq \phi_{\mathrm{mx}}(\mu) \right\}. \tag{5.10}$$

- The base of $\mathcal{T}_\Lambda$ is the interval $[m_1, m_3]$ on the $\mu$-axis.
- The peak of $\mathcal{T}_\Lambda$ is at the point $(m_2, 1)$.
- The height of $\mathcal{T}_\Lambda$ is 1.

# General Portfolio with Three Risky Assets: $\Lambda$ & $\Lambda(\mu)$

Therefore the sets $\Lambda$ and $\Lambda(\mu)$ in $\mathbb{R}^3$ can be visualized as follows.

- The set $\Lambda$ is the triangle in $\mathbb{R}^3$ that is the image of the triangle $\mathcal{T}_\Lambda$ under the affine mapping $(\mu, \phi) \mapsto \mathbf{f}(\mu, \phi)$.
- For every $\mu \in [m_1, m_3]$ the set $\Lambda(\mu)$ is given by

$$\Lambda(\mu) = \Big\{ \mathbf{f}(\mu, \phi) \,:\, 0 \leq \phi \leq \phi_{\mathrm{mx}}(\mu) \Big\} . \tag{5.11}$$

Therefore the set $\Lambda(\mu)$ is the line segment in $\mathbb{R}^3$ that is the image of the interval $[0, \phi_{\mathrm{mx}}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu, \phi)$.

# General Portfolio with Three Risky Assets: $\phi_{\mathrm{mf}}(\mu)$

Hence, the point on the long frontier associated with $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ is $(\sigma_{\mathrm{lf}}(\mu), \mu)$ where $\sigma_{\mathrm{lf}}(\mu)$ solves the constrained minimization problem

$$\begin{aligned}\sigma_{\mathrm{lf}}(\mu)^2 &= \min\left\{\, \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Lambda(\mu) \,\right\} \\ &= \min\left\{\, \mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) \;:\; 0 \le \phi \le \phi_{\mathrm{mx}}(\mu) \,\right\}.\end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\,\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}$$

is a quadratic in $\phi$ and $\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0$, it has a unique global minimizer at

$$\phi = \phi_{\mathrm{mf}}(\mu) = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,. \tag{5.12}$$

Then the Markowitz frontier allocation is $\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}(\mu, \phi_{\mathrm{mf}}(\mu))$.

# General Portfolio with Three Risky Assets: The Minimizers

The global minimizer $\phi_{\mathrm{mf}}(\mu)$ will be the minimizer of our constrained minimization problem for the long frontier if and only if it satifies the constraints $0 \leq \phi_{\mathrm{mf}}(\mu) \leq \phi_{\mathrm{mx}}(\mu)$.

Because the derivative of the objective function with respect to $\phi$ can be written as

$$\partial_\phi \, \mathbf{f}(\mu,\phi)^{\mathrm{T}} \mathbf{V} \mathbf{f}(\mu,\phi) = 2\, \mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{n} \left( \phi - \phi_{\mathrm{mf}}(\mu) \right) ,$$

we can read off the following.

- If $\phi_{\mathrm{mf}}(\mu) \leq 0$ then the objective function is increasing over $[0, \phi_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = 0$.
- If $\phi_{\mathrm{mx}}(\mu) \leq \phi_{\mathrm{mf}}(\mu)$ then the objective function is decreasing over $[0, \phi_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi_{\mathrm{mx}}(\mu)$.

# General Portfolio with Three Risky Assets: $\phi_{\mathrm{lf}}(\mu)$

Hence, the minimizer $\phi_{\mathrm{lf}}(\mu)$ of our constrained minimization problem is

$$\begin{aligned}\phi_{\mathrm{lf}}(\mu) &= \begin{cases} 0 & \text{if } \phi_{\mathrm{mf}}(\mu) \leq 0 \\ \phi_{\mathrm{mf}}(\mu) & \text{if } 0 < \phi_{\mathrm{mf}}(\mu) < \phi_{\mathrm{mx}}(\mu) \\ \phi_{\mathrm{mx}}(\mu) & \text{if } \phi_{\mathrm{mx}}(\mu) \leq \phi_{\mathrm{mf}}(\mu) \end{cases} \\ &= \max\{0\,,\, \min\{\phi_{\mathrm{mf}}(\mu)\,,\, \phi_{\mathrm{mx}}(\mu)\}\} \\ &= \min\{\max\{0\,,\, \phi_{\mathrm{mf}}(\mu)\}\,,\, \phi_{\mathrm{mx}}(\mu)\}\,. \end{aligned} \tag{5.13}$$

Therefore the long frontier is given by

$$\sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}(\mu, \phi_{\mathrm{lf}}(\mu))^{\mathrm{T}} \mathbf{V} \mathbf{f}(\mu, \phi_{\mathrm{lf}}(\mu))}\,. \tag{5.14}$$

# General Portfolio with Three Risky Assets: $\mathcal{T}_\Lambda$ & $\mathcal{L}_{\rm mf}$

Understanding the long frontier thereby reduces to understanding $\phi_{\rm lf}(\mu)$. This can be visualized in the $\mu\phi$-plane by considering the intersection of the triangle $\mathcal{T}_\Lambda$ and the line $\mathcal{L}_{\rm mf}$ given by

$$\mathcal{L}_{\rm mf} = \Big\{(\mu,\phi)\,:\,\phi = \phi_{\rm mf}(\mu)\Big\}\,. \tag{5.15}$$

Because

$$\mathbf{f}_{13}(m_1) = \mathbf{e}_1\,, \qquad \mathbf{f}_{13}(m_2) = \mathbf{e}_2 - \mathbf{n}\,, \quad \text{and} \quad \mathbf{f}_{13}(m_3) = \mathbf{e}_3\,,$$

we see that

$$\begin{aligned}
\phi_{\rm mf}(m_1) &= -\frac{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{f}_{13}(m_1)}{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{n}} = -\frac{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{e}_1}{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{n}}\,,\\
\phi_{\rm mf}(m_2) &= -\frac{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{f}_{13}(m_2)}{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{n}} = 1 - \frac{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{e}_2}{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{n}}\,,\\
\phi_{\rm mf}(m_3) &= -\frac{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{f}_{13}(m_3)}{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{n}} = -\frac{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{e}_3}{\mathbf{n}^{\rm T}\mathbf{V}\mathbf{n}}\,.
\end{aligned}$$

# General Portfolio with Three Risky Assets: $\mathcal{T}_\Lambda$ & $\mathcal{L}_{\rm mf}$

Some geometry can thereby be read off from the signs of the entries of **Vn**.

$$\begin{aligned}
&\mathcal{L}_{\rm mf} \text{ is below vertex } (m_1, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \phi_{\rm mf}(m_1) < 0 \quad \text{iff } \mathbf{e}_1^{\rm T}\mathbf{Vn} > 0\,;\\
&\mathcal{L}_{\rm mf} \text{ is above vertex } (m_1, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \phi_{\rm mf}(m_1) > 0 \quad \text{iff } \mathbf{e}_1^{\rm T}\mathbf{Vn} < 0\,;\\
&\mathcal{L}_{\rm mf} \text{ is below vertex } (m_2, 1) \text{ of } \mathcal{T}_\Lambda \text{ iff } \phi_{\rm mf}(m_2) < 1 \quad \text{iff } \mathbf{e}_2^{\rm T}\mathbf{Vn} > 0\,;\\
&\mathcal{L}_{\rm mf} \text{ is above vertex } (m_2, 1) \text{ of } \mathcal{T}_\Lambda \text{ iff } \phi_{\rm mf}(m_2) > 1 \quad \text{iff } \mathbf{e}_2^{\rm T}\mathbf{Vn} < 0\,;\\
&\mathcal{L}_{\rm mf} \text{ is below vertex } (m_3, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \phi_{\rm mf}(m_3) < 0 \quad \text{iff } \mathbf{e}_3^{\rm T}\mathbf{Vn} > 0\,;\\
&\mathcal{L}_{\rm mf} \text{ is above vertex } (m_3, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \phi_{\rm mf}(m_3) > 0 \quad \text{iff } \mathbf{e}_3^{\rm T}\mathbf{Vn} < 0\,.
\end{aligned} \tag{5.16}$$

By combining this information about each vertex of the triangle $\mathcal{T}_\lambda$ we can work out its intersection with the line $\mathcal{L}_{\rm mf}$.

# General Portfolio with Three Risky Assets: Nine Cases

Below we list all nine cases that arise when $m_1 < m_2 < m_3$.

1. $\phi_{\mathrm{mf}}(m_1) \leq 0$ and $\phi_{\mathrm{mf}}(m_3) \leq 0$ (whereby $\phi_{\mathrm{mf}}(m_2) \leq 0 < 1$).
2. $\phi_{\mathrm{mf}}(m_1) \geq 0$, $\phi_{\mathrm{mf}}(m_2) \geq 1$ and $\phi_{\mathrm{mf}}(m_3) \geq 0$.
3. $\phi_{\mathrm{mf}}(m_1) = 0$, $\phi_{\mathrm{mf}}(m_2) < 1$ and $\phi_{\mathrm{mf}}(m_3) > 0$.
4. $\phi_{\mathrm{mf}}(m_1) > 0$, $\phi_{\mathrm{mf}}(m_2) < 1$ and $\phi_{\mathrm{mf}}(m_3) = 0$.
5. $\phi_{\mathrm{mf}}(m_1) > 0$, $\phi_{\mathrm{mf}}(m_2) < 1$ and $\phi_{\mathrm{mf}}(m_3) > 0$.
6. $\phi_{\mathrm{mf}}(m_1) < 0$, $\phi_{\mathrm{mf}}(m_2) \leq 1$ and $\phi_{\mathrm{mf}}(m_3) > 0$.
7. $\phi_{\mathrm{mf}}(m_1) > 0$, $\phi_{\mathrm{mf}}(m_2) \leq 1$ and $\phi_{\mathrm{mf}}(m_3) < 0$.
8. $\phi_{\mathrm{mf}}(m_1) < 0$, $\phi_{\mathrm{mf}}(m_2) > 1$ and $\phi_{\mathrm{mf}}(m_3) > 0$.
9. $\phi_{\mathrm{mf}}(m_1) > 0$, $\phi_{\mathrm{mf}}(m_2) > 1$ and $\phi_{\mathrm{mf}}(m_3) < 0$.

We present some of these nine cases below. Another ten cases arise when either $m_2 = m_1$ or $m_2 = m_3$.

# General Portfolio with Three Risky Assets: Case 1

**Case 1.** By (5.16) the line $\mathcal{L}_{\mathrm{mf}}$ lies below the interior of $\mathcal{T}_{\Lambda}$ if and only if

$$\mathbf{e}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} \geq 0 \quad \text{and} \quad \mathbf{e}_3^{\mathrm{T}}\mathbf{V}\mathbf{n} \geq 0 \,.$$

Then $\phi_{\mathrm{lf}}(\mu) = 0$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)} \,.$$

This is the long frontier built from assets 1 and 3. It has no nodes and is smooth.

# General Portfolio with Three Risky Assets: Case 2

**Case 2.** By (5.16) the line $\mathcal{L}_{\rm mf}$ lies above the interior of $\mathcal{T}_\Lambda$ if and only if

$$\mathbf{e}_1^{\rm T}\mathbf{V}\mathbf{n} \leq 0\,, \qquad \mathbf{e}_2^{\rm T}\mathbf{V}\mathbf{n} \leq 0\,, \quad \text{and} \quad \mathbf{e}_3^{\rm T}\mathbf{V}\mathbf{n} \leq 0\,.$$

Then $\phi_{\rm lf}(\mu) = \phi_{\rm mx}(\mu)$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\rm lf}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^{\rm T}\mathbf{V}\mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, m_2]\,, \\ \sqrt{\mathbf{f}_{23}(\mu)^{\rm T}\mathbf{V}\mathbf{f}_{23}(\mu)} & \text{for } \mu \in [m_2, m_3]\,. \end{cases}$$

This patches the long frontier built from assets 1 and 2 with the long frontier built from assets 2 and 3. It generally has a jump discontinuity in its first derivative at the node $\mu = m_2$.

# General Portfolio with Three Risky Assets: Case 5

**Case 5.** By (5.16) the line $\mathcal{L}_{\mathrm{mf}}$ lies above the base of $\mathcal{T}_\Lambda$ but intersects the interior of $\mathcal{T}_\Lambda$ if and only if

$$\mathbf{e}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,, \qquad \mathbf{e}_2^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0\,, \quad \text{and} \quad \mathbf{e}_3^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,.$$

Then there exists

- $\mu_1 \in [m_1, m_2]$ where $\phi_{\mathrm{mf}}(\mu)$ intersects $\dfrac{\mu - m_1}{m_2 - m_1}$, and
- $\mu_2 \in [m_2, m_3]$ where $\phi_{\mathrm{mf}}(\mu)$ intersects $\dfrac{m_3 - \mu}{m_3 - m_2}$.

Then

$$\phi_{\mathrm{lf}}(\mu) = \begin{cases} \dfrac{\mu - m_1}{m_2 - m_1} & \text{for } \mu \in [m_1, \mu_1]\,, \\ \phi_{\mathrm{mf}}(\mu) & \text{for } \mu \in (\mu_1, \mu_2)\,, \\ \dfrac{m_3 - \mu}{m_3 - m_2} & \text{for } \mu \in [\mu_2, m_3]\,. \end{cases}$$

# General Portfolio with Three Risky Assets: Case 5

The long frontier for this case is

$$\sigma = \sigma_{\mathrm{lf}}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, \mu_1]\,, \\ \sigma_{\mathrm{mf}}(\mu) & \text{for } \mu \in (\mu_1, \mu_2)\,, \\ \sqrt{\mathbf{f}_{23}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{23}(\mu)} & \text{for } \mu \in [\mu_2, m_3]\,. \end{cases}$$

Because

$$\sigma_{\mathrm{mf}}(\mu) \leq \sqrt{\mathbf{f}_{12}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{12}(\mu)} \qquad \text{for every } \mu \in \mathbb{R}\,,$$

$$\sigma_{\mathrm{mf}}(\mu) \leq \sqrt{\mathbf{f}_{23}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{12}(\mu)} \qquad \text{for every } \mu \in \mathbb{R}\,,$$

with equaility at $\mu = \mu_1$ and $\mu = \mu_2$ respectively, we see that the first derivative of $\sigma_{\mathrm{lf}}(\mu)$ is continuous at the nodes $\mu = \mu_1$ and $\mu = \mu_2$, but its second derivative will generally have jump discontinuities at those points.