Portfolios that Contain Risky Assets
3.4. Capital Asset Pricing Model

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

3. Models for Portfolios with Risk-Free Assets

Capital Asset Pricing Model

Introduction

In 1963 William Sharpe published a paper based upon his 1961 Doctoral dissertation at UCLA. It initiated a series of works that built upon the foundational work of Harry Markowitz (who was his unofficial advisor), James Tobin and others and introduced what became known as the *Capital Asset Pricing Model (CAPM)*. These works brought many fresh ideas into the new field of portfolio theory. For example, they standardized the use of the $\sigma\mu$-plane and introduced the Sharpe ratio. They also introduced far stronger assumptions than those made by earlier work. One consequence of these assumptions was that the predictions of CAPM were not always supported by observations. However by the 1970s CAPM had inspired more flexible theories that are still in use today, most notably *Arbitrage Pricing Theory*. In 1990 Sharpe was awarded the Nobel Prize in Economics (with Markowitz) for this influential body of work.

Here we present a development of CAPM and explore its consequences.

Introduction

Our development of CAPM differs from that of Sharpe. It has two steps.

Step 1. In the setting of Markowitz portfolios with a one-rate model of risk-free assets, we add some very strong assumptions that allow us to identify the tangent allocation with the market capitalization allocation. This step is the heart of CAPM.

Step 2. We develop some relations between the return means and volatilities of individual assets with those of the tangent portfolio. These do not require the very strong assumptions of the first step. We then see what they say given the conclusion of **Step 1**.

We will close with a summary and critique.

Remark. There is another part of CAPM that relates returns of individual assets to those of market capitalization allocation. We will present it after the needed tools from probability have been covered.

Sharpe One-Rate Model: CAPM Assumptions

CAPM is built upon five basic assumptions.

1. The market consists of N risky assets and risk-free assets with a common return rate μ_{rf}.
2. There are K investors, each of which holds a solvent Markowitz portfolio governed by the one-rate model for risk-free assets.
3. The market capitalization of each asset is equal to the sum of its value of that asset held in each portfolio.
4. The distribution of daily return vectors is known and is stationary. Let $\mathbf{m}$ and $\mathbf{V}$ be its known mean vector and covariance matrix. Then μ_{rf}, $\mathbf{m}$ and $\mathbf{V}$ are the same for all investors and are constant in time.
5. Each investor holds a portfolio on the efficient Tobin frontier.

Sharpe One-Rate Model: Colloquial Descriptions

The last two assumptions are often described more colloquially as follows.

- **Assumption 4** is described as that of an equilibrium or steady market because it asserts that the distribution of return vectors is stationary. It is also described as that of a rational market because it asserts that the distribution of return vectors is known to all investors.
- **Assumption 5** is described as that of an efficient or rational market because all investors act efficiently on complete information.

The difficulty with such descriptions is that they are imprecise. The same words can mean different things in other settings. For example, here "equilibrium" does not mean that prices are statistically stationary, but in other settings it would. There is also a general notion that markets can be efficient without all of its investors being efficient. We will focus on more precise language that can be quantified unambiguously.

Sharpe One-Rate Model: Concerns

The assumptions go far beyond those made by Markowitz and Tobin. We will see that they lead to predictions that are unsupported by observation. Some immediate concerns are the following.

- The one-rate model for risk-free assets is too simple.
- Investors generally do not hold positions in every risky asset. Rather, they hold positions in much smaller collections of assets.
- No investor knows $\mathbf{m}$ or $\mathbf{V}$. Rather, these are estimated from data.
- Even investors that hold the same risky assets might use different values for $\mathbf{m}$ and $\mathbf{V}$ because different return histories or different weights or even other methods were used to estimate them.
- Tobin frontiers are not constant in time.
- Not all investors are efficient.

We will put our concerns aside for now and proceed with the development.

Sharpe One-Rate Model: Investor Portfolios

Assumption 1 and **Assumption 2** imply that the k^{th} investor holds a solvent portfolio that at the openning of day d is governed by

- its value $\pi_k(d-1) > 0$ at the close of day $d-1$, and
- its allocation $\left(\mathbf{f}_k\,,\, f_k^{\mathrm{rf}}\right) \in \mathcal{M}_1$,

where each allocation satisfies

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_k + f_k^{\mathrm{rf}} = 1\,.$$

Specifically, the value that the portfolio holds at the openning of day d in the risky assets and in the risk-free assets is

$$\pi_k(d-1)\,\mathbf{f}_k\,, \qquad \pi_k(d-1)\,f_k^{\mathrm{rf}}\,. \tag{2.1}$$

Sharpe One-Rate Model: Market Capitalization

The *market capitalization* of the i^{th} risky asset at the openning of day d is $W_i(d)$ given by

$$W_i(d) = s_i(d-1) \times \#\{\text{outstanding shares of asset } i\}\,,$$

where $s_i(d-1)$ is the closing share price of asset i on day $d-1$. Because $s_i(d-1) > 0$ for every i and d, we see that $W_i(d) > 0$ for every i and d. The *market capitalization vector* is defined by

$$\mathbf{W}(d) = \begin{pmatrix} W_1(d) & W_2(d) & \cdots & W_N(d) \end{pmatrix}^{\mathrm{T}}\,.$$

Assumption 3 and equations (2.1) say that

$$\sum_{k=1}^{K} \pi_k(d-1)\,\mathbf{f}_k = \mathbf{W}(d)\,, \qquad \sum_{k=1}^{K} \pi_k(d-1)\,f_k^{\mathrm{rf}} = W_{\mathrm{rf}}(d)\,, \qquad (2.2)$$

where $W_{\mathrm{rf}}(d) > 0$ is the net worth of the risk-free assets.

Sharpe One-Rate Model: Efficient Tobin Frontier

The Tobin frontier depends upon μ_{rf}, $\mathbf{m}$ and $\mathbf{V}$, so **Assumption 4** says that it is constant in time. There will be three cases to consider.

- If $\mu_{\mathrm{rf}} = \mu_{\mathrm{mv}}$ then there is no tangent allocation.
- If $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ then there is an efficient tangent allocation $\mathbf{f}_{\mathrm{tg}} \in \mathcal{M}$.
- If $\mu_{\mathrm{rf}} > \mu_{\mathrm{mv}}$ then there is an inefficient tangent allocation $\mathbf{f}_{\mathrm{tg}} \in \mathcal{M}$.

Assumption 5 implies that each allocation must take one of two forms.

- If $\mu_{\mathrm{rf}} = \mu_{\mathrm{mv}}$ then the allocation for the k^{th} investor has the form

$$\left(\mathbf{f}_k\,,\; f_k^{\mathrm{rf}}\right) = \left(\phi_k\,\mathbf{g}_{\mathrm{mv}}\,,\; 1\right) \qquad \text{for some } \phi_k \geq 0\,. \tag{2.3a}$$

- If $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ then the allocation for the k^{th} investor has the form

$$\left(\mathbf{f}_k\,,\; f_k^{\mathrm{rf}}\right) = \left(\phi_k\,\mathbf{f}_{\mathrm{tg}}\,,\; 1 - \phi_k\right) \qquad \text{for some } \phi_k \in \mathbb{R}\,, \tag{2.3b}$$

  where $\phi_k \geq 0$ when $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ and $\phi_k \leq 0$ when $\mu_{\mathrm{rf}} > \mu_{\mathrm{mv}}$.

Sharpe One-Rate Model: Case $\mu_{\rm rf} = \mu_{\rm mv}$

First recall that

$$\mathbf{g}_{\rm mv} = \frac{1}{\nu_{\rm mv}^2}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm mv}\mathbf{1})\,.$$

Then placing the allocation form (2.3a) into equations (2.2) yields

$$\sum_{k=1}^{K}\pi_k(d-1)\,\phi_k\,\mathbf{g}_{\rm mv} = \mathbf{W}(d)\,, \qquad \sum_{k=1}^{K}\pi_k(d-1) = W_{\rm rf}(d)\,.$$

Upon multipying the first equation on the left by $\mathbf{1}^{\rm T}$ and using the fact that $\mathbf{1}^{\rm T}\mathbf{g}_{\rm mv} = 0$, we obtain

$$0 = \mathbf{1}^{\rm T}\mathbf{W}(d) = \sum_{i=1}^{N} W_i(d)\,.$$

But each $W_i(d) > 0$, so this yields a contradiction. Therefore the case $\mu_{\rm rf} = \mu_{\rm mv}$ is excluded and a tangent allocation must exist.

Sharpe One-Rate Model: Case $\mu_{\rm rf} \neq \mu_{\rm mv}$

Now placing the allocation form (2.3b) into equations (2.2) yields

$$\sum_{k=1}^{K} \pi_k (d-1)\, \phi_k \, \mathbf{f}_{\rm tg} = \mathbf{W}(d)\,, \tag{2.4a}$$

$$\sum_{k=1}^{K} \pi_k (d-1)\, (1-\phi_k) = W_{\rm rf}(d)\,. \tag{2.4b}$$

Upon multipying (2.4a) on the left by $\mathbf{1}^{\rm T}$ and using the fact that $\mathbf{f}_{\rm tg} \in \mathcal{M}$, whereby $\mathbf{1}^{\rm T}\mathbf{f}_{\rm tg} = 1$, we obtain

$$\sum_{k=1}^{K} \pi_k (d-1)\, \phi_k = \mathbf{1}^{\rm T}\mathbf{W}(d) = \sum_{i=1}^{N} W_i(d)\,. \tag{2.5}$$

Sharpe One-Rate Model: Tangent Efficiency

But (2.5) cannot be satisfied if $\mathbf{f}_{\mathrm{tg}}$ is inefficient because in that case

- its left-hand side is nonpositive because every $\pi_k(d-1) > 0$ and every $\phi_k \leq 0$, while
- its right-hand side is positive because every $W_i(d) > 0$.

Therefore the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ must be efficient, every ϕ_k in (2.3b) must be nonnegative, and equation (2.5) holds.

Upon placing (2.5) into equations (2.4), they become

$$W_{\mathrm{M}}(d)\,\mathbf{f}_{\mathrm{tg}} = \mathbf{W}(d)\,, \qquad \sum_{k=1}^{K} \pi_k(d-1) = W_{\mathrm{M}}(d) + W_{\mathrm{rf}}(d)\,, \tag{2.6}$$

where $W_{\mathrm{M}}(d) > 0$ is the *total market capitalization* given by

$$W_{\mathrm{M}}(d) = \mathbf{1}^{\mathrm{T}}\mathbf{W}(d) = \sum_{i=1}^{N} W_i(d)\,.$$

Sharpe One-Rate Model: Market Allocation

The *market allocation* is defined by for each d by

$$\mathbf{f}_{\mathrm{M}}(d) = \frac{1}{W_{\mathrm{M}}(d)} \mathbf{W}(d) . \tag{2.7}$$

Then the first equation of (2.6) takes the form

$$\mathbf{f}_{\mathrm{tg}} = \mathbf{f}_{\mathrm{M}}(d) \qquad \text{for every } d , \tag{2.8a}$$

while equation (2.5) and the second equation of (2.6) are

$$\sum_{k=1}^{K} \pi_k(d-1)\, \phi_k = W_{\mathrm{M}}(d) , \tag{2.8b}$$

$$\sum_{k=1}^{K} \pi_k(d-1) = W_{\mathrm{M}}(d) + W_{\mathrm{rf}}(d) . \tag{2.8c}$$

Sharpe One-Rate Model: Summary So Far

Equations (2.8) contain all of our conclusions thus far about CAPM. The most significant is equation (2.8a), which shows the following.

- The market allocation $\mathbf{f}_{\mathrm{M}}$ is constant in time.
- The tangent allocation is the market allocation ($\mathbf{f}_{\mathrm{tg}} = \mathbf{f}_{\mathrm{M}}$).
- The tangent allocation $\mathbf{f}_{\mathrm{tg}}$ is long.

Also important is equation (2.8b), which showed the following.

- The tangent allocation $\mathbf{f}_{\mathrm{tg}}$ is efficient.

The trouble with CAPM is that these predictions are not always supported by observations. Therefore the five assumptions upon which CAPM is built will need to be critically examined.

Before such a critical examination of its underlying assumptions, we will complete our development of CAPM.

Tangent Portfolio Relations: Tangent Portfolios

Tangent portfolio relations are relations between the return mean and volatility of any individual asset with those of the tangent portfolio. CAPM asserts that the tangent portfolio exists and is the market portfolio.

The tangent portfolio exists for any given μ_{rf}, $\mathbf{m}$ and $\mathbf{V}$ if and only if $\mu_{\text{rf}} \neq \mu_{\text{mv}}$, in which case its allocation is given by

$$\mathbf{f}_{\text{tg}} = \frac{\mu_{\text{tg}} - \mu_{\text{rf}}}{\nu_{\text{rf}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\text{rf}}\mathbf{1}) \,, \tag{3.9a}$$

where $\nu_{\text{rf}} > 0$ is determined by

$$\nu_{\text{rf}}^2 = (\mathbf{m} - \mu_{\text{rf}}\mathbf{1})^{\text{T}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\text{rf}}\mathbf{1}) \,, \tag{3.9b}$$

and the return mean μ_{tg} of the tangent portfolio is determined by

$$1 = \frac{(\mu_{\text{tg}} - \mu_{\text{rf}})\,(\mu_{\text{mv}} - \mu_{\text{rf}})}{\nu_{\text{rf}}^2\,\sigma_{\text{mv}}^2} \,. \tag{3.9c}$$

Tangent Portfolio Relations: Volatility Formulation

The volatility $\sigma_{\mathrm{tg}} \geq 0$ of the tangent portfolio is determined by

$$\begin{aligned}\sigma_{\mathrm{tg}}^2 = \mathbf{f}_{\mathrm{tg}}^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{tg}} &= \frac{(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}})^2}{\nu_{\mathrm{rf}}^4} (\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1})^{\mathrm{T}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1}) \\ &= \frac{(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}})^2}{\nu_{\mathrm{rf}}^4} \nu_{\mathrm{rf}}^2 = \frac{(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}})^2}{\nu_{\mathrm{rf}}^2} \,. \end{aligned} \tag{3.10}$$

Because (3.9c) implies $\mu_{\mathrm{tg}} \neq \mu_{\mathrm{rf}}$, we see from (3.10) that $\sigma_{\mathrm{tg}} > 0$ and that

$$\frac{\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2} = \frac{\sigma_{\mathrm{tg}}^2}{\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}} \,,$$

whereby the tangent allocation (3.9a) can be expressed as

$$\mathbf{f}_{\mathrm{tg}} = \frac{\sigma_{\mathrm{tg}}^2}{\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1}) \,. \tag{3.11}$$

Tangent Portfolio Relations: Return Mean Relations

We see from (3.11) that

$$\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1} = \frac{\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{tg}}^2}\,\mathbf{V}\mathbf{f}_{\mathrm{tg}}\,. \tag{3.12}$$

The return mean $\mu(\mathbf{f})$ for any $\mathbf{f} \in \mathcal{M}_+$ is given by

$$\mu(\mathbf{f}) = \mu_{\mathrm{rf}} + (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f}\,. \tag{3.13}$$

We obtain from (3.12) the *general return mean relation*

$$\mu(\mathbf{f}) - \mu_{\mathrm{rf}} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{\mathrm{tg}}}{\sigma_{\mathrm{tg}}^2}\left(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}\right). \tag{3.14}$$

Taking $\mathbf{f} = \mathbf{e}_i$ gives the *return mean relation for the i^{th} asset*

$$m_i - \mu_{\mathrm{rf}} = \frac{\mathbf{e}_i^{\mathrm{T}}\mathbf{V}\mathbf{f}_{\mathrm{tg}}}{\sigma_{\mathrm{tg}}^2}\left(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}\right). \tag{3.15}$$

Return Mean Relations: Volatility

The volatility $\sigma(\mathbf{f})$ for any $\mathbf{f} \in \mathcal{M}_+$ is given by

$$\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} = \|\mathbf{f}\|_{\mathbf{V}} . \tag{3.16}$$

Because $\sigma_{\mathrm{tg}} = \|\mathbf{f}_{\mathrm{tg}}\|_{\mathbf{V}}$ while for every $\mathbf{f} \in \mathcal{M}_+$ we have

$$(\mathbf{f}_{\mathrm{tg}} \,|\, \mathbf{f})_{\mathbf{V}} = \mathbf{f}_{\mathrm{tg}}^{\mathrm{T}}\mathbf{V}\mathbf{f} ,$$

the $\mathbf{V}$-othogonal projection of $\mathbf{f}$ onto $\mathrm{Span}\{\mathbf{f}_{\mathrm{tg}}\}$ is

$$\mathbf{P}\mathbf{f} = \mathbf{f}_{\mathrm{tg}} \frac{(\mathbf{f}_{\mathrm{tg}} \,|\, \mathbf{f})_{\mathbf{V}}}{\|\mathbf{f}_{\mathrm{tg}}\|_{\mathbf{V}}^2} = \mathbf{f}_{\mathrm{tg}} \frac{1}{\sigma_{\mathrm{tg}}^2} \mathbf{f}_{\mathrm{tg}}^{\mathrm{T}}\mathbf{V}\mathbf{f} .$$

Then by (3.16) we see that for every $\mathbf{f} \in \mathcal{M}_+$ we have

$$0 \leq \|\mathbf{f}\|_{\mathbf{V}}^2 - \|\mathbf{P}\mathbf{f}\|_{\mathbf{V}}^2 = \sigma(\mathbf{f})^2 - \frac{1}{\sigma_{\mathrm{tg}}^2} \left(\mathbf{f}_{\mathrm{tg}}^{\mathrm{T}}\mathbf{V}\mathbf{f}\right)^2 , \tag{3.17}$$

with equality if and only if $\mathbf{f}$ is proportional to $\mathbf{f}_{\mathrm{tg}}$.

Return Mean Relations: Volatility

Letting

$$\beta_i = \frac{1}{\sigma_{\mathrm{tg}}^2} \mathbf{e}_i^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{tg}} \,, \tag{3.18a}$$

we see that (3.15) becomes

$$m_i - \mu_{\mathrm{rf}} = \beta_i \left(\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}} \right) , \tag{3.18b}$$

while (3.17) becomes

$$\sigma_i^2 = \beta_i^2 \sigma_{\mathrm{tg}}^2 + \eta_i^2 \,, \tag{3.18c}$$

for some $\eta_i \geq 0$ with $\eta_i = 0$ if and only if $\mathbf{e}_i$ is proportional to $\mathbf{f}_{\mathrm{tg}}$. Because $\|\mathbf{P}\mathbf{e}_i\|_{\mathbf{V}}$ is the systemic volatility of the allocation $\mathbf{e}_i$, we see that (3.18c) is a docomposition where $\beta_i \, \sigma_{\mathrm{tg}}$ is the systemic risk and η_i is the diversifiable risk.

Tangent Portfolio Relations: Application to CAPM

Because the market portfolio is the tangent portfolio for CAPM, by setting $\mathbf{f}_{\mathrm{tg}} = \mathbf{f}_{\mathrm{M}}$ in (3.18) we obtain the following.

The return mean of the i^{th} asset satisfies the *CAPM return mean relation*

$$m_i - \mu_{\mathrm{rf}} = \beta_i \left(\mu_{\mathrm{M}} - \mu_{\mathrm{rf}} \right) , \tag{3.19a}$$

where

$$\beta_i = \frac{1}{\sigma_{\mathrm{M}}^2} \, \mathbf{e}_i^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{M}} , \tag{3.19b}$$

and the variance of the i^{th} asset has the *CAPM variance decomposition*

$$\sigma_i{}^2 = \beta_i^2 \sigma_{\mathrm{M}}^2 + \eta_i^2 , \tag{3.19c}$$

where $\beta_i \, \sigma_{\mathrm{M}}$ is the systemic risk and η_i is the diversifiable risk.

Summary and Critique: CAPM Assumptions

In summary, CAPM is built upon five assumptions.

1. The market consists of N risky assets and risk-free assets with a common return rate $\mu_{\rm rf}$.
2. There are K investors, each of which holds a solvent Markowitz portfolio governed by the one-rate model for risk-free assets.
3. The market capitalization of each asset is equal to the sum of its value of that asset held in each portfolio.
4. The distribution of daily return vectors is known and is stationary. Let $\mathbf{m}$ and $\mathbf{V}$ be its known mean vector and covariance matrix. Then $\mu_{\rm rf}$, $\mathbf{m}$ and $\mathbf{V}$ are the same for all investors and are constant in time.
5. Each investor holds a portfolio on the efficient Tobin frontier.

Summary and Critique: CAPM Main Conclusions

The main conclusions of CAPM are as follows.

- The tangent portfolio exists and is efficient $(\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}})$.
- The tangent allocation $\mathbf{f}_{\mathrm{tg}}$ is the market capitalization allocation $\mathbf{f}_{\mathrm{M}}$, and is thereby long.
- The market capitalization allocation $\mathbf{f}_{\mathrm{M}}$ does not depend on time.
- The return mean of the i^{th} asset satisfies

$$m_i - \mu_{\mathrm{rf}} = \beta_i \left(\mu_{\mathrm{M}} - \mu_{\mathrm{rf}}\right), \quad \text{where} \quad \beta_i = \frac{1}{\sigma_{\mathrm{M}}^2}\, \mathbf{e}_i^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{M}}\,. \tag{4.20a}$$

- The variance of the i^{th} asset has the decomposition

$$\sigma_i^2 = \beta_i^2 \sigma_{\mathrm{M}}^2 + \eta_i^2\,, \tag{4.20b}$$

where $\beta_i\, \sigma_{\mathrm{M}}$ is the systemic risk and η_i is the diversifiable risk.

Summary and Critique: CAPM Impacts

CAPM suggests that the optimal portfolio allocation is always some combination of a long position in the market capitalization allocation with either a long, short or neutral position in risk-free assets. By the late 1960s research indicated that the predictions of CAPM held approximately over the preceding decade. This created a demand for products that would give investors an easy way to hold the market capitalization portfolio.

In practice the market capitalization allocation $\mathbf{f}_{\mathrm{M}}$ is hard to compile, so equity indices are used as proxies for it. The S&P 500 index is used by VFINX, which was the first index fund. It was offered to the public in 1970 by the Vanguard Group led by John C. Bogle. Since then index funds have grown to make up between 20 to 30% of the U.S. equities market. (The range here is due to the uncertainty about what is an index fund.)

Summary and Critique: Equity Indices

Two proxies for the market capitalization allocation are widely used.

- The **S&P 500** index is based upon the top 500 U.S. firms by market capitalization.
- The **Russell 1000** index is based upon the top 1000 U.S. firms by market capitalization.

The total market capitalization of U.S. equities was about 52 trillion$ in January 2022.

- The S&P 500 was about 40 trillion$ (about 77%) of this total.
- The Russell 1000 was about 48 trillion$ (about 92%) of this total.

Let $\mathbf{f}_{\mathrm{SP}}$ and $\mathbf{f}_{\mathrm{RS}}$ be the market capitalization allocations of these indicies. The question is how well do $(\sigma_{\mathrm{SP}}, \mu_{\mathrm{SP}})$ and $(\sigma_{\mathrm{RS}}, \mu_{\mathrm{RS}})$ track $(\sigma_{\mathrm{M}}, \mu_{\mathrm{M}})$?

Summary and Critique: Continuity Bounds

Upper bounds can be found using Lipshitz bounds for the functions

$$\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \qquad \mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,.$$

From the percentages given on the previous slide it can be shown that

$$\left\|\mathbf{f}_{\mathrm{M}} - \mathbf{f}_{\mathrm{SP}}\right\|_1 \approx .46\,, \qquad \left\|\mathbf{f}_{\mathrm{M}} - \mathbf{f}_{\mathrm{RS}}\right\|_1 \approx .16\,, \qquad \left\|\mathbf{f}_{\mathrm{RS}} - \mathbf{f}_{\mathrm{SP}}\right\|_1 \approx .33\,.$$

For any $\mathbf{f}_0$, $\mathbf{f}_1 \in \mathcal{M}$ we can derive the Lipshitz bounds

$$\begin{aligned} \left|\sigma(\mathbf{f}_1) - \sigma(\mathbf{f}_0)\right| &\leq \sigma_{\mathrm{mx}} \left\|\mathbf{f}_1 - \mathbf{f}_0\right\|_1\,, \\ \left|\mu(\mathbf{f}_1) - \mu(\mathbf{f}_0)\right| &\leq \tfrac{1}{2}(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}) \left\|\mathbf{f}_1 - \mathbf{f}_0\right\|_1\,. \end{aligned}$$

Putting these together we obtain the approximate rough bounds

$$\begin{aligned} \left|\sigma_{\mathrm{M}} - \sigma_{\mathrm{SP}}\right| &\leq .46\,\sigma_{\mathrm{mx}}\,, & \left|\mu_{\mathrm{M}} - \mu_{\mathrm{SP}}\right| &\leq .23\,(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,, \\ \left|\sigma_{\mathrm{M}} - \sigma_{\mathrm{RS}}\right| &\leq .16\,\sigma_{\mathrm{mx}}\,, & \left|\mu_{\mathrm{M}} - \mu_{\mathrm{RS}}\right| &\leq .08\,(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,, \\ \left|\sigma_{\mathrm{RS}} - \sigma_{\mathrm{SP}}\right| &\leq .33\,\sigma_{\mathrm{mx}}\,, & \left|\mu_{\mathrm{RS}} - \mu_{\mathrm{SP}}\right| &\leq .17\,(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,. \end{aligned}$$

Summary and Critique: Index Tracking

Considering that $\sigma_{\rm mx}$ and $(\mu_{\rm mx} - \mu_{\rm mn})$ are based upon all the risky assets in the entire market, the foregoing bounds are huge. They do not offer much support for the idea of using an equity index to track $(\sigma_{\rm M}, \mu_{\rm M})$. However, the uniform Lipshitz bounds over $\mathcal{M}$ from which they were derived are not refined. There is reason to hope for better.

A practical way to test the idea of using an equity index to track $(\sigma_{\rm M}, \mu_{\rm M})$ is to check how well two equity indices track each other. For example, it can be checked how well $(\sigma_{\rm SP}, \mu_{\rm SP})$ and $(\sigma_{\rm RS}, \mu_{\rm RS})$ track each other. One finds that they track each other fairly well most of the time, but can drift apart at critical times. More importantly, both of these indices can track so far from the efficient frontier that they approach the inefficient frontier. This is not what CAPM predicts!

Summary and Critique: Critique

By the 1970s reseach had showed that the predictions of CAPM did not even approximately hold at all times. This stimulated the development of more flexible models, most notably *Arbitrage Pricing Theory (APT)* by Stephen Ross in 1976. This theory is still in use today.

The problem with CAPM is that some of its assuumptions are too strong. **Assumptions 4** is the most problematic. It states that the distributions of returns for the entire market is known to all investors. In fact no entry of either $\mathbf{m}$ or $\mathbf{V}$ is known to any investor! At best each investor who cares to do it to will estimate $\mathbf{m}$ and $\mathbf{V}$ for a subset of the assets either from return histories or from some other data set by some other method. Moreover, these investors might use different risk-free rates or use the Two-Rate model or a more complicated model for risk-free assets. Consequently, there will be no univerisal indentification of the efficient frontier.