Portfolios that Contain Risky Assets
3.3. Metrics with Risk-Free Assets

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

3. Models for Portfolios with Risk-Free Assets

Metrics with Risk-Free Assets

Introduction to Metrics with Risk-Free Assets

Now we enlarge the set of metrics that we will explore with new metrics that involve the risk-free rates $\mu_{\rm si}$ and $\mu_{\rm cl}$. This will be done in three ways.

- We introduce two new metrics that depend only upon the Tobin frontier parameters ($\mu_{\rm rf}$ and $\nu_{\rm rf}$) and Markowitz frontier parameters ($\sigma_{\rm mv}$, $\mu_{\rm mv}$, and $\nu_{\rm mv}$). We call these *stability metrics*.
- We consider applying the four portfolio metric functions developed earlier ($\omega^\lambda(\mathbf{f})$, $\omega^\delta(\mathbf{f})$, $\omega^\mu(\mathbf{f})$, and $\omega^\sigma(\mathbf{f})$) to tangent portfolio allocations that depend upon a risk-free rate $\mu_{\rm rf}$.
- We develop a new portfolio metric function $\omega^\rho(\mathbf{f})$ build from a function $\rho(\mathbf{f})$ called the *Sharpe ratio* that depends upon the safe investment rate $\mu_{\rm si}$ as well as an allocation $\mathbf{f} \in \mathcal{M}$. We also identify allocations to which this new metric function will be applied. We call these *Sharpe metrics*.

Stability Ratio and Metrics: Introduction

Recall that if μ_{rf} is a risk-free rate then its Tobin frontier in the $\sigma\mu$-plane is $\sigma = \sigma_{\mathrm{tf}}(\mu)$, where

$$\sigma_{\mathrm{tf}}(\mu) = \frac{|\mu - \mu_{\mathrm{rf}}|}{\nu_{\mathrm{rf}}}\,, \tag{2.1}$$

where $\nu_{\mathrm{rf}} > 0$ is determined by

$$\nu_{\mathrm{rf}}^2 = (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,. \tag{2.2}$$

The Tobin frontier parameters (μ_{rf} and ν_{rf}) and the Markowitz frontier parameters (σ_{mv}, μ_{rf} and ν_{rf}) satisfy the frontier parameter relation

$$\nu_{\mathrm{rf}}^2 = \nu_{\mathrm{mv}}^2 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}}\right)^2\,. \tag{2.3}$$

Stability Ratio and Metrics: Tangency and Intersection

When $\mu_{\rm rf} \neq \mu_{\rm mv}$ the Tobin Frontier becomes tangent to the Markowitz frontier, $\sigma = \sigma_{\rm mf}(\mu)$, at the unique tangency point $(\sigma_{\rm tg}, \mu_{\rm tg})$ that is determined by

$$\sigma_{\rm tg} = \frac{|\mu_{\rm tg} - \mu_{\rm rf}|}{\nu_{\rm rf}}, \qquad \frac{(\mu_{\rm tg} - \mu_{\rm rf})(\mu_{\rm mv} - \mu_{\rm rf})}{\nu_{\rm rf}^2\, \sigma_{\rm mv}^2} = 1\,. \tag{2.4}$$

We will develop a metric that are measures of the certainty with which we know this tangency point.

We see from (2.3) that $\nu_{\rm rf}^2 > \nu_{\rm mv}^2$ when $\mu_{\rm rf} \neq \mu_{\rm mv}$. Because the slopes of the Markowitz asymptotes are $\pm\nu_{\rm mv}$ and the slopes of the Tobin frontier branches are $\pm\nu_{\rm rf}$, the Tobin frontier must cross a Markowitz asymptotes at a unique intersection point $(\sigma_{\rm as}, \mu_{\rm as})$. We will develop a metric that are measures of the certainty with which we know this intersection point.

Stability Ratio and Metrics: Ratios

The first is based upon the ratio

$$\frac{\mu_{\mathrm{tg}} - \mu_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}} . \tag{2.5}$$

The second is based upon the ratio

$$\frac{\mu_{\mathrm{as}} - \mu_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}} , \tag{2.6}$$

These ratios quantify the stability of μ_{tg} and μ_{as} to changes in $\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}$. A smaller ratio means greater stability.

Both metrics will be simple functions of the ratio

$$\frac{\nu_{\mathrm{mv}}}{\nu_{\mathrm{rf}}} = \frac{1}{\sqrt{1 + \frac{(\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})^2}{\nu_{\mathrm{mv}}^2 \, \sigma_{\mathrm{mv}}^2}}} , \tag{2.7}$$

which we dub the *stability ratio*.

Stability Ratio and Metrics: Tangency Point Ratio

We begin by exploring the stability of $\mu_{\rm tg} - \mu_{\rm mv}$ to changes in $\mu_{\rm mv} - \mu_{\rm rf}$. We see from (2.4) and (2.3) that

$$\mu_{\rm tg} - \mu_{\rm rf} = \frac{\nu_{\rm rf}^2 \, \sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} = \frac{\nu_{\rm mv}^2 \, \sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} + \left(\mu_{\rm mv} - \mu_{\rm rf}\right),$$

whereby

$$\mu_{\rm tg} - \mu_{\rm mv} = \frac{\nu_{\rm mv}^2 \, \sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} .$$

Therfore the tangency point ratio (2.5) is

$$\frac{\mu_{\rm tg} - \mu_{\rm mv}}{\mu_{\rm mv} - \mu_{\rm rf}} = \frac{\nu_{\rm mv}^2 \, \sigma_{\rm mv}^2}{(\mu_{\rm mv} - \mu_{\rm rf})^2} . \tag{2.8}$$

This ratio is always positive.

Stability Ratio and Metrics: Tangency Point Metric

To convert this positive quantity to a value in $(0,1)$ we use the function

$$x \;\mapsto\; \frac{x}{1+x}\,,$$

whereby the right-hand side of (2.8) converts as

$$\frac{\nu_{\mathrm{mv}}^2\,\sigma_{\mathrm{mv}}^2}{(\mu_{\mathrm{mv}}-\mu_{\mathrm{rf}})^2} \;\mapsto\; \frac{1}{1+\dfrac{(\mu_{\mathrm{mv}}-\mu_{\mathrm{rf}})^2}{\nu_{\mathrm{mv}}^2\,\sigma_{\mathrm{mv}}^2}}\,.$$

Therefore, we define the tangency point metric for the rate μ_{rf} by

$$\omega_{\mathrm{rf}}^{\mathrm{tg}} = \frac{1}{1+\dfrac{(\mu_{\mathrm{mv}}-\mu_{\mathrm{rf}})^2}{\nu_{\mathrm{mv}}^2\,\sigma_{\mathrm{mv}}^2}}\,. \qquad (2.9)$$

Stability Ratio and Metrics: Alternative Derivation

Remark. Here is an alternative derivation of the tangency point metric. We can express μ_{tg} in terms of the Markowitz frontier parameters μ_{mv}, σ_{mv}, and ν_{mv}, and the risk-free rate μ_{rf} as

$$\mu_{\text{tg}} - \mu_{\text{mv}} = \frac{\nu_{\text{mv}}^2 \, \sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{rf}}} \,.$$

The partial derivative of μ_{tg} with respect to μ_{rf} is

$$\frac{\partial \mu_{\text{tg}}}{\partial \mu_{\text{rf}}} = \frac{\nu_{\text{mv}}^2 \, \sigma_{\text{mv}}^2}{(\mu_{\text{mv}} - \mu_{\text{rf}})^2} \,.$$

This partial derivative is a reasonable measure of the stability of μ_{tg} with restect to changes in μ_{rf}. However, it is equal to the tangency point ratio (2.8), so using it to generate a metric would also generate the tangency point metric $\omega_{\text{rf}}^{\text{tg}}$ given by (2.9).

Stability Ratio and Metrics: Asymptote Intersection Point

Next we explore the stability of $\mu_{\rm as} - \mu_{\rm mv}$ to changes in $\mu_{\rm mv} - \mu_{\rm rf}$. We begin by determining the unique intersection point $(\sigma_{\rm as}, \mu_{\rm as})$ of the Tobin frontier with the Markowitz frontier asymptotes when $\mu_{\rm rf} \neq \mu_{\rm mv}$. The Markowitz frontier asymptotes are given by

$$\sigma = \frac{|\mu - \mu_{\rm mv}|}{\nu_{\rm mv}}, \qquad (2.10)$$

where $\nu_{\rm mv} > 0$ is determined by

$$\nu_{\rm mv}^2 = (\mathbf{m} - \mu_{\rm mv}\mathbf{1})^{\rm T}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm mv}\mathbf{1}). \qquad (2.11)$$

The Tobin frontier (2.1) intersects the Markowitz frontier asymptotes when

$$\frac{(\mu - \mu_{\rm rf})^2}{\nu_{\rm rf}^2} = \frac{(\mu - \mu_{\rm mv})^2}{\nu_{\rm mv}^2}. \qquad (2.12)$$

Stability Ratio and Metrics: Asymptote Intersection Ratio

Because $\nu_{\mathrm{rf}} > \nu_{\mathrm{mf}}$, the unique solution of equation (2.12) is $\mu = \mu_{\mathrm{as}}$ where μ_{as} satisfies the linear equation

$$\mu_{\mathrm{as}} - \mu_{\mathrm{rf}} = \frac{\nu_{\mathrm{rf}}}{\nu_{\mathrm{mv}}} \left(\mu_{\mathrm{as}} - \mu_{\mathrm{mv}}\right) .$$

It follows that

$$\mu_{\mathrm{as}} - \mu_{\mathrm{mv}} = \left(\mu_{\mathrm{as}} - \mu_{\mathrm{rf}}\right) - \left(\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}\right) = \frac{\nu_{\mathrm{rf}}}{\nu_{\mathrm{mv}}} \left(\mu_{\mathrm{as}} - \mu_{\mathrm{mv}}\right) - \left(\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}\right) ,$$

whereby

$$\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}} = \left(\frac{\nu_{\mathrm{rf}}}{\nu_{\mathrm{mv}}} - 1\right) \left(\mu_{\mathrm{as}} - \mu_{\mathrm{mv}}\right) .$$

Therefore the asymptote intersection ratio is

$$\frac{\mu_{\mathrm{as}} - \mu_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}} = \frac{\nu_{\mathrm{mv}}}{\nu_{\mathrm{rf}} - \nu_{\mathrm{mv}}} . \tag{2.13}$$

Stability Ratio and Metrics: Asymptote Intersection Metric

To convert this positive quantity to a value in $(0,1)$ we use the function

$$x \;\mapsto\; \frac{x}{1+x}\,,$$

whereby the right-hand side of (2.13) converts into

$$\frac{\nu_{\mathrm{mv}}}{\nu_{\mathrm{rf}}-\nu_{\mathrm{mv}}} \;\mapsto\; \frac{\nu_{\mathrm{mv}}}{\nu_{\mathrm{rf}}}\,.$$

Therefore, we define the *asymptote intersection metric* for the rate μ_{rf} by

$$\omega_{\mathrm{rf}}^{\mathrm{as}} = \frac{\nu_{\mathrm{mv}}}{\nu_{\mathrm{rf}}}\,. \tag{2.14}$$

This is just the stability ratio (2.7).

Stability Ratio and Metrics: Comparison

By using the frontier parameter relation (2.3), we may express the tangency point metric (2.9) as

$$\omega_{\mathrm{rf}}^{\mathrm{tg}} = \frac{\nu_{\mathrm{mv}}^2}{\nu_{\mathrm{rf}}^2} \,.$$

This is just the square of the stability ratio (2.7).

Upon comparing this with asymptote intersection metric (2.14), we see that $\omega_{\mathrm{rf}}^{\mathrm{tg}}$ and $\omega_{\mathrm{rf}}^{\mathrm{as}}$ are simply related by

$$\omega_{\mathrm{rf}}^{\mathrm{as}} = \sqrt{\omega_{\mathrm{rf}}^{\mathrm{tg}}} \,. \tag{2.15}$$

We see that the asymptote intersection metric is more sensitive at lower values and the tangency point metric is more sensitive at higher values. **Project One** explores which, if any, of these sensitivities is more useful.

Stability Ratio and Metrics: Summary

In summary,

$$\begin{aligned} \omega_{\rm rf}^{\rm tg} &= \frac{\nu_{\rm mv}^2}{\nu_{\rm rf}^2} = \frac{1}{1 + \left(\dfrac{\mu_{\rm mv} - \mu_{\rm rf}}{\nu_{\rm mv}\,\sigma_{\rm mv}}\right)^2}\,, \\ \omega_{\rm rf}^{\rm as} &= \frac{\nu_{\rm mv}}{\nu_{\rm rf}} = \frac{1}{\sqrt{1 + \left(\dfrac{\mu_{\rm mv} - \mu_{\rm rf}}{\nu_{\rm mv}\,\sigma_{\rm mv}}\right)^2}}\,, \end{aligned} \tag{2.16}$$

For a Two-Rate Model these should be applied to $\mu_{\rm rf} = \mu_{\rm si}$ and $\mu_{\rm rf} = \mu_{\rm cl}$, yielding the four metrics

$$\omega_{\rm si}^{\rm tg}\,, \qquad \omega_{\rm cl}^{\rm tg}\,, \qquad \omega_{\rm si}^{\rm as}\,, \qquad \omega_{\rm cl}^{\rm as}\,.$$

Portfolio Metrics: Leverage

We have already introduced four portfolio metric functions

$$\omega^\lambda(\mathbf{f}), \qquad \omega^\delta(\mathbf{f}), \qquad \omega^\mu(\mathbf{f}), \qquad \omega^\sigma(\mathbf{f}),$$

built from four functions of $\mathbf{f} \in \mathcal{M}$: the leverage ratio $\lambda(\mathbf{f})$, the downside potential $\delta(\mathbf{f})$, the return mean $\mu(\mathbf{f})$ and the volatility $\sigma(\mathbf{f})$.

The leverage ratio is given by

$$\lambda(\mathbf{f}) = \tfrac{1}{2}\left(\|\mathbf{f}\|_1 - 1\right).$$

It was used to define the leverge metric function by

$$\omega^\lambda(\mathbf{f}) = \frac{\lambda(\mathbf{f})}{1 + \lambda(\mathbf{f})}. \qquad (3.17)$$

The leverage ratio and metric do not have an explicit dependence upon the return history. They can gain statistical meaning by applying them to allocations $\mathbf{f}$ that do depend on the return history.

Portfolio Metrics: Leverage Metrics

The leverage metric function $\omega^\lambda(\mathbf{f})$ given by (3.17) had been applied to

- the minimum volatility allocation $\mathbf{f}_{\mathrm{mv}}$,
- the safe tangent allocation $\mathbf{f}_{\mathrm{st}}$,
- the credit tangent allocation $\mathbf{f}_{\mathrm{ct}}$.

Because $\lambda(\mathbf{f}) = 0$ for every $\mathbf{f} \in \Lambda$, applying the leverage metric to any long portfolio yields no information. In particular, it makes no sense to apply it to individual assets. We are thereby left with three leverage metrics:

$$\omega^\lambda_{\mathrm{mv}} = \omega^\lambda(\mathbf{f}_{\mathrm{mv}}) \,, \qquad \omega^\lambda_{\mathrm{st}} = \omega^\lambda(\mathbf{f}_{\mathrm{st}}) \,, \qquad \omega^\lambda_{\mathrm{ct}} = \omega^\lambda(\mathbf{f}_{\mathrm{ct}}) \,. \tag{3.18}$$

Remark. Markowitz frontier allocations lie on the line $\mathbf{f} = \mathbf{f}_{\mathrm{mf}}(\mu)$ in $\mathcal{M}$. Hence, because $\lambda(\mathbf{f})$ is a convex function over $\mathcal{M}$, we can bound $\lambda(\mathbf{f})$ above by convex combinations of its value at $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$ and $\mathbf{f}_{\mathrm{ct}}$.

Portfolio Metrics: Liquidity

The return mean, $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, and downside potential,

$$\delta(\mathbf{f}) = \max\left\{ -\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \,:\, d = 1, \cdots, D \right\},$$

were used to define the liquidity metric function by

$$\omega^{\delta}(\mathbf{f}) = \begin{cases} \dfrac{\delta(\mathbf{f}) + \mu(\mathbf{f})}{1 + \mu(\mathbf{f})} & \text{if } \delta(\mathbf{f}) < 1\,, \\ 1 & \text{if } \delta(\mathbf{f}) \geq 1\,. \end{cases} \tag{3.19}$$

The downside potential (and through it, the liquidity metric function) has an explicit dependence on the return history. Moreover, this dependence goes beyond the mean-variance statistics of $\mathbf{m}$ and $\mathbf{V}$.

Portfolio Metrics: Liquidity Metrics

The liquidity metric function, $\omega^\delta(\mathbf{f})$, given by (3.19) can be applied to any $\mathbf{f} \in \mathcal{M}$. However, it is most useful when applied to leveraged portfolio allocations that have the potential of not being solvent. It had been applied only to

- the minimum volatility allocation $\mathbf{f}_{\mathrm{mv}}$,

We now enlarge this set to include

- the safe tangent allocation $\mathbf{f}_{\mathrm{st}}$,
- the credit tangent allocation $\mathbf{f}_{\mathrm{ct}}$.

This gives a total of three liquidity metrics:

$$\omega^\delta_{\mathrm{mv}} = \omega^\delta(\mathbf{f}_{\mathrm{mv}}) \,, \qquad \omega^\delta_{\mathrm{st}} = \omega^\delta(\mathbf{f}_{\mathrm{st}}) \,, \qquad \omega^\delta_{\mathrm{ct}} = \omega^\delta(\mathbf{f}_{\mathrm{ct}}) \,. \tag{3.20}$$

Remark. Markowitz frontier allocations lie on the line $\mathbf{f} = \mathbf{f}_{\mathrm{mf}}(\mu)$ in $\mathcal{M}$. Hence, because $\delta(\mathbf{f})$ is a convex function over $\mathcal{M}$, we can bound $\delta(\mathbf{f})$ above by convex combinations of its value at $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$ and $\mathbf{f}_{\mathrm{ct}}$.

Portfolio Metrics: Efficiency and Proximity

The volatility, $\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$, and return mean, $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, were used to define the efficiency and proximity metric functions by

$$\begin{aligned} \omega^{\mu}(\mathbf{f}) &= \frac{\mu_{\mathrm{emf}}(\sigma(\mathbf{f})) - \mu(\mathbf{f})}{\mu_{\mathrm{emf}}(\sigma(\mathbf{f})) - \mu_{\mathrm{imf}}(\sigma(\mathbf{f}))}\,, \\ \omega^{\sigma}(\mathbf{f}) &= \sqrt{1 - \frac{\sigma_{\mathrm{mf}}(\mu(\mathbf{f}))^2}{\sigma(\mathbf{f})^2}}\,. \end{aligned} \tag{3.21}$$

The efficiency and proximity metric functions have explicit dependence upon the return history through the Markowitz frontier parameters μ_{mv}, σ_{mv} and ν_{mv}, which determine $\mu_{\mathrm{emf}}(\sigma)$, $\mu_{\mathrm{imf}}(\sigma)$ and $\sigma_{\mathrm{mf}}(\mu)$.

Portfolio Metrics: Efficiency and Proximity Metrics

The efficiency and proximity metric functions, $\omega^\mu(\mathbf{f})$ and $\omega^\sigma(\mathbf{f})$ given by (3.21), had been applied to

- the equity index fund $\mathbf{e}_{\mathrm{EI}}$,
- the total bond index fund $\mathbf{e}_{\mathrm{BI}}$.

They were not applied to portfolios on the Markowitz frontier because

- $\omega^\mu(\mathbf{f}) = \omega^\sigma(\mathbf{f}) = 0$ for all portfolios on the efficient frontier,
- $\omega^\mu(\mathbf{f}) = 1$ and $\omega^\sigma(\mathbf{f}) = 0$ for all portfolios on the inefficient frontier.

In particular, it makes no sense to apply them to $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$, or $\mathbf{f}_{\mathrm{ct}}$.

This leaves the original two efficiency metrics and two proximity metrics:

$$\begin{aligned} \omega^\mu_{\mathrm{EI}} &= \omega^\mu(\mathbf{e}_{\mathrm{EI}})\,, &\qquad \omega^\mu_{\mathrm{BI}} &= \omega^\mu(\mathbf{e}_{\mathrm{BI}})\,, \\ \omega^\sigma_{\mathrm{MI}} &= \omega^\sigma(\mathbf{e}_{\mathrm{EI}})\,, &\qquad \omega^\sigma_{\mathrm{BI}} &= \omega^\sigma(\mathbf{e}_{\mathrm{BI}})\,. \end{aligned} \tag{3.22}$$

Sharpe Ratio and Metrics: Ratio

In the early 1960s William Sharpe introduce several ideas into portfolio theory. One of these became known as the *Sharpe ratio*, $\rho(\mathbf{f})$. which is defined for every $\mathbf{f} \in \mathcal{M}$ in terms of the volatility $\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$, the return mean $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$, and the safe investment rate μ_{si} by

$$\rho(\mathbf{f}) = \frac{\mu(\mathbf{f}) - \mu_{\mathrm{si}}}{\sigma(\mathbf{f})} \,. \tag{4.23}$$

This is just the slope of the line in the $\sigma\mu$-plane between the the points $(0, \mu_{\mathrm{si}})$ and $(\sigma(\mathbf{f}), \mu(\mathbf{f}))$.

Recalling our construction of the efficient frontier for the One-Rate and Two-Rate models, the Sharpe ratio is bounded by $\rho(\mathbf{f}) \leq \nu_{\mathrm{si}}$. Sharpe argued that the closer this ratio is to this upper bound, the better.

Sharpe Ratio and Metrics: Metrics

This suggests introducing the *Sharpe metric fuction* defined by

$$\omega^{\rho}(\mathbf{f}) = 1 - \max\left\{ \frac{\rho(\mathbf{f})}{\nu_{\mathrm{si}}} \,, \, 0 \right\} . \tag{4.24}$$

The Sharpe metric function $\omega^{\rho}(\mathbf{f})$ can be applied to any $\mathbf{f} \in \mathcal{M}$. It vanishes only for $\mathbf{f} = \mathbf{f}_{\mathrm{st}}$ when $\mathbf{f}_{\mathrm{st}}$ lies on the efficient Markowitz frontier. Recall that $\mathbf{f}_{\mathrm{st}}$ lies on the efficient Markowitz frontier if and only if $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}}$. When $\mu_{\mathrm{si}} > \mu_{\mathrm{mv}}$ we have $\omega^{\rho}(\mathbf{f}_{\mathrm{st}}) = 1$. Hence, $\omega^{\rho}(\mathbf{f}_{\mathrm{st}})$ can take just two values, 0 or 1. It will be more informative for other portfolios.

We will apply the Sharpe metric function to the frontier portfolio $\mathbf{f}_{\mathrm{ct}}$, and to the equity index fund $\mathbf{e}_{\mathrm{EI}}$. This gives the two Sharpe metrics:

$$\omega^{\rho}_{\mathrm{ct}} = \omega^{\rho}(\mathbf{f}_{\mathrm{ct}}) \,, \qquad \omega^{\rho}_{\mathrm{EI}} = \omega^{\rho}(\mathbf{e}_{\mathrm{EI}}) \,. \tag{4.25}$$

Sharpe Ratio and Metrics: Return Correlations

The Sharpe ratio emerges by asking how the returns for a portfolio with allocation $\mathbf{f} \in \mathcal{M}$ correlate with those for a portfolio with allocation $\mathbf{f}_* \in \mathcal{M}_+$ that lies on the efficient Tobin frontier of the safe investment. Such an $\mathbf{f}_*$ has the form

$$\mathbf{f}_* = \mathbf{f}_{\text{sf}}(\mu_*) = (\mu_* - \mu_{\text{si}})\,\mathbf{g}_{\text{si}} \qquad \text{for some } \mu_* > \mu_{\text{si}}\,, \tag{4.26}$$

where

$$\mathbf{g}_{\text{si}} = \frac{1}{\nu_{\text{si}}^2}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\text{si}}\mathbf{1})\,, \qquad \nu_{\text{si}}^2 = \nu_{\text{mv}}^2 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{si}}}{\sigma_{\text{mv}}}\right)^2 .$$

Because $\mu_* > \mu_{\text{si}}$ and $\nu_{\text{si}} > 0$, its volatility is given by

$$\sigma_* = \sigma_{\text{sf}}(\mu_*) = \frac{\mu_* - \mu_{\text{si}}}{\nu_{\text{si}}}\,. \tag{4.27}$$

Sharpe Ratio and Metrics: Return Correlations

By using (4.26) we see that the return covariance for the portfolios with allocations $\mathbf{f}_*$ and any $\mathbf{f} \in \mathcal{M}$ is

$$\begin{aligned}\mathbf{f}_*^{\mathrm{T}}\mathbf{V}\mathbf{f} &= (\mu_* - \mu_{\mathrm{si}})\,\mathbf{g}_{\mathrm{si}}^{\mathrm{T}}\mathbf{V}\mathbf{f} \\ &= \frac{\mu_* - \mu_{\mathrm{si}}}{\nu_{\mathrm{si}}^2}\,(\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1})^{\mathrm{T}}\mathbf{f} = \frac{(\mu_* - \mu_{\mathrm{si}})\,(\mathbf{m}^{\mathrm{T}}\mathbf{f} - \mu_{\mathrm{si}})}{\nu_{\mathrm{si}}^2}\,.\end{aligned}$$

Therefore from (4.27) we see the return correlation of these portfolios is

$$\begin{aligned}\frac{\mathbf{f}_*^{\mathrm{T}}\mathbf{V}\mathbf{f}}{\sigma_*\,\sigma(\mathbf{f})} &= \frac{(\mu_* - \mu_{\mathrm{si}})\,(\mathbf{m}^{\mathrm{T}}\mathbf{f} - \mu_{\mathrm{si}})}{\nu_{\mathrm{si}}^2}\,\frac{\nu_{\mathrm{si}}}{\mu_* - \mu_{\mathrm{si}}}\,\frac{1}{\sigma(\mathbf{f})} \\ &= \frac{\mu(\mathbf{f}) - \mu_{\mathrm{si}}}{\nu_{\mathrm{si}}\,\sigma(\mathbf{f})} = \frac{\rho(\mathbf{f})}{\nu_{\mathrm{si}}}\,,\end{aligned} \tag{4.28}$$

where $\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$ is the volatility, $\mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}$ is the return mean, and $\rho(\mathbf{f})$ given by (4.23) is the Sharpe ratio for the portfolio allocation $\mathbf{f} \in \mathcal{M}$.