Portfolios that Contain Risky Assets
3.2. Two-Rate Model for Risk-Free Assets

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

3. Models for Portfolios with Risk-Free Assets

Two-Rate Model for Risk-Free Assets

Two-Rate Model Portfolios: Allocations

Now we extend the notion of Markowitz portfolios to portfolios that might include a safe investment with rate $\mu_{\rm si}$ and a credit line with rate $\mu_{\rm si}$ where $\mu_{\rm si} < \mu_{\rm cl}$.

Let $b_{\rm si}(d)$ and $b_{\rm cl}(d)$ be the balances in these assets at the start of day d. We require that $b_{\rm si}(d) \geq 0$ and $b_{\rm cl}(d) \leq 0$.

A *Markowitz portfolio* containing these risk-free assets and N risky assets is uniquely determined by real numbers $f_{\rm si}$, $f_{\rm cl}$, and $\{f_i\}_{i=1}^N$ that satisfy

$$f_{\rm si} + f_{\rm cl} + \sum_{i=1}^{N} f_i = 1\,.$$

Here $f_{\rm si}$ is the allocation of the portfolio in the safe investment, $f_{\rm cl}$ is the allocation of the portfolio in the credit line, and f_i is the allocation of the portfolio in the $i^{\rm th}$ risky asset.

Two-Rate Model Portfolios: Values

The portfolio is rebalanced at the start of each day so that

$$\frac{b_{\mathrm{si}}(d)}{\pi(d-1)} = f_{\mathrm{si}}\,, \qquad \frac{b_{\mathrm{cl}}(d)}{\pi(d-1)} = f_{\mathrm{cl}}\,,$$

$$\frac{n_i(d)\, s_i(d-1)}{\pi(d-1)} = f_i \qquad \text{for } i = 1, \cdots, N\,.$$

Its value at the start of day d is

$$\pi(d-1) = b_{\mathrm{si}}(d) + b_{\mathrm{cl}}(d) + \sum_{i=1}^{N} n_i(d)\, s_i(d-1)\,,$$

while its value at the end of day d is approximately

$$\pi(d) = b_{\mathrm{si}}(d)\,(1+\mu_{\mathrm{si}}) + b_{\mathrm{cl}}(d)\,(1+\mu_{\mathrm{cl}}) + \sum_{i=1}^{N} n_i(d)\, s_i(d)\,.$$

Two-Rate Model Portfolios: Returns

We can thereby approximate the return for day d as

$$\begin{aligned} r(d) &= \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} \\ &= \frac{b_{\mathrm{si}}(d)\,\mu_{\mathrm{si}}}{\pi(d-1)} + \frac{b_{\mathrm{cl}}(d)\,\mu_{\mathrm{cl}}}{\pi(d-1)} + \sum_{i=1}^{N} \frac{n_i(d)}{\pi(d-1)} \left(s_i(d) - s_i(d-1)\right) \\ &= \frac{b_{\mathrm{si}}(d)\,\mu_{\mathrm{si}}}{\pi(d-1)} + \frac{b_{\mathrm{cl}}(d)\,\mu_{\mathrm{cl}}}{\pi(d-1)} + \sum_{i=1}^{N} \frac{n_i(d)s_i(d-1)}{\pi(d-1)} \, \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} \\ &= f_{\mathrm{si}}\,\mu_{\mathrm{si}} + f_{\mathrm{cl}}\,\mu_{\mathrm{cl}} + \sum_{i=1}^{N} f_i\, r_i(d) = f_{\mathrm{si}}\,\mu_{\mathrm{si}} + f_{\mathrm{cl}}\,\mu_{\mathrm{cl}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\,. \end{aligned}$$

We thereby obtain the formula

$$r(d) = f_{\mathrm{si}}\,\mu_{\mathrm{si}} + f_{\mathrm{cl}}\,\mu_{\mathrm{cl}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\,.$$

Two-Rate Model Portfolios: Means and Variances

The portfolio return mean μ and variance v are then given by

$$\begin{aligned}
\mu &= \sum_{d=1}^{D} w(d)\left(f_{\mathrm{si}}\,\mu_{\mathrm{si}} + f_{\mathrm{cl}}\,\mu_{\mathrm{cl}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\right) \\
&= f_{\mathrm{si}}\,\mu_{\mathrm{si}} + f_{\mathrm{cl}}\,\mu_{\mathrm{cl}} + \mathbf{f}^{\mathrm{T}}\mathbf{m}\,, \\
v &= \sum_{d=1}^{D} w(d)\left(r(d)-\mu\right)^2 = \sum_{d=1}^{D} w(d)\left(\mathbf{f}^{\mathrm{T}}\mathbf{r}(d) - \mathbf{f}^{\mathrm{T}}\mathbf{m}\right)^2 \\
&= \mathbf{f}^{\mathrm{T}}\left(\sum_{d=1}^{D} w(d)\left(\mathbf{r}(d)-\mathbf{m}\right)\left(\mathbf{r}(d)-\mathbf{m}\right)^{\mathrm{T}}\right)\mathbf{f} = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,.
\end{aligned}$$

We thereby obtain the formulas

$$\mu = f_{\mathrm{si}}\,\mu_{\mathrm{si}} + f_{\mathrm{cl}}\,\mu_{\mathrm{cl}} + \mathbf{f}^{\mathrm{T}}\mathbf{m}\,, \qquad v = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,.$$

Two-Rate Model Portfolios: $\mathcal{M}_2$

The set of allocations of all assets in the Two-Rate Model is

$$\mathcal{M}_2 = \Big\{ (\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \ : \ \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{si}} + f_{\mathrm{cl}} = 1 \,,\ f_{\mathrm{si}} \geq 0 \,,\ f_{\mathrm{cl}} \leq 0 \Big\} \,. \tag{1.1}$$

The volatility and return mean for the allocation $(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathcal{M}_2$ are

$$\sigma(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \,,$$
$$\mu(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{si}} f_{\mathrm{si}} + \mu_{\mathrm{cl}} f_{\mathrm{cl}} \,.$$

The fundamental difference between this Two-Rate model and the One-Rate model treated earlier is the presence of the two inquality constraints in definition (1.1) of $\mathcal{M}_2$.

Two-Rate Model Frontiers: Introduction

The frontier for this model is found by seeking a minimizer of $\sigma(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}})$ over $(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathcal{M}_2$ while holding $\mu(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}})$ fixed. This becomes the constrained minimization problem

$$\min \left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathcal{M}_2\,,\ \sigma(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) = \mu \right\}. \qquad (2.2)$$

Because of the inequality constraints in definition (1.1) of $\mathcal{M}_2$, this problem can not be solved using the method of Lagrange multipliers. Rather, we will use capital allocation lines (CAL) to construct the set

$$\Sigma(\mathcal{M}_2) = \left\{ \big(\sigma(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}), \mu(\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}})\big) \;:\; (\mathbf{f}, f_{\mathrm{si}}, f_{\mathrm{cl}}) \in \mathcal{M}_2 \right\}. \qquad (2.3)$$

Two-Rate Model Frontiers: CAL Constructions

Our construction has two steps.

- We first enlarge $\mathcal{M}$ to $\mathcal{M}_{\rm si}$ by including safe investment allocations. More specifically, we start from $\Sigma(\mathcal{M})$ and use CAL to construct

$$\Sigma(\mathcal{M}_{\rm si}) = \Big\{ \big(\sigma(\mathbf{f}, f_{\rm si}), \mu(\mathbf{f}, f_{\rm si})\big) \,:\, (\mathbf{f}, f_{\rm si}) \in \mathcal{M}_{\rm si} \Big\}, \tag{2.4}$$

where

$$\mathcal{M}_{\rm si} = \Big\{ (\mathbf{f}, f_{\rm si}) \in \mathbb{R}^N \times \mathbb{R} \,:\, \mathbf{1}^{\rm T}\mathbf{f} + f_{\rm si} = 1\,,\ f_{\rm si} \geq 0 \Big\}. \tag{2.5}$$

- We then enlarge $\mathcal{M}_{\rm si}$ to $\mathcal{M}_2$ by including credit line allocations. More specifically, we start from $\Sigma(\mathcal{M}_{\rm si})$ and use CAL to construct $\Sigma(\mathcal{M}_2)$.

Two-Rate Model Frontiers: $\Sigma(\mathcal{M}_{\rm si})$

For every $(\tilde{\sigma}, \tilde{\mu}) \in \Sigma(\mathcal{M})$ the CAL construction yields points $(\sigma, \mu) \in \Sigma(\mathcal{M}_{\rm si})$ given by

$$\sigma = |\phi|\,\tilde{\sigma}\,, \qquad \mu = \phi\,\tilde{\mu} + (1-\phi)\,\mu_{\rm si}\,,$$

for every $\phi \in (-\infty, 1]$. The condition $\phi \leq 1$ insures that there is no short position in the safe investment.

There are three cases to consider:

- $\mu_{\rm si} = \mu_{\rm mv}$,
- $\mu_{\rm si} < \mu_{\rm mv}$,
- $\mu_{\rm si} > \mu_{\rm mv}$.

The first case is easy because the Tobin frontier allocation holds a long position in the safe investment, and thereby is in $\mathcal{M}_{\rm si}$.

Two-Rate Model Frontiers: $\mu_{\mathrm{si}} \neq \mu_{\mathrm{mv}}$

When $\mu_{\mathrm{si}} \neq \mu_{\mathrm{mv}}$ the tangency point associated with μ_{si} is $(\sigma_{\mathrm{st}}, \mu_{\mathrm{st}})$ where

$$\sigma_{\mathrm{st}} = \frac{|\mu_{\mathrm{st}} - \mu_{\mathrm{si}}|}{\nu_{\mathrm{si}}}, \qquad \mu_{\mathrm{st}} = \mu_{\mathrm{si}} + \frac{\nu_{\mathrm{si}}^2\, \sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}},$$

$$\nu_{\mathrm{si}}^2 = (\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1}) = \nu_{\mathrm{mv}}^2 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}{\sigma_{\mathrm{mv}}}\right)^2.$$

- When $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}}$ this point is on the efficient Markowitz frontier.
- When $\mu_{\mathrm{si}} > \mu_{\mathrm{mv}}$ this point is on the inefficient Markowitz frontier.

The tangent portfolio allocation associated with μ_{si} is

$$\mathbf{f}_{\mathrm{st}} = \frac{\mu_{\mathrm{st}} - \mu_{\mathrm{si}}}{\nu_{\mathrm{si}}^2}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1}) = \frac{\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1}).$$

Two-Rate Model Frontiers: $\mu_{\text{si}} \neq \mu_{\text{mv}}$

The Tobin frontier and frontier allocation associated with μ_{si} are given by

$$\sigma_{\text{sf}}(\mu) = \frac{|\mu - \mu_{\text{si}}|}{\nu_{\text{si}}}, \qquad \mathbf{f}_{\text{sf}}(\mu) = \frac{\mu - \mu_{\text{si}}}{\nu_{\text{si}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\text{si}}\mathbf{1}).$$

The allocation f_{si} in the safe investment of $\mathbf{f}_{\text{sf}}(\mu)$ is

$$f_{\text{si}} = 1 - \mathbf{1}^{\text{T}}\mathbf{f}_{\text{sf}}(\mu) = 1 - \frac{(\mu - \mu_{\text{si}})(\mu_{\text{mv}} - \mu_{\text{si}})}{\nu_{\text{si}}^2 \, \sigma_{\text{mv}}^2}.$$

- When $\mu_{\text{si}} < \mu_{\text{mv}}$ the constraint $f_{\text{si}} \geq 0$ is equivalent to $\mu \leq \mu_{\text{st}}$. Therefore when $\mu \leq \mu_{\text{st}}$ this Tobin frontier allocation does not hold a short position in the safe investment, and thereby is in $\mathcal{M}_{\text{si}}$.
- When $\mu_{\text{si}} > \mu_{\text{mv}}$ the constraint $f_{\text{si}} \geq 0$ is equivalent to $\mu \geq \mu_{\text{st}}$. Therefore when $\mu \geq \mu_{\text{st}}$ this Tobin frontier allocation does not hold a short position in the safe investment, and thereby is in $\mathcal{M}_{\text{si}}$.

Two-Rate Model Frontiers: $\mu_{\rm si} < \mu_{\rm mv}$

When $\mu_{\rm si} < \mu_{\rm mv}$ and $\mu > \mu_{\rm st}$ the CAL construction only yields portfolios with $\sigma \geq \sigma_{\rm mf}(\mu)$.

Therefore when $\mu_{\rm si} < \mu_{\rm mv}$ the frontier of $\Sigma(\mathcal{M}_{\rm si})$ is $\sigma = \sigma_{\rm f}(\mu)$ where

$$\sigma_{\rm f}(\mu) = \begin{cases} \sigma_{\rm sf}(\mu) & \text{for } \mu \leq \mu_{\rm st} \,, \\ \sigma_{\rm mf}(\mu) & \text{for } \mu > \mu_{\rm st} \,. \end{cases}$$

Because $\mu_{\rm si} < \mu_{\rm mv} < \mu_{\rm st}$ the efficient frontier of $\Sigma(\mathcal{M}_{\rm si})$ is $\mu = \mu_{\rm ef}(\sigma)$ where

$$\mu_{\rm ef}(\sigma) = \begin{cases} \mu_{\rm si} + \nu_{\rm si}\sigma & \text{for } \sigma \in [0, \sigma_{\rm st}) \,, \\ \mu_{\rm emf}(\sigma) & \text{for } \sigma \in [\sigma_{\rm st}, \infty) \,. \end{cases}$$

Two-Rate Model Frontiers: $\mu_{\rm si} > \mu_{\rm mv}$

When $\mu_{\rm si} > \mu_{\rm mv}$ and $\mu < \mu_{\rm st}$ the CAL construction only yields portfolios with $\sigma \geq \sigma_{\rm mf}(\mu)$.

Therefore when $\mu_{\rm si} > \mu_{\rm mv}$ the frontier of $\Sigma(\mathcal{M}_{\rm si})$ is $\sigma = \sigma_{\rm f}(\mu)$ where

$$\sigma_{\rm f}(\mu) = \begin{cases} \sigma_{\rm mf}(\mu) & \text{for } \mu < \mu_{\rm st} \,, \\ \sigma_{\rm sf}(\mu) & \text{for } \mu \geq \mu_{\rm st} \,. \end{cases}$$

Because $\mu_{\rm st} < \mu_{\rm mv} < \mu_{\rm si}$ the efficient frontier of $\Sigma(\mathcal{M}_{\rm si})$ is $\mu = \mu_{\rm ef}(\sigma)$ where

$$\mu_{\rm ef}(\sigma) = \mu_{\rm si} + \nu_{\rm si}\sigma \qquad \text{for } \sigma \in [0, \infty) \,.$$

Two-Rate Model Frontiers: $\Sigma(\mathcal{M}_{\mathrm{si}})$ Summary

In summary, the efficient frontier of $\Sigma(\mathcal{M}_{\mathrm{si}})$ has two cases.

- If $\mu_{\mathrm{mv}} \leq \mu_{\mathrm{si}}$ then the efficient frontier of $\Sigma(\mathcal{M}_{\mathrm{si}})$ is given by $\mu = \mu_{\mathrm{ef}}(\sigma)$ where

$$\mu_{\mathrm{ef}}(\sigma) = \mu_{\mathrm{si}} + \nu_{\mathrm{si}}\,\sigma \qquad \text{for } \sigma \in [0, \infty)\,. \tag{2.6}$$

- If $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}}$ then the efficient frontier of $\Sigma(\mathcal{M}_{\mathrm{si}})$ is given by $\mu = \mu_{\mathrm{ef}}(\sigma)$ where

$$\mu_{\mathrm{ef}}(\sigma) = \begin{cases} \mu_{\mathrm{si}} + \nu_{\mathrm{si}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathrm{st}})\,, \\ \mu_{\mathrm{emf}}(\sigma) & \text{for } \sigma \in [\sigma_{\mathrm{st}}, \infty)\,. \end{cases} \tag{2.7}$$

This is all that we need to go onto the second step of our construction.

Two-Rate Model Frontiers: $\Sigma(\mathcal{C}_2)$

For every $(\tilde{\mathbf{f}}, \tilde{f}_{\rm si}) \in \mathcal{M}_{\rm si}$ and every $\phi \in [1, \infty)$ the capital allocation line is

$$(\mathbf{f}, f_{\rm si}, f_{\rm cl}) = \left(\phi\, \tilde{\mathbf{f}}\,,\ \phi\, \tilde{f}_{\rm si}\,,\ 1 - \phi\right).$$

The condition $\phi \geq 1$ insures there is no long position in the credit line. Verifying that $(\mathbf{f}, f_{\rm si}, f_{\rm cl}) \in \mathcal{M}_2$ is easy.

This construction yields the point (σ, μ) in the $\sigma\mu$-plane given by

$$\sigma = \phi\, \tilde{\sigma}\,, \qquad \mu = \phi\, \tilde{\mu} + (1 - \phi)\, \mu_{\rm cl}\,, \tag{2.8}$$

where $(\tilde{\sigma}, \tilde{\mu}) \in \Sigma(\mathcal{M}_{\rm si})$ is given by

$$\tilde{\sigma} = \sqrt{\tilde{\mathbf{f}}^{\rm T}\mathbf{V}\tilde{\mathbf{f}}}\,, \qquad \tilde{\mu} = \mathbf{m}^{\rm T}\tilde{\mathbf{f}} + \mu_{\rm si}\tilde{f}_{\rm si}\,.$$

Because $(\mathbf{f}, f_{\rm si}, f_{\rm cl}) \in \mathcal{C}_2 \subset \mathcal{M}_2$ we have $(\sigma, \mu) \in \Sigma(\mathcal{C}_2) \subset \Sigma(\mathcal{M}_2)$.

Two-Rate Model Frontiers: $\Sigma(\mathcal{C}_2)$

For each $(\tilde{\sigma}, \tilde{\mu}) \in \Sigma(\mathcal{M}_{\mathrm{si}})$ the construction (2.8) is a ray on the line through the points $(0, \mu_{\mathrm{cl}})$ and $(\tilde{\sigma}, \tilde{\mu})$ that starts at $(\tilde{\sigma}, \tilde{\mu})$ and moves away from $(0, \mu_{\mathrm{cl}})$.

Two-Rate Model Frontiers: $\mu_{\text{cl}} \neq \mu_{\text{mv}}$

When $\mu_{\text{cl}} \neq \mu_{\text{mv}}$ the tangency point associated with μ_{cl} is $(\sigma_{\text{ct}}, \mu_{\text{ct}})$ where

$$\sigma_{\text{ct}} = \frac{|\mu_{\text{ct}} - \mu_{\text{cl}}|}{\nu_{\text{cl}}}, \qquad \mu_{\text{ct}} = \mu_{\text{cl}} + \frac{\nu_{\text{cl}}^2 \, \sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{cl}}},$$

$$\nu_{\text{cl}}^2 = (\mathbf{m} - \mu_{\text{cl}}\mathbf{1})^{\text{T}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\text{cl}}\mathbf{1}) = \nu_{\text{mv}}^2 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{cl}}}{\sigma_{\text{mv}}} \right)^2 .$$

- When $\mu_{\text{cl}} < \mu_{\text{mv}}$ this point is on the efficient Markowitz frontier.
- When $\mu_{\text{cl}} > \mu_{\text{mv}}$ this point is on the inefficient Markowitz frontier.

The tangent portfolio allocation associated with μ_{cl} is

$$\mathbf{f}_{\text{ct}} = \frac{\mu_{\text{ct}} - \mu_{\text{cl}}}{\nu_{\text{cl}}^2} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\text{cl}}\mathbf{1}) = \frac{\sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{cl}}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\text{cl}}\mathbf{1}) .$$

Two-Rate Model Frontiers: $\Sigma(\mathcal{M}_2)$ Summary

In summary, the efficient frontier of $\Sigma(\mathcal{M}_2)$ has three cases.

- If $\mu_{\mathrm{mv}} \leq \mu_{\mathrm{si}} < \mu_{\mathrm{cl}}$ then there is no tangency point on the efficient Markowitz frontier and the efficient frontier of $\Sigma(\mathcal{M}_2)$ is given by $\mu = \mu_{\mathrm{ef}}(\sigma)$ where

$$\mu_{\mathrm{ef}}(\sigma) = \mu_{\mathrm{si}} + \nu_{\mathrm{si}}\,\sigma \qquad \text{for } \sigma \in [0,\infty)\,. \tag{2.9}$$

- If $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}} \leq \mu_{\mathrm{cl}}$ then $(\sigma_{\mathrm{st}}, \mu_{\mathrm{st}})$ is the only tangency point on the efficient Markowitz frontier and the efficient frontier of $\Sigma(\mathcal{M}_2)$ is given by $\mu = \mu_{\mathrm{ef}}(\sigma)$ where

$$\mu_{\mathrm{ef}}(\sigma) = \begin{cases} \mu_{\mathrm{si}} + \nu_{\mathrm{si}}\,\sigma & \text{for } \sigma \in [0,\sigma_{\mathrm{st}})\,, \\ \mu_{\mathrm{emf}}(\sigma) & \text{for } \sigma \in [\sigma_{\mathrm{st}},\infty)\,. \end{cases} \tag{2.10}$$

Two-Rate Model Frontiers: $\Sigma(\mathcal{M}_2)$ Summary

- If $\mu_{\text{si}} < \mu_{\text{cl}} < \mu_{\text{mv}}$ then both $(\sigma_{\text{st}}, \mu_{\text{st}})$ and $(\sigma_{\text{ct}}, \mu_{\text{ct}})$ are tangencey points on the efficient Markowitz frontier and the efficient frontier of $\Sigma(\mathcal{M}_2)$ is given by $\mu = \mu_{\text{ef}}(\sigma)$ where

$$\mu_{\text{ef}}(\sigma) = \begin{cases} \mu_{\text{si}} + \nu_{\text{si}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\text{st}})\,, \\ \mu_{\text{emf}}(\sigma) & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{ct}})\,, \\ \mu_{\text{cl}} + \nu_{\text{cl}}\,\sigma & \text{for } \sigma \in [\sigma_{\text{ct}}, \infty)\,. \end{cases} \tag{2.11}$$

In all three cases

$$\Sigma(\mathcal{M}_2) = \Big\{ (\sigma, \mu) \in [0, \infty) \times \mathbb{R} \,:\, \mu \le \mu_{\text{ef}}(\sigma) \Big\}\,. \tag{2.12}$$

Two-Rate Model Frontiers: Efficient Allocations Summary

The efficient portfolio allocations of $\mathcal{M}_2$ also has three cases.

- If $\mu_{\rm mv} \leq \mu_{\rm si} < \mu_{\rm cl}$ then there is no tangency point on the efficient Markowitz frontier and the efficient portfolio allocations of $\mathcal{M}_2$ are

$$\left(-\frac{\sigma}{\sigma_{\rm st}} \mathbf{f}_{\rm st} \,,\, 1 + \frac{\sigma}{\sigma_{\rm st}} \,,\, 0 \right) \qquad \text{for } \sigma \in [0, \infty) \,. \tag{2.13}$$

- If $\mu_{\rm si} < \mu_{\rm mv} \leq \mu_{\rm cl}$ then $(\sigma_{\rm st}, \mu_{\rm st})$ is the only tangency point on the efficient Markowitz frontier and the efficient portfolio allocations of $\mathcal{M}_2$ are

$$\begin{aligned} &\left(\frac{\sigma}{\sigma_{\rm st}} \mathbf{f}_{\rm st} \,,\, 1 - \frac{\sigma}{\sigma_{\rm st}} \,,\, 0 \right) && \text{for } \sigma \in [0, \sigma_{\rm st}) \,, \\ &\left(\mathbf{f}_{\rm emf}(\sigma) \,,\, 0 \,,\, 0 \right) && \text{for } \sigma \in [\sigma_{\rm st}, \infty) \,, \end{aligned} \tag{2.14}$$

Two-Rate Model Frontiers: Efficient Allocations Summary

- If $\mu_{\rm si} < \mu_{\rm cl} < \mu_{\rm mv}$ then both $(\sigma_{\rm st}, \mu_{\rm st})$ and $(\sigma_{\rm ct}, \mu_{\rm ct})$ are tangencey points on the efficient Markowitz frontier and the efficient portfolio allocations of $\mathcal{M}_2$ are

$$\begin{aligned}
&\left(\frac{\sigma}{\sigma_{\rm st}}\mathbf{f}_{\rm st}\,,\ 1-\frac{\sigma}{\sigma_{\rm st}}\,,\ 0\right) && \text{for } \sigma \in [0, \sigma_{\rm st})\,, \\
&\left(\ \mathbf{f}_{\rm emf}(\sigma)\ ,\ 0\ ,\ 0\ \right) && \text{for } \sigma \in [\sigma_{\rm st}, \sigma_{\rm ct}]\,, \\
&\left(\frac{\sigma}{\sigma_{\rm ct}}\mathbf{f}_{\rm ct}\,,\ 0\,,\ 1-\frac{\sigma}{\sigma_{\rm ct}}\right) && \text{for } \sigma \in (\sigma_{\rm ct}, \infty)\,.
\end{aligned} \tag{2.15}$$