# Portfolios that Contain Risky Assets
# 3.1. Portfolios with Risk-Free Assets

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# Portfolios that Contain Risky Assets
# Part I: Portfolio Models

1. Preliminary Topics
2. Markowitz Portfolio Model
3. Models for Portfolios with Risk-Free Assets
4. Models for Long Portfolios
5. Models for Limited-Leverage Portfolios

# Portfolios that Contain Risky Assets
# Part I: Portfolio Models

## 3. Models for Portfolios with Risk-Free Assets

# 3.1. Portfolios with Risk-Free Assets

# Risk-Free Assets: Safe Investments and Credit Lines

Until now we have considered portfolios that contain only risky assets. We now consider two kinds of *risk-free* assets (assets that have no volatility associated with them) that can play a major role in portfolio management.

The first is a *safe investment* that pays dividends at a prescribed annual rate $\mu_{\rm si}^{\rm an}$. This can be an FDIC insured bank account, or safe securities such as US Treasury Bills, Notes, or Bonds. (U.S. Treasury Bills are used most commonly.) We can only hold a long position in such an asset.

The second is a *credit line* from which you can borrow at a prescribed annual rate $\mu_{\rm cl}^{\rm an}$ up to your credit limit. Such a credit line should require you to put up assets like real estate or part of your portfolio (a *margin*) as collateral from which the borrowed money can be recovered if need be. We can only hold a short position in such an asset.

# Risk-Free Assets: Relationship Between the Them

We will assume that $\mu_{\rm cl}^{\rm an} \geq \mu_{\rm si}^{\rm an}$, because otherwise investors would make money by borrowing at rate $\mu_{\rm cl}^{\rm an}$ and invest it at the greater yield rate $\mu_{\rm si}^{\rm an}$. (Here we are again neglecting transaction costs.) Because free money does not sit around for long, market forces would quickly adjust the returns so that $\mu_{\rm cl}^{\rm an} \geq \mu_{\rm si}^{\rm an}$.

We will employ two kinds of models.

- Our One-Rate Models set $\mu_{\rm cl}^{\rm an} = \mu_{\rm si}^{\rm an}$.
- Our Two-Rate Models set $\mu_{\rm cl}^{\rm an} > \mu_{\rm si}^{\rm an}$.

In practice, $\mu_{\rm cl}^{\rm an}$ is about three percentage points higher than $\mu_{\rm si}^{\rm an}$.

**Remark.** When $\mu_{\rm cl}^{\rm an} > \mu_{\rm si}^{\rm an}$ it is not Markowitz efficient for a portfolio to hold positions in both the safe investment and the credit line. To do so would be borrowing at rate $\mu_{\rm cl}^{\rm an}$ in order to invest at the lesser yield $\mu_{\rm si}^{\rm an}$. While there can be cash-flow management reasons for holding such a position for a short time, it is not a smart long term position.

## Risk-Free Assets: Getting Risk-Free Annual Rates

The first step is to find the annual rates $\mu_{\mathrm{si}}^{\mathrm{an}}$ and $\mu_{\mathrm{cl}}^{\mathrm{an}}$ that are available to us at the start of the period over which we are investing. If we are using a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ to guide us and are planning to invest over the year following this history (say, the year that starts on day $D+1$) then we can select $\mu_{\mathrm{si}}^{\mathrm{an}}$ and $\mu_{\mathrm{cl}}^{\mathrm{an}}$ as follows.

- Let $\mu_{\mathrm{si}}^{\mathrm{an}}$ be the rate for the 13 week U.S. Treasury Bill on day $D$. (Return histories for U.S. Treasury Bills, Notes, and Bonds can be found on the U.S. Treasury website. Use the "coupon" rate. Because these rates are given in percentages, they must be divided by 100.)
- Our One-Rate Models set $\mu_{\mathrm{cl}}^{\mathrm{an}} = \mu_{\mathrm{si}}^{\mathrm{an}}$.
- Our Two-Rate Models set $\mu_{\mathrm{cl}}^{\mathrm{an}} = \mu_{\mathrm{si}}^{\mathrm{an}} + .03$. (Recall that .03 is 3%. While many sources of credit are available, the best run about 3% above the Treasury Bill rate.)

# Risk-Free Assets: Conversion to Daily Rates

The next step is to convert the annual rate $\mu_{\mathrm{rf}}^{\mathrm{an}}$ of each risk-free asset to a return $\mu_{\mathrm{rf}}$ per trading day. If there are $D_{\mathrm{y}}$ trading days in a year then we can approximate $\mu_{\mathrm{rf}}$ by the relation

$$(1+\mu_{\mathrm{rf}}^{\mathrm{an}})^{\frac{1}{D_{\mathrm{y}}}} = 1 + \mu_{\mathrm{rf}} \,.$$

Here we take the growth of the risk-free asset to be the same between any two successive trading days — namely, the factor $(1+\mu_{\mathrm{rf}})$. Whether this is true or not depends upon the asset, but for most risk-free assets it depends upon the number of calendar days between successive trading days.

Because $\frac{1}{D_{\mathrm{y}}}$ and $\mu_{\mathrm{rf}}^{\mathrm{an}}$ are small, we can make the further approximation

$$(1+\mu_{\mathrm{rf}}^{\mathrm{an}})^{\frac{1}{D_{\mathrm{y}}}} \approx 1 + \frac{1}{D_{\mathrm{y}}}\mu_{\mathrm{rf}}^{\mathrm{an}} \,,$$

whereby $\mu_{\mathrm{rf}} \approx \frac{1}{D_{\mathrm{y}}}\mu_{\mathrm{rf}}^{\mathrm{an}}$.

# Risk-Free Assets: Trading Days Per Year

There are several ways to choose $D_{\mathrm{y}}$. Because there are typically 252 trading days in a year, we might simply set $D_{\mathrm{y}} = 252$. However, it is better to set $D_{\mathrm{y}} = D/h$ when our history is over a multiple $h$ of a full year. For example, if our history is over two full years and $D = 505$ then we set $D_{\mathrm{y}} = 252.5$. Typical values of $h$ are $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 5 and 10.

**Remark.** The value $D_{\mathrm{y}} = 252$ is correct when there are 365 calendar days, 104 weekend days, and 9 holidays, none of which fall on a weekend. For such a year we have $252 = 365 - 104 - 9$. Of course, leap years have 366 days, some years have 105 or 106 weekend days, and holidays fall on weekends in some years. Some years will have unscheduled closures due to bad weather or some other emergency. The day after Thanksgiving (always a Friday) plus July 3 and December 24 when they do not fall on a weekend will be half-days. We will treat half-days the same as other trading days.

# One-Rate Model Portfolios: Allocations

First we extend the notion of Markowitz portfolios to portfolios that might include a single risk-free asset with return $\mu_{\mathrm{rf}}$ where $\mu_{\mathrm{rf}} = \mu_{\mathrm{si}} = \mu_{\mathrm{cl}}$.

Let $b_{\mathrm{rf}}(d)$ denote the balance in the risk-free asset at the start of day $d$. This position is long when $b_{\mathrm{rf}}(d) > 0$ and short when $b_{\mathrm{rf}}(d) < 0$.

A *Markowitz portfolio* containing this risk-free asset and $N$ risky assets is uniquely determined by real numbers $f_{\mathrm{rf}}$ and $\{f_i\}_{i=1}^N$ that satisfy

$$f_{\mathrm{rf}} + \sum_{i=1}^{N} f_i = 1\,.$$

Here $f_{\mathrm{rf}}$ is the allocation of the portfolio in the risk-free asset while $f_i$ is the allocation of the portfolio in the $i^{\mathrm{th}}$ risky asset.

# One-Rate Model Portfolios: Values

The portfolio is rebalanced at the start of each day so that

$$\frac{b_{\mathrm{rf}}(d)}{\pi(d-1)} = f_{\mathrm{rf}}\,, \qquad \frac{n_i(d)\, s_i(d-1)}{\pi(d-1)} = f_i \qquad \text{for } i = 1, \cdots, N\,.$$

Its value at the start of day $d$ is

$$\pi(d-1) = b_{\mathrm{rf}}(d) + \sum_{i=1}^{N} n_i(d)\, s_i(d-1)\,,$$

while its value at the end of day $d$ is approximately

$$\pi(d) = b_{\mathrm{rf}}(d)\,(1+\mu_{\mathrm{rf}}) + \sum_{i=1}^{N} n_i(d)\, s_i(d)\,.$$

The approximation arises here because the value of risk-free assets generally depends upon calendar days rather than upon trading days. However, the error introduced by this is tiny.

# One-Rate Model Portfolios: Returns

We can thereby approximate the return for day $d$ as

$$\begin{aligned} r(d) &= \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} \\ &= \frac{b_{\mathrm{rf}}(d)\,\mu_{\mathrm{rf}}}{\pi(d-1)} + \sum_{i=1}^{N} \frac{n_i(d)\big(s_i(d) - s_i(d-1)\big)}{\pi(d-1)} \\ &= \frac{b_{\mathrm{rf}}(d)\,\mu_{\mathrm{rf}}}{\pi(d-1)} + \sum_{i=1}^{N} \frac{n_i(d) s_i(d-1)}{\pi(d-1)}\,\frac{s_i(d) - s_i(d-1)}{s_i(d-1)} \\ &= f_{\mathrm{rf}}\,\mu_{\mathrm{rf}} + \sum_{i=1}^{N} f_i\, r_i(d) = f_{\mathrm{rf}}\,\mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\,. \end{aligned}$$

We thereby obtain the formula

$$r(d) = f_{\mathrm{rf}}\,\mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\,.$$

# One-Rate Model Portfolios: Mean and Variance

The portfolio return mean $\mu$ and variance $v$ are then given by

$$\mu = \sum_{d=1}^{D} w(d)\, r(d) = \sum_{d=1}^{D} w(d) \left( f_{\mathrm{rf}}\, \mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}} \mathbf{r}(d) \right)$$

$$= f_{\mathrm{rf}}\, \mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}} \left( \sum_{d=1}^{D} w(d)\, \mathbf{r}(d) \right) = f_{\mathrm{rf}}\, \mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}} \mathbf{m}\,,$$

$$v = \sum_{d=1}^{D} w(d) \left( r(d) - \mu \right)^2 = \sum_{d=1}^{D} w(d) \left( \mathbf{f}^{\mathrm{T}} \mathbf{r}(d) - \mathbf{f}^{\mathrm{T}} \mathbf{m} \right)^2$$

$$= \mathbf{f}^{\mathrm{T}} \left( \sum_{d=1}^{D} w(d) \left( \mathbf{r}(d) - \mathbf{m} \right) \left( \mathbf{r}(d) - \mathbf{m} \right)^{\mathrm{T}} \right) \mathbf{f} = \mathbf{f}^{\mathrm{T}} \mathbf{V} \mathbf{f}\,.$$

We thereby obtain the formulas

$$\mu = \mathbf{m}^{\mathrm{T}} \mathbf{f} + \mu_{\mathrm{rf}} \left( 1 - \mathbf{1}^{\mathrm{T}} \mathbf{f} \right), \qquad v = \mathbf{f}^{\mathrm{T}} \mathbf{V} \mathbf{f}\,.$$

# One-Rate Model Portfolios: $\mathcal{M}_1$ and $\mathcal{M}_+$

The set of allocations of all assets in the One-Rate Model is

$$\mathcal{M}_1 = \left\{ (\mathbf{f}, f_{\mathrm{rf}}) \in \mathbb{R}^N \times \mathbb{R} \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} + f_{\mathrm{rf}} = 1 \right\} . \qquad (2.1)$$

The volatility and return mean for the portfolio allocation $(\mathbf{f}, f_{\mathrm{rf}}) \in \mathcal{M}_1$ are

$$\sigma(\mathbf{f}, f_{\mathrm{rf}}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \,, \qquad \mu(\mathbf{f}, f_{\mathrm{rf}}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{rf}} f_{\mathrm{rf}} \,.$$

The set of allocations of all risky assets is $\mathcal{M}_+ = \mathbb{R}^N$. The volatility and return mean for the portfolio with allocation $\mathbf{f} \in \mathcal{M}_+$ are

$$\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \,, \qquad \mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{rf}}\left(1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}\right) . \qquad (2.2)$$

Here we have used the constraint in (2.1) to eliminate $f_{\mathrm{rf}}$.

**Remark.** Notice that $\mathcal{M} \subset \mathcal{M}_+$ and that when the expressions (2.2) for $\sigma(\mathbf{f})$ and $\mu(\mathbf{f})$ are restricted to $\mathcal{M}$ they agree with our previous expressions.

# One-Rate Model Frontiers: Introduction

Risk-free assets can be included in the original Markowitz theory as a singular limit that lets some eigenvalues of $\mathbf{V}$ go to 0. However, the important role of such assets was first clarified by James Tobin in 1958. He showed that adding a risk-free asset to a portfolio of risky assets can increase its efficiency. We now illustrate this with the One-Rate Model.

The frontier for this model is found by seeking a minimizer of $\sigma(\mathbf{f})$ over $\mathbf{f} \in \mathcal{M}_+$ while holding $\mu(\mathbf{f})$ fixed. Because $\mathcal{M}_+ = \mathbb{R}^N$ this becomes the constrained minimization problem

$$\min_{\mathbf{f}\in\mathbb{R}^N} \left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \ : \ (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{rf}} = \mu \right\}. \tag{3.3}$$

Because its only constraint is an equality constraint, this problem can be solved by the method of Lagrange multipliers

# One-Rate Model Frontiers: Lagrange

Because there is one equality constraint, we need just one Lagrange multiplier $\beta$. Then the Lagrangian is

$$\Phi(\mathbf{f}, \beta) = \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \beta \left( (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f} - (\mu - \mu_{\mathrm{rf}}) \right) .$$

By setting the partial derivatives of $\Phi(\mathbf{f}, \beta)$ equal to zero we obtain

$$\begin{aligned} \mathbf{0} &= \nabla_{\mathbf{f}}\Phi(\mathbf{f}, \beta) = \mathbf{V}\mathbf{f} - \beta\,(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,, \\ 0 &= \partial_{\beta}\Phi(\mathbf{f}, \beta) = -(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f} + (\mu - \mu_{\mathrm{rf}})\,. \end{aligned}$$

Because $\mathbf{V}$ is positive definite we can solve the first equation for $\mathbf{f}$ as

$$\mathbf{f} = \beta\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,.$$

# One-Rate Model Frontiers: Minimizer

By setting this expression for **f** into the second equation we obtain

$$(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,\beta = \mu - \mu_{\mathrm{rf}}\,,$$

whereby

$$\beta = \frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\,,$$

where $\nu_{\mathrm{rf}} > 0$ is determined by

$$\nu_{\mathrm{rf}}^2 = (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,. \tag{3.4}$$

Hence, for each $\mu \in \mathbb{R}$ the unique minimizer is given by

$$\mathbf{f}(\mu) = \frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,. \tag{3.5}$$

# One-Rate Model Frontiers: Minimum Volatility

The associated minimum volatility $\sigma$ satisfies

$$\sigma^2 = \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) = \left(\frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\right)^2 (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})$$
$$= \left(\frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\right)^2 \nu_{\mathrm{rf}}^2 = \left(\frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}}\right)^2 .$$

Upon solving this for $\mu$ we obtain

$$\mu = \mu_{\mathrm{rf}} \pm \nu_{\mathrm{rf}}\,\sigma \qquad \text{for } \sigma \geq 0 .$$

These equations describe two half-lines in the $\sigma\mu$-plane that share the $\mu$-intercept $(0, \mu_{\mathrm{rf}})$ and have slopes $\pm\nu_{\mathrm{rf}}$.

# One-Rate Model Frontiers: Tobin Frontier

To honor his contribution, we will call these half-lines the *Tobin frontier*. By (3.5) the *Tobin frontier allocations* are given by $\mathbf{f} = \mathbf{f}_{\mathrm{tf}}(\mu)$, where

$$\mathbf{f}_{\mathrm{tf}}(\mu) = \frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}) \qquad \text{for } \mu \in \mathbb{R}\,. \tag{3.6}$$

The *Tobin frontier* is given by $\sigma = \sigma_{\mathrm{tf}}(\mu)$, where

$$\sigma_{\mathrm{tf}}(\mu) = \frac{|\mu - \mu_{\mathrm{rf}}|}{\nu_{\mathrm{rf}}} \qquad \text{for } \mu \in \mathbb{R}\,. \tag{3.7}$$

The *efficient Tobin frontier* is given by $\mu = \mu_{\mathrm{etf}}(\sigma)$, where

$$\mu_{\mathrm{etf}}(\sigma) = \mu_{\mathrm{rf}} + \nu_{\mathrm{rf}}\,\sigma \qquad \text{for } \sigma \in [0, \infty)\,. \tag{3.8}$$

The *inefficient Tobin frontier* is given by $\mu = \mu_{\mathrm{itf}}(\sigma)\mu$, where

$$\mu_{\mathrm{itf}}(\sigma) = \mu_{\mathrm{rf}} - \nu_{\mathrm{rf}}\,\sigma \qquad \text{for } \sigma \in (0, \infty)\,. \tag{3.9}$$

# One-Rate Model Frontiers: Frontier Parameters

The Tobin frontier (3.7) is completely described by the so-called *Tobin frontier parameters*, $\mu_{\mathrm{rf}}$ and $\nu_{\mathrm{rf}}$. Because

$$\mathbf{m} - \nu_{\mathrm{rf}}\mathbf{1} = (\mathbf{m} - \nu_{\mathrm{mv}}\mathbf{1}) - (\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})\mathbf{1}\,,$$

and $(\mathbf{m} - \nu_{\mathrm{mv}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} = 0$, we see that

$$\begin{aligned}\nu_{\mathrm{rf}}^2 &= (\mathbf{m} - \nu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \nu_{\mathrm{rf}}\mathbf{1}) \\ &= (\mathbf{m} - \nu_{\mathrm{mv}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \nu_{\mathrm{mv}}\mathbf{1}) + (\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})^2\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} \\ &= \nu_{\mathrm{mv}}^2 + (\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})^2\frac{1}{\sigma_{\mathrm{mv}}^2}\,.\end{aligned}$$

Therefore the Tobin frontier parameters and the Markowitz frontier parameters satisfy the *frontier parameter relation*

$$\nu_{\mathrm{rf}}^2 = \nu_{\mathrm{mv}}^2 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}}\right)^2\,. \tag{3.10}$$

# One-Rate Model Frontiers: Orthogonal Projection

The Tobin frontier allocations (3.6) can be expressed as

$$\mathbf{f}_{\mathrm{tf}}(\mu) = (\mu - \mu_{\mathrm{rf}})\, \mathbf{g}_{\mathrm{rf}}\,, \qquad \text{where} \quad \mathbf{g}_{\mathrm{rf}} = \frac{1}{\nu_{\mathrm{rf}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,.$$

We have

$$\|\mathbf{g}_{\mathrm{rf}}\|_{\mathbf{V}}^2 = \frac{1}{\nu_{\mathrm{rf}}^4} (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}) = \frac{1}{\nu_{\mathrm{rf}}^2}\,.$$

For every $\mathbf{f} \in \mathcal{M}_+$ if we set $\mu(\mathbf{f}) = \mu_{\mathrm{rf}} + (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f}$ then

$$(\mathbf{g}_{\mathrm{rf}} \,|\, \mathbf{f})_{\mathbf{V}} = \frac{1}{\nu_{\mathrm{rf}}^2} (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f} = \frac{\mu(\mathbf{f}) - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\,,$$

and we see that the $\mathbf{V}$-othogonal projection of $\mathbf{f}$ onto $\mathrm{Span}\{\mathbf{g}_{\mathrm{rf}}\}$ is

$$\mathbf{P}\mathbf{f} = \frac{(\mathbf{g}_{\mathrm{rf}} \,|\, \mathbf{f})_{\mathbf{V}}}{\|\mathbf{g}_{\mathrm{rf}}\|_{\mathbf{V}}^2}\, \mathbf{g}_{\mathrm{rf}} = (\mu(\mathbf{f}) - \mu_{\mathrm{rf}})\, \mathbf{g}_{\mathrm{rf}} = \mathbf{f}_{\mathrm{tf}}(\mu(\mathbf{f}))\,.$$

# One-Rate Model Frontiers: When is $\mathbf{f}_{\mathrm{tf}}(\mu) \in \mathcal{M}$?

Because

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{\mathrm{tf}}(\mu) = \frac{\mu - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^2}\,\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}) = \frac{(\mu - \mu_{\mathrm{rf}})(\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})}{\nu_{\mathrm{rf}}^2\,\sigma_{\mathrm{mv}}^2}\,,$$

we see that $\mathbf{f}_{\mathrm{tf}}(\mu) \in \mathcal{M}$ if and only if

$$\frac{(\mu - \mu_{\mathrm{rf}})(\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}})}{\nu_{\mathrm{rf}}^2\,\sigma_{\mathrm{mv}}^2} = 1\,.$$

If $\mu_{\mathrm{rf}} = \mu_{\mathrm{mv}}$ this has no solution. If $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ it has the unique solution $\mu = \mu_{\mathrm{tg}}$ where

$$\mu_{\mathrm{tg}} = \mu_{\mathrm{rf}} + \frac{\nu_{\mathrm{rf}}^2\,\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}\,. \tag{3.11a}$$

By (3.7) the associated volatility is

$$\sigma_{\mathrm{tg}} = \sigma_{\mathrm{tf}}(\mu_{\mathrm{tg}}) = \frac{|\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}|}{\nu_{\mathrm{rf}}}\,. \tag{3.11b}$$

# One-Rate Model Frontiers: Frontier Tangency

Because when $\mu_{\text{rf}} \neq \mu_{\text{mv}}$ we have

- $\sigma_{\text{tf}}(\mu) \leq \sigma_{\text{mf}}(\mu)$ for every $\mu \in \mathbb{R}$,
- and $\sigma_{\text{tg}} = \sigma_{\text{tf}}(\mu_{\text{tg}}) = \sigma_{\text{mf}}(\mu_{\text{tg}})$,

we see that the Tobin frontier is tangent to the Markowitz frontier at the point $(\sigma_{\text{tg}}, \mu_{\text{tg}})$.

We see from (3.11a) and the frontier parameter relation (3.10) that

$$\mu_{\text{tg}} = \mu_{\text{rf}} + \frac{\nu_{\text{rf}}^2\, \sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{rf}}} = \mu_{\text{mv}} + \frac{\nu_{\text{mv}}^2\, \sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{rf}}} .$$

This shows that

- $\mu_{\text{tg}} > \mu_{\text{mv}}$ when $\mu_{\text{rf}} < \mu_{\text{mv}}$, so $(\sigma_{\text{tg}}, \mu_{\text{tg}})$ is on the efficient frontier,
- $\mu_{\text{tg}} < \mu_{\text{mv}}$ when $\mu_{\text{rf}} > \mu_{\text{mv}}$, so $(\sigma_{\text{tg}}, \mu_{\text{tg}})$ is on the inefficient frontier.

# One-Rate Model Allocations: Introduction

We now describe the allocations of the efficient Tobin portfolios as a function of $\sigma$. This is the optimal allocation suggested by the One-Rate Model for an investor who wants to build a portfolio with risk $\sigma$. This description will be broken down into three cases.

- When $\mu_{\mathrm{rf}} = \mu_{\mathrm{mv}}$ there is no tangent portfolio.
- When $\mu_{\mathrm{rf}} > \mu_{\mathrm{mv}}$ there is an inefficient tangent portfolio.
- When $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ there is an efficient tangent portfolio.

The tangent portfolio allocation $\mathbf{f}_{\mathrm{tg}}$ plays a leading role in the last two cases, so before treating them we will show how to express efficient Tobin allocations in terms of it. Finally, we will describe how all of these efficient Tobin portfolios are built.

# One-Rate Model Allocations: No Tangent

When $\mu_{\rm rf} = \mu_{\rm mv}$ we have $\mathbf{g}_{\rm rf} = \mathbf{g}_{\rm mv}$ and $\nu_{\rm rf} = \nu_{\rm mv}$, so the Tobin frontier allocation (3.6) becomes

$$\mathbf{f}_{\rm tf}(\mu) = (\mu - \mu_{\rm mv})\, \mathbf{g}_{\rm mv} \,,$$

while the efficient Tobin frontier (3.8) becomes

$$\mu - \mu_{\rm mv} = \nu_{\rm mv}\, \sigma \,.$$

Hence, the efficient Tobin allocations in $\mathcal{M}_+$ become

$$\mathbf{f}_{\rm etf}(\sigma) = \sigma\, \nu_{\rm mv}\, \mathbf{g}_{\rm mv} \,.$$

Therefore the efficient Tobin allocations in $\mathcal{M}_1$ are

$$\left(\sigma\, \nu_{\rm mv}\, \mathbf{g}_{\rm mv} \,,\, 1\right) \qquad \text{for } \sigma \in [0, \infty) \,. \tag{4.12}$$

# One-Rate Model Allocations: Tangent Allocation

When $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ the *tangent portfolio allocation* $\mathbf{f}_{\mathrm{tg}}$ associated with the unique tangency point $(\sigma_{\mathrm{tg}}, \mu_{\mathrm{tg}})$ is given by

$$
\begin{aligned}
\mathbf{f}_{\mathrm{tg}} = \mathbf{f}_{\mathrm{tf}}(\mu_{\mathrm{tg}}) = (\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}})\,\mathbf{g}_{\mathrm{rf}} &= \frac{\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}}{\nu_{\mathrm{rf}}^{2}}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}) \\
&= \frac{\sigma_{\mathrm{mv}}^{2}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})\,.
\end{aligned} \tag{4.13}
$$

This is the unique portfolio on the Tobin frontier that is comprised solely of risky assets. The allocation of every other portfolio on the Tobin frontier can be expressed in terms of $\mathbf{f}_{\mathrm{tg}}$ as

$$
\mathbf{f}_{\mathrm{tf}}(\mu) = (\mu - \mu_{\mathrm{rf}})\,\mathbf{g}_{\mathrm{rf}} = \frac{\mu - \mu_{\mathrm{rf}}}{\mu_{\mathrm{tg}} - \mu_{\mathrm{rf}}}\,\mathbf{f}_{\mathrm{tg}}\,. \tag{4.14}
$$

# One-Rate Model Allocations: Efficient Allocations in $\mathcal{M}_+$

When $\mu_{\rm rf} \neq \mu_{\rm mv}$ we see from (3.8) and (3.9) we see that

$$\mu_{\rm tg} - \mu_{\rm rf} = \begin{cases} \nu_{\rm rf}\,\sigma_{\rm tg} & \text{if } \mu_{\rm rf} < \mu_{\rm mv}\,, \\ -\nu_{\rm rf}\,\sigma_{\rm tg} & \text{if } \mu_{\rm rf} > \mu_{\rm mv}\,. \end{cases} \tag{4.15}$$

Because $\mu - \mu_{\rm rf} = \nu_{\rm rf}\,\sigma$ on the efficient Tobin frontier, we see from (4.13) that the Tobin allocations on this frontier are given by

$$\mathbf{f}_{\rm tf}(\mu) = \frac{\mu - \mu_{\rm rf}}{\mu_{\rm tg} - \mu_{\rm rf}}\,\mathbf{f}_{\rm tg} = \frac{\nu_{\rm rf}\,\sigma}{\mu_{\rm tg} - \mu_{\rm rf}}\,\mathbf{f}_{\rm tg}\,.$$

Hence, by (4.15) the efficient Tobin allocations in $\mathcal{M}_+$ are given by

$$\mathbf{f}_{\rm etf}(\sigma) = \begin{cases} \dfrac{\sigma}{\sigma_{\rm tg}}\,\mathbf{f}_{\rm tg} & \text{if } \mu_{\rm rf} < \mu_{\rm mv}\,, \\ -\dfrac{\sigma}{\sigma_{\rm tg}}\,\mathbf{f}_{\rm tg} & \text{if } \mu_{\rm rf} > \mu_{\rm mv}\,. \end{cases} \tag{4.16}$$

# One-Rate Model Allocations: Efficient Allocations in $\mathcal{M}_1$

When $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ the efficient Tobin allocations in $\mathcal{M}_1$ can be read off from (4.16).

- When $\mu_{\mathrm{mv}} < \mu_{\mathrm{rf}}$ the tangent portfolio is inefficient and the efficient Tobin allocations in $\mathcal{M}_1$ are

$$\left( -\frac{\sigma}{\sigma_{\mathrm{tg}}} \mathbf{f}_{\mathrm{tg}} \,, \, 1 + \frac{\sigma}{\sigma_{\mathrm{tg}}} \right) \qquad \text{for } \sigma \in [0, \infty) \,. \tag{4.17a}$$

- When $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ the tangent portfolio is efficient and the efficient Tobin allocations in $\mathcal{M}_1$ are

$$\left( \frac{\sigma}{\sigma_{\mathrm{tg}}} \mathbf{f}_{\mathrm{tg}} \,, \, 1 - \frac{\sigma}{\sigma_{\mathrm{tg}}} \right) \qquad \text{for } \sigma \in [0, \infty) \,. \tag{4.17b}$$

# One-Rate Model Allocations: No and Inefficient Tangent

When $\mu_{\rm rf} = \mu_{\rm mv}$ the efficient allocations (4.12) are built as follows.

1. If $\sigma = 0$ then the investor holds only the risk-free asset;
2. If $\sigma \in (0, \infty)$ then the investor places
   - the entire portfolio value in the risk-free asset,
   - and holds an allocation of $\sigma \, \nu_{\rm mv} \, \mathbf{g}_{\rm mv}$ in risky assets.

When $\mu_{\rm rf} > \mu_{\rm mv}$ the efficient allocations (4.17a) are built as follows.

1. If $\sigma = 0$ then the investor holds only the risk-free asset;
2. If $\sigma \in (0, \infty)$ then the investor places
   - $\dfrac{\sigma_{\rm tg} + \sigma}{\sigma_{\rm tg}}$ of the portfolio value in the risk-free asset,
   - by shorting $\dfrac{\sigma}{\sigma_{\rm tg}}$ of its value from the tangency portfolio $\mathbf{f}_{\rm tg}$.

# One-Rate Model Allocations: Efficient Tangent

When $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ the efficient allocations (4.17b) are built as follows.

1. If $\sigma = 0$ then the investor holds only the risk-free asset;
2. If $\sigma \in (0, \sigma_{\mathrm{tg}})$ then the investor places
    - $\dfrac{\sigma_{\mathrm{tg}} - \sigma}{\sigma_{\mathrm{tg}}}$ of the portfolio value in the risk-free asset,
    - $\dfrac{\sigma}{\sigma_{\mathrm{tg}}}$ of the portfolio value in the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$.
3. If $\sigma = \sigma_{\mathrm{tg}}$ then the investor holds only the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$.
4. If $\sigma \in (\sigma_{\mathrm{tg}}, \infty)$ then the investor places
    - $\dfrac{\sigma}{\sigma_{\mathrm{tg}}}$ of the portfolio value in the tangency portfolio $\mathbf{f}_{\mathrm{tg}}$,
    - by borrowing $\dfrac{\sigma - \sigma_{\mathrm{tg}}}{\sigma_{\mathrm{tg}}}$ of this value from the risk-free asset.

# Capital Allocation Lines: Motivation

Formulas (2.2) for $\sigma(\mathbf{f})$ and $\mu(\mathbf{f})$ for every $\mathbf{f} \in \mathcal{M}_+$ are

$$\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \qquad \mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu_{\mathrm{rf}}\left(1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}\right). \tag{5.18}$$

The Markowitz frontier $\sigma = \sigma_{\mathrm{mf}}(\mu)$ is the boundary of the set

$$\Sigma(\mathcal{M}) = \left\{ \left(\sigma(\mathbf{f})\,,\, \mu(\mathbf{f})\right) \,:\, \mathbf{f} \in \mathcal{M} \right\}. \tag{5.19}$$

The Tobin frontier $\sigma = \sigma_{\mathrm{tf}}(\mu)$ is the boundary of the set

$$\Sigma(\mathcal{M}_+) = \left\{ \left(\sigma(\mathbf{f})\,,\, \mu(\mathbf{f})\right) \,:\, \mathbf{f} \in \mathcal{M}_+ \right\}. \tag{5.20}$$

Here we give a geometric construction of the set $\Sigma(\mathcal{M}_+)$ from the set $\Sigma(\mathcal{M})$ and the risk-free return $\mu_{\mathrm{rf}}$. The virtue of this construction is that it also applies to the Two-Rate Model and to some other models.

# Capital Allocation Lines: Construction

For every $\tilde{\mathbf{f}} \in \mathcal{M}$ we construct the line in $\mathcal{M}_+$ given by

$$\mathbf{f} = \phi \tilde{\mathbf{f}} \qquad \text{for } \phi \in \mathbb{R} . \tag{5.21}$$

Because $\mathbf{1}^{\mathrm{T}}\tilde{\mathbf{f}} = 1$, we see that

$$\mathbf{1}^{\mathrm{T}}\mathbf{f} = \phi , \qquad 1 - \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 - \phi .$$

This the portfolio with allocation $\mathbf{f} \in \mathcal{M}$ holds $\phi$ of its value in risky assets and $1 - \phi$ of its value in the risk-free asset. Because $\phi$ determines this allocation of $\mathbf{f}$, we call it the *capital allocation parameter* and we call the line in $\mathcal{M}_+$ given by (5.21) the *capital allocation line*.

Let $\mathcal{C}_+$ denote the union of all the capital allocation lines (5.21), so that

$$\mathcal{C}_+ = \left\{ \phi \tilde{\mathbf{f}} \, : \, \tilde{\mathbf{f}} \in \mathcal{M} \, , \, \phi \in \mathbb{R} \right\} . \tag{5.22}$$

It is clear that $\mathcal{C}_+ \subset \mathcal{M}_+$, but this inclusion is proper.

# Capital Allocation Lines: $\mathcal{C}_+$ Characterization

**Fact 1.** $\mathcal{C}_+ = \left\{ \mathbf{f} \in \mathcal{M}_+ \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} \neq 0 \right\} \cup \{\mathbf{0}\}$.

**Proof.** Clearly $\mathbf{0} \in \mathcal{C}_+$. Let $\mathbf{f} \in \mathcal{M}_+$ with $\mathbf{1}^{\mathrm{T}}\mathbf{f} \neq 0$. Set $\phi = \mathbf{1}^{\mathrm{T}}\mathbf{f}$ and set

$$\tilde{\mathbf{f}} = \frac{1}{\phi}\,\mathbf{f}\,.$$

Because $\tilde{\mathbf{f}} \in \mathcal{M}$, it follows from the definition (5.22) of $\mathcal{C}_+$ that $\mathbf{f} \in \mathcal{C}_+$, whereby the containment $\supset$ is proved.

Now let $\mathbf{f} \in \mathcal{C}_+$. Then by definition (5.22) there exists $\tilde{\mathbf{f}} \in \mathcal{M}$ and $\phi \in \mathbb{R}$ such that $\mathbf{f} = \phi\,\tilde{\mathbf{f}}$. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = \phi$, we see that:

- if $\phi \neq 0$ then $\mathbf{1}^{\mathrm{T}}\mathbf{f} \neq 0$;
- if $\phi = 0$ then $\mathbf{f} = \phi\,\tilde{\mathbf{f}} = \mathbf{0}$.

Therefore the containment $\subset$ is also proved. $\square$

# Capital Allocation Lines: Image in the $\sigma\mu$-Plane

The image of $\mathcal{C}_+$ in the $\sigma\mu$-plane is

$$\Sigma(\mathcal{C}_+) = \left\{ \left(\sigma(\mathbf{f}) \,,\, \mu(\mathbf{f})\right) \,:\, \mathbf{f} \in \mathcal{C}_+ \right\}. \tag{5.23}$$

By (5.22) every $\mathbf{f} \in \mathcal{C}_+$ has the form $\mathbf{f} = \phi\,\tilde{\mathbf{f}}$ for some $\tilde{\mathbf{f}} \in \mathcal{M}$ and $\phi \in \mathbb{R}$. Let $\sigma = \sigma(\mathbf{f})$ and $\mu = \mu(\mathbf{f})$. We see from (5.18) that $\sigma$ and $\mu$ satisfy

$$\sigma = |\phi|\,\tilde{\sigma}\,, \qquad \mu = \phi\,\tilde{\mu} + (1-\phi)\,\mu_{\text{rf}}\,, \tag{5.24}$$

where $\tilde{\sigma} = \sigma(\tilde{\mathbf{f}})$ and $\tilde{\mu} = \mu(\tilde{\mathbf{f}})$. Clearly $(\tilde{\sigma}, \tilde{\mu}) \in \Sigma(\mathcal{M})$ by (5.19).

For every $(\tilde{\sigma}, \tilde{\mu}) \in \Sigma(\mathcal{M})$ equations (5.24) are parametric equations in $\phi$ describing two half-lines that share the $\mu$-intercept $(0, \mu_{\text{rf}})$ and have slopes

$$\pm\frac{\tilde{\mu} - \mu_{\text{rf}}}{\tilde{\sigma}}\,.$$

## Capital Allocation Lines: Construction of $\Sigma(\mathcal{M}_+)$

Therefore the half-lines described by (5.24) satisfy the explicit equations

$$\mu = \mu_{\mathrm{rf}} \pm \frac{\tilde{\mu} - \mu_{\mathrm{rf}}}{\tilde{\sigma}}\, \sigma \qquad \text{over } \sigma \geq 0 \,. \tag{5.25}$$

The point $(\tilde{\sigma}, \tilde{\mu}) \in \Sigma(\mathcal{M})$ lies on the half-line with the $+$ sign.
The set $\Sigma(\mathcal{C}_+)$ defined by (5.23) is the union of these half-lines.

Because $\mathcal{C}_+ \subset \mathcal{M}_+$, we see from (5.20) and (5.23) that $\Sigma(\mathcal{C}_+) \subset \Sigma(\mathcal{M}_+)$.
More is true. Let

- $\overline{\mathcal{C}}_+$ denote the closure of $\mathcal{C}_+$ in $\mathbb{R}^N$, and
- $\overline{\Sigma(\mathcal{C}_+)}$ denote the closure of $\Sigma(\mathcal{C}_+)$ in the $\sigma\mu$-plane.

We have the following facts, the proofs of which are left as exercises.

- $\overline{\mathcal{C}}_+ = \mathcal{M}_+$. Hint: Use **Fact 1**.
- $\overline{\Sigma(\mathcal{C}_+)} = \Sigma(\mathcal{M}_+)$. Hint: $\sigma(\mathbf{f})$ and $\mu(\mathbf{f})$ are continuous over $\mathcal{M}_+$.

# Capital Allocation Lines: Examples

*Hence, we can construct the set $\Sigma(\mathcal{M}_+)$ with the capital allocation line construction (5.25) from the point $(0, \mu_{\mathrm{rf}})$ and the set $\Sigma(\mathcal{M})$.*

**Example.** If $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ then the capital allocation line construction (5.25) applied to the point $(0, \mu_{\mathrm{rf}})$ and the set $\Sigma(\mathcal{M})$ yields the set

$$\Sigma(\mathcal{C}_+) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq \sigma_{\mathrm{tf}}(\mu) \right\} .$$

Because $\Sigma(\mathcal{C}_+) = \Sigma(\mathcal{M}_+)$, its boundary is the Tobin frontier $\sigma = \sigma_{\mathrm{tf}}(\mu)$.

**Example.** If $\mu_{\mathrm{rf}} = \mu_{\mathrm{mv}}$ then the capital allocation line construction (5.25) applied to the point $(0, \mu_{\mathrm{rf}})$ and the set $\Sigma(\mathcal{M})$ yields the set

$$\Sigma(\mathcal{C}_+) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma > \sigma_{\mathrm{tf}}(\mu) \right\} \cup \left\{ (0, \mu_{\mathrm{rf}}) \right\} ,$$

Because $\overline{\Sigma(\mathcal{C}_+)} = \Sigma(\mathcal{M}_+)$, its boundary is the Tobin frontier $\sigma = \sigma_{\mathrm{tf}}(\mu)$.