Portfolios that Contain Risky Assets
2.3. Portfolio Functions and Metrics

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Part I: Portfolio Models

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2. Markowitz Portfolio Model

2.3. Portfolio Functions and Metrics

Introduction: Portfolio Functions and Metrics

For **Project One** each team will be given a suite of metrics, each of which is designed to capture some aspect of the market. A different suite will be assigned to each team. The goal of each project will be to determine which (if any) of the assigned metrics is a leading indicator of possible systemic market downturns.

In order to better compare the metrics, each metric will be designed to take values in $[0, 1]$, with higher values being worse. Some of these metrics will be functions of $\mathbf{f} \in \mathcal{M}$ that are applied to particular allocations $\mathbf{f}$ that have market significance. Such metrics are called *portfolio metrics*, and the functions from which they come are called *portfolio functions*.

Introduction: Outline

In these slides we develop four such portfolio metric functions.

- We introduce the notion of Markowitz efficiency and develop the metric functions of efficiency and proximity based upon it.
- We develop a metric function of leverage based upon upon Markowitz efficiency and the leverage ratio.
- We introduce the downside and upside potentials and develop metrics of liquidity based upon Markowitz efficiency and the downside potential.

In subsequent slides we will identify portfolios to which these metric functions will be applied, and will develop additional metrics.

Markowitz Efficiency: Definition

In his 1952 paper Harry Markowitz introduced a notion of *efficiency*.

Definition. Given two portfolios with volatilities and means given by (σ_1, μ_1) and (σ_2, μ_2). We say the first is *more efficient* than the second when

$$\sigma_1 \leq \sigma_2 \,, \quad \mu_1 \geq \mu_2 \,, \quad \text{and} \quad (\sigma_1, \mu_1) \neq (\sigma_2, \mu_2) \,.$$

Similarly, we say the first is *less efficient* than the second when

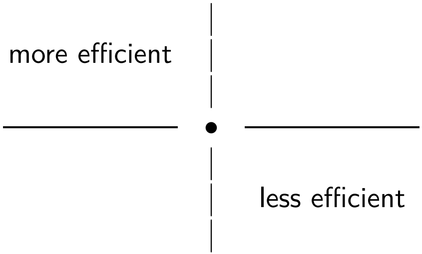
$$\sigma_1 \geq \sigma_2 \,, \quad \mu_1 \leq \mu_2 \,, \quad \text{and} \quad (\sigma_1, \mu_1) \neq (\sigma_2, \mu_2) \,.$$

Remark. Clearly if the first is more efficient the second then the second is less efficient than the first.

Remark. Because volatility and mean are the proxies for risk and reward, a more efficient portfolio offers either greater reward for no greater risk or no less reward for less risk. Similarly, a less efficient portfolio offers either less reward for no less risk or no greater reward for greater risk.

Markowitz Efficiency: Graphical Representation

Given a portfolio with volatility and return mean given by the point • in the $\sigma\mu$-plane then the points that correspond to more efficient and less efficient portfolios are shown below.



Remark. Markowitz introduced the notion of efficiency in his 1952 paper, but its graphical representation in the $\sigma\mu$-plane came later.

Markowitz Efficiency: Efficient and Inefficient Portfolios

Markowitz went on to define efficient and inefficient portfolios.

Definition. The part of the Markowitz frontier with return mean $\mu \geq \mu_{\mathrm{mv}}$ is called the *efficient frontier* because every portfolio that it represents has no portfolio that is more efficient than it.

Definition. Every portfolio represented by the efficient frontier is called *efficient* because no portfolio is more efficient than it. Every other portfolio is called *inefficient* because some portfolios are more efficient than it.

Definition. The part of the Markowitz frontier with return mean $\mu < \mu_{\mathrm{mv}}$ is called the *inefficient frontier* because every portfolio that it represents is less efficient than every other portfolio with no greater volatility.

Markowitz Efficiency: Efficient Frontier

The efficient Markowitz frontier is the upper branch of the frontier hyperbola in the right-half $\sigma\mu$-plane. It is given as a function of σ by

$$\mu = \mu_{\text{emf}}(\sigma) = \mu_{\text{mv}} + \nu_{\text{mv}} \sqrt{\sigma^2 - \sigma_{\text{mv}}^2}\,, \qquad \text{for } \sigma \geq \sigma_{\text{mv}}\,.$$

This curve is increasing and concave and emerges vertically upward from the point $(\sigma_{\text{mv}}, \mu_{\text{mv}})$. As $\sigma \to \infty$ it becomes asymptotic to the line

$$\mu = \mu_{\text{mv}} + \nu_{\text{mv}}\,\sigma\,.$$

Remark. The efficient Markowitz frontier quantifies the relationship between risk and reward mentioned in the slides entitled *Risk and Reward*.

Markowitz Efficiency: Inefficient Frontier

The inefficient Markowitz frontier is the lower branch of the frontier hyperbola in the right-half $\sigma\mu$-plane. It is given as a function of σ by

$$\mu = \mu_{\mathrm{imf}}(\sigma) = \mu_{\mathrm{mv}} - \nu_{\mathrm{mv}} \sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2}\,, \qquad \text{for } \sigma > \sigma_{\mathrm{mv}}\,.$$

This curve is decreasing and convex and emerges vertically downward from the point $(\sigma_{\mathrm{mv}}, \mu_{\mathrm{mv}})$. As $\sigma \to \infty$ it becomes asymptotic to the line

$$\mu = \mu_{\mathrm{mv}} - \nu_{\mathrm{mv}}\,\sigma\,.$$

Remark. The minimum volatility point $(\sigma_{\mathrm{mv}}, \mu_{\mathrm{mv}})$ is not on the inefficient frontier because it is efficient.

Efficiency and Proximity Functions: Introduction

Within the framework of Morkowitz Portfolio Theory (MPT) an investor should want to be as efficient as possible. Recall that for portfolios with three or more assets all of the points (σ_i, m_i) generally lie to the right of the frontier hyperbola in the $\sigma\mu$-plane. This means that generally no single asset is efficient! This implies that a wise investor should build a portfolio with multiple assets that is close to being efficient.

Remark. This conclusion of MPT led to a revolution in investment planning, which had previously focused on picking a few individual assets rather than on building a balanced portfolio.

We now develop two metric functions that quantify how useful assets might be for building a portfolio that is as efficient as possible.

Efficiency and Proximity Functions: Efficiency Function

Definition. We define the *efficiency function* for any $\mathbf{f} \in \mathcal{M}$ by

$$\omega^{\mu}(\mathbf{f}) = \frac{\mu_{\mathrm{emf}}(\sigma) - \mu}{\mu_{\mathrm{emf}}(\sigma) - \mu_{\mathrm{imf}}(\sigma)}, \qquad (3.1)$$

where

$$\sigma = \sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}, \qquad \mu = \mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f}.$$

The efficiency function has the following properties.

- $\omega^{\mu}(\mathbf{f})$ is nondimensional.
- $\omega^{\mu}(\mathbf{f})$ takes values in $[0, 1]$.
- $\mathbf{f}$ is an efficient frontier portfolio if and only if $\omega^{\mu}(\mathbf{f}) = 0$.
- $\mathbf{f}$ is an inefficient frontier portfolio if and only if $\omega^{\mu}(\mathbf{f}) = 1$.

Efficiency and Proximity Functions: Efficiency Function

It makes sense to apply the efficiency function $\omega^\mu(\mathbf{f})$ only to allocations $\mathbf{f} \in \mathcal{M}$ that are not frontier portfolios. For example, setting $\mathbf{f} = \mathbf{e}_i$ yields

$$\omega_i^\mu \equiv \omega^\mu(\mathbf{e}_1) = \frac{\mu_{\mathrm{emf}}(\sigma_i) - m_i}{\mu_{\mathrm{emf}}(\sigma_i) - \mu_{\mathrm{imf}}(\sigma_i)} \,.$$

Setting $\mathbf{f} = \frac{1}{N}\mathbf{1}$ yeilds the efficiency of the equidistributed portfolio.

It is tempting to think that we should consider investing in only in assets whose efficiency ω_i^μ is close to 0. However, the Two Fund Property shows that the entire efficient frontier can realized by a portfolio in any two assets that lie on the frontier. This suggests that a reasonable portfolio can be built from just two assets: one with high return that lies near the efficient frontier, and one with modest return that lies near the frontier.

Efficiency and Proximity Functions: Volatilities

Recall the decomposition of the variance

$$\sigma^2 = \sigma_{\mathrm{mf}}(\mu)^2 + \sigma_{\mathrm{dv}}^2 \,, \tag{3.2}$$

where the volatility σ, return mean μ and diversifiable volatility σ_{dv} are given by

$$\sigma = \sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \,, \qquad \mu = \mu(\mathbf{f}) = \mathbf{m}^{\mathrm{T}}\mathbf{f} \,,$$
$$\sigma_{\mathrm{dv}} = \|\mathbf{g}_{\mathrm{mv}}\|_{\mathbf{V}} = \|(\mathbf{I} - \mathbf{P})\mathbf{f}\|_{\mathbf{V}} \,.$$

Because $\sigma_{\mathrm{mf}}(\mu) \geq \sigma_{\mathrm{mv}} > 0$, we see that

- $\sigma_{\mathrm{dv}} < \sigma$,
- $\mathbf{f}$ is a frontier portfolio if and only if $\sigma_{\mathrm{dv}} = 0$.

These suggest that a good metric of proximity to the frontier is $\sigma_{\mathrm{dv}}/\sigma$.

Efficiency and Proximity Functions: Proximity Function

We can solve for $\sigma_{\text{dv}}/\sigma$ using (3.2).

Definition. We define the *proximity function* for any $\mathbf{f} \in \mathcal{M}$ by

$$\omega^\sigma(\mathbf{f}) = \frac{\sigma_{\text{dv}}}{\sigma} = \sqrt{1 - \frac{\sigma_{\text{mf}}(\mu)^2}{\sigma^2}} \,. \tag{3.3}$$

The proximity function has the following properties.

- $\omega^\sigma(\mathbf{f})$ is nondimensional.
- $\omega^\sigma(\mathbf{f})$ takes values in $[0, 1)$.
- $\mathbf{f}$ is a frontier portfolio if and only if $\omega^\sigma(\mathbf{f}) = 0$.
- $\omega^\sigma(\mathbf{f}) \to 1$ as $\sigma(\mathbf{f}) \to \infty$ while $\mu(\mathbf{f})$ is fixed.

Efficiency and Proximity Functions: Proximity Function

It makes sense to apply the efficiency function $\omega^\mu(\mathbf{f})$ only to allocations $\mathbf{f} \in \mathcal{M}$ that are not frontier portfolios. For example, setting $\mathbf{f} = \mathbf{e}_i$ yields

$$\omega_i^\sigma \equiv \omega^\sigma(\mathbf{e}_1) = \sqrt{1 - \frac{\sigma_{\mathrm{mf}}(m_i)^2}{\sigma_i^2}} \,.$$

Setting $\mathbf{f} = \frac{1}{N}\mathbf{1}$ yeilds the proximity of the equidistributed portfolio.

Remark. We must still identify allocations $\mathbf{f}$ to which $\omega^\mu(\mathbf{f})$ and $\omega^\sigma(\mathbf{f})$ will be applied.

Efficient Market Hypothesis: Weak to Strong

The *efficient-market hypothesis* (EMH) was framed by Eugene Fama in the early 1960's in his University of Chicago doctoral dissertation, which was published in 1965. It has several versions, the most basic is the following.

Given the information available when an investment is made, no investor will consistently beat market returns on a risk-adjusted basis over long periods except by chance.

This version of the EMH is called the *weak EMH*. The *semi-strong* and *strong* versions of the EMH make bolder claims that markets reflect information instantly, even information that is not publicly available in the case of the strong EMH. While it is true that some investors react quickly, most investors do not act instantly to every piece of news. Consequently, there is little evidence supporting these stronger versions of the EMH.

Efficient Market Hypothesis: Index Funds

The EMH is an assertion about markets, not about investors. If the weak EMH is true then the only way for an actively-managed fund to beat the market is by chance. Of course, there is some debate regarding the truth of the weak EMH. It can be recast in the language of MPT as follows.

Markets for broad classes of assets will lie on or near the efficient frontier.

Therefore we can test the weak EMH with MPT! If we understand "market" to mean a capitalization weighted collection of assets (i.e. a broad index fund) then the EMH can be tested by checking whether index funds lie on or near the efficient frontier. You will see that this is often the case, but not always.

Remark. It is a common misconception that MPT assumes the weak EMH. It does not, which is why the EMH can be checked with MPT!

Efficient Market Hypothesis: Index Funds

Given an index fund with volatility σ_I and return mean μ_I its efficiency is

$$\omega_I^{\mu} = \frac{\mu_{\mathrm{emf}}(\sigma_I) - \mu_I}{\mu_{\mathrm{emf}}(\sigma_I) - \mu_{\mathrm{imf}}(\sigma_I)} .$$

while its proximity is

$$\omega_I^{\sigma} = \sqrt{1 - \frac{\sigma_{\mathrm{mf}}(\mu_I)^2}{\sigma_I^2}} .$$

The EMH claims that both of these metrics should be small for an S&P 500 index fund and that the proximity should be small for a total bond index fund. These claims can be easily checked.

Efficient Market Hypothesis: Rational Markets

Remark. If two such index funds do lie on the frontier then by the Two Fund Property we would be able to take any position on the efficient frontier simply by investing in those funds. This idea underpins the investment ideas advanced in *A Random Walk Down Wall Street* by Burton G. Malkiel.

Remark. It is often asserted that the EMH holds in *rational markets*. Such a market is one for which information regarding its assets is freely available to all investors. *This does not mean that investors will act rationally based on this information! Nor does it mean that markets price assets correctly.* Rational markets are subject to the greed and fear of its investors. That is why we have bubbles and crashes. Rational markets can behave irrationally because information is not knowledge!

Efficient Market Hypothesis: Free Markets

Remark. It is sometimes asserted that the EMH holds in *free markets.* Such markets

- have many agents,
- are rational, and
- are subject to regulatory and legal oversight.

These are all elements in Adam Smith's notion of free market, which refers to the freedom of its agents to act, not to the freedom from any government role. Indeed, his radical idea was that government should nurture free markets by playing the role of empowering individual agents. He had to write his book because free markets do not arise spontaneously, even though his *invisible hand* of agents pursuing their self interest insures that markets do.

Leverage Ratio and Function: Ratio

Recall that the leverage ratio of a Markowitz portfolio with allocation $\mathbf{f} \in \mathcal{M}$ is given by

$$\lambda(\mathbf{f}) = \tfrac{1}{2}\left(\|\mathbf{f}\|_1 - 1\right), \tag{5.4}$$

where $\|\mathbf{f}\|_1$ is the ℓ^1-norm of $\mathbf{f}$, which is given by

$$\|\mathbf{f}\|_1 = \sum_{i=1}^{N} |f_i| \,.$$

Recall too that $\mathbf{f} \in \Lambda$ if and only if $\lambda(\mathbf{f}) = 0$.

Remark. Both $\lambda(\mathbf{f})$ and $\|\mathbf{f}\|_1$ are convex functions of $\mathbf{f}$.

Remark. Recall that for $\lambda(\mathbf{f})$ to be the true leverage ratio of the portfolio, the portfolio must be solvent — i.e. $\mathbf{f} \in \Omega$. This should be verified first.

Leverage Ratio and Function: Function Definition

The leverage function is defined for any $\mathbf{f} \in \mathcal{M}$ by

$$\omega^\lambda(\mathbf{f}) = \frac{\lambda(\mathbf{f})}{1 + \lambda(\mathbf{f})} \,. \tag{5.5}$$

By (5.4) this can be expressed as

$$\omega^\lambda(\mathbf{f}) = \frac{\|\mathbf{f}\|_1 - 1}{\|\mathbf{f}\|_1 + 1} \,.$$

The leverage function has the following properties.

- $\omega^\lambda(\mathbf{f})$ is nondimensional.
- $\omega^\lambda(\mathbf{f})$ takes values in $[0, 1)$.
- $\mathbf{f}$ is a long portfolio if and only if $\omega^\lambda(\mathbf{f}) = 0$.
- $\omega^\lambda(\mathbf{f}) \to 1$ as $\lambda(\mathbf{f}) \to \infty$.

Leverage Ratio and Function: Frontier Portfolios

Markowitz efficiency suggests that frontier portfolios play an essential role in any investment strategy. Recall that frontier allocations are given by

$$\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}_{\mathrm{mv}} + (\mu - \mu_{\mathrm{mv}})\,\mathbf{g}_{\mathrm{mv}}\,,$$

where $\mathbf{f}_{\mathrm{mv}}$ is the minimum volatilty allocation, $\mathbf{g}_{\mathrm{mv}}$ is the associated slope, and μ_{mv} is the return mean of $\mathbf{f}_{\mathrm{mv}}$. Here

$$\mathbf{f}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^2 \mathbf{V}^{-1}\mathbf{1}\,, \qquad \mathbf{g}_{\mathrm{mv}} = \frac{1}{\nu_{\mathrm{mv}}^2}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1})\,,$$

where $\mu_{\mathrm{mv}} = \mathbf{m}^{\mathrm{T}}\mathbf{f}_{\mathrm{mv}}$,

$$\frac{1}{\sigma_{\mathrm{mv}}^2} = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}\,, \qquad \nu_{\mathrm{mv}}^2 = (\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1})\,.$$

Leverage Ratio and Function: Frontier Upper Bound

By the triangle inequality we have the upper bound

$$\|\mathbf{f}_{\mathrm{mf}}(\mu)\|_1 \leq \|\mathbf{f}_{\mathrm{mv}}\|_1 + |\mu - \mu_{\mathrm{mv}}| \, \|\mathbf{g}_{\mathrm{mv}}\|_1 \,.$$

By using (5.4) this leads to an upper bound for the leverage ratio along the Markowitz frontier of

$$\lambda(\mathbf{f}_{\mathrm{mf}}(\mu)) \leq \lambda(\mathbf{f}_{\mathrm{mv}}) + \tfrac{1}{2}|\mu - \mu_{\mathrm{mv}}| \, \|\mathbf{g}_{\mathrm{mv}}\|_1 \,.$$

By using (5.5) this leads to an upper bound for leverage function along the Markowitz frontier of

$$\omega^{\lambda}(\mathbf{f}_{\mathrm{mf}}(\mu)) \leq \frac{\lambda(\mathbf{f}_{\mathrm{mv}}) + \tfrac{1}{2}|\mu - \mu_{\mathrm{mv}}| \, \|\mathbf{g}_{\mathrm{mv}}\|_1}{1 + \lambda(\mathbf{f}_{\mathrm{mv}}) + \tfrac{1}{2}|\mu - \mu_{\mathrm{mv}}| \, \|\mathbf{g}_{\mathrm{mv}}\|_1} \,.$$

These upper bounds are sharp when $\mu = \mu_{\mathrm{mv}}$, and so likely remain good when $|\mu - \mu_{\mathrm{mv}}|$ is not too large.

Leverage Ratio and Function: Frontier Lower Bound

By the reverse triangle inequality we have the lower bound

$$\|\mathbf{f}_{\mathrm{mf}}(\mu)\|_1 \geq |\mu - \mu_{\mathrm{mv}}| \, \|\mathbf{g}_{\mathrm{mv}}\|_1 - \|\mathbf{f}_{\mathrm{mv}}\|_1 \, .$$

By using (5.4) this leads to a lower bound for the leverage ratio along the frontier of

$$\lambda(\mathbf{f}_{\mathrm{mf}}(\mu)) \geq \tfrac{1}{2}|\mu - \mu_{\mathrm{mv}}| \, \|\mathbf{g}_{\mathrm{mv}}\|_1 - 1 - \lambda(\mathbf{f}_{\mathrm{mv}}) \, .$$

This lower bound is poor when $\mu = \mu_{\mathrm{mv}}$, and so remains poor when $|\mu - \mu_{\mathrm{mv}}|$ is small. However, it does show that frontier portfolios are leveraged when $|\mu - \mu_{\mathrm{mv}}|$ is big enough to make the lower bound positive. It also shows that their leverage ratios become unbounded as $|\mu| \to \infty$.

Downside and Upside Potentials: Definitions

For any Markowitz portfolio with allocation $\mathbf{f} \in \mathcal{M}$ we define its downside potential $\delta(\mathbf{f})$ and upside potential $\upsilon(\mathbf{f})$ by

$$\begin{aligned} \delta(\mathbf{f}) &= \max\left\{ -\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \,:\, d = 1, \cdots, D \right\}, \\ \upsilon(\mathbf{f}) &= \max\left\{ \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \,:\, d = 1, \cdots, D \right\}. \end{aligned} \tag{6.6}$$

Clearly we have the bounds

$$-\delta(\mathbf{f}) \leq \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \leq \upsilon(\mathbf{f}) \qquad \text{for every } d = 1, \cdots, D. \tag{6.7}$$

Remark. The letter υ appearing above is the Greek letter upsilon.

Remark. Both $\delta(\mathbf{f})$ and $\upsilon(\mathbf{f})$ are convex functions of $\mathbf{f}$ because they are each a maximum over a finite family of linear functions of $\mathbf{f}$.

Downside and Upside Potentials: Individual Assets

The downside and upside potentials for the i^{th} risky asset are obtained by setting $\mathbf{f} = \mathbf{e}_i$, which gives

$$\delta(\mathbf{e}_i) = \max\left\{-r_i(d) \,:\, d = 1, \cdots, D\right\} ,$$
$$\upsilon(\mathbf{e}_i) = \max\left\{r_i(d) \,:\, d = 1, \cdots, D\right\} .$$

Because $-\delta(\mathbf{e}_i)$ is the biggest downside return demonstrated by asset i over the return history $\{r_i(d)\}_{d=1}^{D}$, it is a measure of risk associated with asset i different than σ_i.

Because $1 + r_i(d) > 0$ for every $d = 1, \cdots, D$, the lower bound in (6.6) implies that

$$\delta(\mathbf{e}_i) < 1 .$$

Downside and Upside Potentials: Solvency Characterization

Solvency of $\mathbf{f} \in \mathcal{M}$ is characterized by $\delta(\mathbf{f}) < 1$.

Fact 1. For every $\mathbf{f} \in \mathcal{M}$ we have

$$\mathbf{f} \in \Omega \qquad \Longleftrightarrow \qquad \delta(\mathbf{f}) < 1\,.$$

Proof. Recall that $\mathbf{f} \in \Omega$ if and only if

$$1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} > 0 \qquad \text{for every } d = 1, \cdots, D\,.$$

Because

$$\min\left\{1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} \,:\, 1 = 1, \cdots, D\right\} = 1 - \delta(\mathbf{f})\,,$$

we see that $\mathbf{f} \in \Omega$ if and only if

$$1 - \delta(\mathbf{f}) > 0\,,$$

which is equivalent to $\delta(\mathbf{f}) < 1$. □

Liquidity Function: Definition

Because for every $\mathbf{f} \in \mathcal{M}$

$$\sum_{d=1}^{D} w(d) = 1\,, \qquad \sum_{d=1}^{D} w(d)\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} = \mu(\mathbf{f})\,,$$

the bounds (6.7) imply

$$-\,\delta(\mathbf{f}) \leq \mu(\mathbf{f}) \leq \upsilon(\mathbf{f})\,. \tag{7.8}$$

Definition. We define the *liquidity function* for any $\mathbf{f} \in \mathcal{M}$ by

$$\omega^{\delta}(\mathbf{f}) = \begin{cases} 1 & \text{for } \delta(\mathbf{f}) \geq 1 \\ \dfrac{\delta(\mathbf{f}) + \mu(\mathbf{f})}{1 + \mu(\mathbf{f})} & \text{for } \delta(\mathbf{f}) < 1\,. \end{cases} \tag{7.9}$$

Liquidity Function: Properties

The liquidity function has the following properties.

- $\omega^\delta(\mathbf{f})$ is nondimensional.
- $\omega^\delta(\mathbf{f})$ takes values in $(0, 1]$.
- $\mathbf{f}$ is an insolvent portfolio if and only if $\omega^\delta(\mathbf{f}) = 1$.
- $\omega^\delta(\mathbf{f}) \to 0$ as $\sigma(\mathbf{f}) \to 0$.

It makes sense to apply the liquidity function $\omega^\delta(\mathbf{f})$ to any allocations $\mathbf{f} \in \mathcal{M}$. For example, setting $\mathbf{f} = \mathbf{e}_i$ and recalling that all long portfolios are solvent (so that $\delta(\mathbf{e}_i) < 1$) yields

$$\omega_i^\delta \equiv \omega^\delta(\mathbf{e}_i) = \frac{\delta(\mathbf{e}_i) + m_i}{1 + m_i} .$$

Setting $\mathbf{f} = \frac{1}{N}\mathbf{1}$ yeilds the δ-liquidity of the equidistributed portfolio.