Portfolios that Contain Risky Assets
2.2. Markowitz Frontiers

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

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2. Markowitz Portfolio Model

2.2. Markowitz Frontiers

Introduction: Reward and Risk Proxies

The 1952 paper by Markowitz initiated what evolved into *Modern Portfolio Theory* (MPT). Because 1952 was long ago, and because some of the later additions to that theory are wrong, we will use the name *Markowitz Portfolio Theory* (still MPT), to distinguish the orginal work from what came later. (Markowitz simply called it portfolio theory, and often made fun of the name it aquired.)

Portfolio theories strive to maximize reward for a given risk — or what is related, mimimize risk for a given reward. They do this by quantifying the notions of reward and risk, and identifying a class of idealized portfolios for which an analysis is tractable. Here we present MPT, the first such theory. *Markowitz chose to use the return mean μ as the proxy for the reward of a portfolio, and the volatility $\sigma = \sqrt{v}$ as the proxy for its risk. He also chose to analyze the class of Markowitz portfolios, the allocations of which comprise the set $\mathcal{M}$.*

Introduction: Minimizing Risk

The simplest setting is to use the set of all Markowitz portfolios. Then for a portfolio of N risky assets characterized by $\mathbf{m}$ and $\mathbf{V}$ the problem of minimizing risk for a given reward becomes the problem of minimizing

$$\sigma(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$$

over $\mathbf{f} \in \mathbb{R}^N$ subject to the constraints

$$\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,, \qquad \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\,,$$

where μ is given. Here $\mathbf{1}$ is the N-vector that has every entry equal to 1.

Remark. Additional constraints can be added. For example, we can restrict to long portfolios by adding the inequality constraints $\mathbf{f} \geq \mathbf{0}$. Such constraints will be treated later.

Constrained Minimization Problem: Lagrange Multipliers

Because $\sigma > 0$, minimizing σ is equivalent to minimizing σ^2. Because σ^2 is a quadratic function of $\mathbf{f}$, it is easier to minimize than σ. We therefore choose to solve the constrained minimization problem

$$\min_{\mathbf{f}\in\mathbb{R}^N}\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}. \tag{2.1}$$

Because there are two equality constraints, we introduce the *Lagrange multipliers* α and β, and define

$$\Phi(\mathbf{f},\alpha,\beta) = \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \alpha\left(\mathbf{1}^{\mathrm{T}}\mathbf{f} - 1\right) - \beta\left(\mathbf{m}^{\mathrm{T}}\mathbf{f} - \mu\right).$$

By setting the partial derivatives of $\Phi(\mathbf{f},\alpha,\beta)$ equal to zero we obtain

$$\begin{aligned}
\mathbf{0} &= \nabla_{\mathbf{f}}\Phi(\mathbf{f},\alpha,\beta) = \mathbf{V}\mathbf{f} - \alpha\mathbf{1} - \beta\mathbf{m}\,,\\
0 &= \partial_{\alpha}\Phi(\mathbf{f},\alpha,\beta) = -\mathbf{1}^{\mathrm{T}}\mathbf{f} + 1\,,\\
0 &= \partial_{\beta}\Phi(\mathbf{f},\alpha,\beta) = -\mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu\,.
\end{aligned}$$

Constrained Minimization Problem: Two-by-Two System

Because $\mathbf{V}$ is positive definite we may solve the first equation for $\mathbf{f}$ as

$$\mathbf{f} = \alpha\,\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{V}^{-1}\mathbf{m}\,.$$

By setting this into the second and third equations we obtain the system

$$\alpha\,\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} = 1\,,$$
$$\alpha\,\mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} = \mu\,.$$

If we introduce a, b, and c by

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,, \qquad c = \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,,$$

then the above two-by-two system can be expressed as

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}\,.$$

Constrained Minimization Problem: Invertibility

The foregoing linear algebraic system has a unique solution if and only if

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{is invertible}\,,$$

where

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,, \qquad c = \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,.$$

For every $x, y \in \mathbb{R}$ we have

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x\mathbf{1} + y\mathbf{m})^{\mathrm{T}}\mathbf{V}^{-1}(x\mathbf{1} + y\mathbf{m})\,.$$

Because $\mathbf{V}^{-1}$ is positive definite, we see that the above 2×2 matrix is positive definite if and only if the vectors $\mathbf{1}$ and $\mathbf{m}$ are not co-linear ($\mathbf{m} \neq \mu\mathbf{1}$ for every μ). This is usually the case in practice.

Constrained Minimization Problem: The Minimizer

When $\mathbf{1}$ and $\mathbf{m}$ are not co-linear, we find that α and β are

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ \mu \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} c - b\mu \\ a\mu - b \end{pmatrix} .$$

Hence, for each μ there is a unique minimizer given by

$$\mathbf{f}(\mu) = \frac{c - b\mu}{ac - b^2} \mathbf{V}^{-1}\mathbf{1} + \frac{a\mu - b}{ac - b^2} \mathbf{V}^{-1}\mathbf{m} . \tag{2.2}$$

The associated minimum value of σ^2 is

$$\begin{aligned}
\sigma^2 &= \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) = \left(\alpha\,\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{V}^{-1}\mathbf{m}\right)^{\mathrm{T}}\mathbf{V}\left(\alpha\,\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{V}^{-1}\mathbf{m}\right) \\
&= \begin{pmatrix} \alpha & \beta \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{ac - b^2} \begin{pmatrix} 1 & \mu \end{pmatrix} \begin{pmatrix} c & -b \\ -b & a \end{pmatrix} \begin{pmatrix} 1 \\ \mu \end{pmatrix} \\
&= \frac{1}{a} + \frac{a}{ac - b^2}\left(\mu - \frac{b}{a}\right)^2 .
\end{aligned}$$

Constrained Minimization Problem: In Practice

Remark. When calculating formulas such as (2.2) on a computer, we never compute the inverse of a matrix! Rather, we solve the linear algebraic systems

$$\mathbf{V}\mathbf{y} = \mathbf{1}\,, \qquad \mathbf{V}\mathbf{z} = \mathbf{m}\,.$$

We then generate a, b, and c by the formulas

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{y}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{z}\,, \qquad c = \mathbf{m}^{\mathrm{T}}\mathbf{z}\,.$$

Then formula (2.2) becomes

$$\mathbf{f}(\mu) = \frac{c - b\mu}{ac - b^2}\,\mathbf{y} + \frac{a\mu - b}{ac - b^2}\,\mathbf{z}\,.$$

Therefore when you see $\mathbf{V}^{-1}\mathbf{1}$ and $\mathbf{V}^{-1}\mathbf{m}$ in what follows, think of them as symbols for the vectors $\mathbf{y}$ and $\mathbf{z}$ which indicate that $\mathbf{y}$ and $\mathbf{z}$ are the solutions of certain linear algebraic systems!

Constrained Minimization Problem: Co-Linear Case

Remark. When $\mathbf{1}$ and $\mathbf{m}$ are co-linear we have $\mathbf{m} = \mu\mathbf{1}$ for some μ. If we minimize $\frac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ subject to the constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ then, by Lagrange multipliers, we find that

$$\mathbf{V}\mathbf{f} = \alpha\mathbf{1}\,, \quad \text{for some } \alpha \in \mathbb{R}\,.$$

Because $\mathbf{V}$ is invertible, for every μ the unique minimizer is given by

$$\mathbf{f}(\mu) = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}\,\mathbf{V}^{-1}\mathbf{1}\,.$$

The associated minimum value of σ^2 is

$$\sigma^2 = \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}} = \frac{1}{a}\,.$$

Remark. The mathematics used above is fairly elementary. Markowitz had cleverly indentified a class of portfolios which is analytically tractable by elemetary methods, and which provides a framework for useful models.

Frontier Portfolios: Risk-Reward Relationship

We have seen that for every μ there exists a unique Markowitz portfolio with mean μ that minimizes σ^2. This minimum value is

$$\sigma^2 = \frac{1}{a} + \frac{a}{ac - b^2}\left(\mu - \frac{b}{a}\right)^2 .$$

This is the equation of a hyperbola in the $\sigma\mu$-plane. Because volatility is nonnegative, we only consider the right half-plane $\sigma \geq 0$.

The volatility σ and mean μ of any Markowitz portfolio will be a point (σ, μ) in this half-plane that lies either on or to the right of this hyperbola.

Every point (σ, μ) on the hyperbola branch in this half-plane represents a unique Markowitz portfolio. These portfolios are called *frontier portfolios*, or *Markowitz frontier portfolios* when we want to distinguish them from other frontier portfolios that arise later.

Frontier Portfolios: Frontier Parameters

We now replace a, b, and c with the more meaningful *frontier parameters*

$$\sigma_{\rm mv} = \frac{1}{\sqrt{a}}\,, \qquad \mu_{\rm mv} = \frac{b}{a}\,, \qquad \nu_{\rm mv} = \sqrt{\frac{ac - b^2}{a}}\,.$$

The volatility σ for the Markowitz frontier portfolio with mean μ is then

$$\sigma = \sigma_{\rm mf}(\mu) \equiv \sqrt{\sigma_{\rm mv}^2 + \left(\frac{\mu - \mu_{\rm mv}}{\nu_{\rm mv}}\right)^2}\,.$$

We see that this hyperbola branch has center $(0, \mu_{\rm mv})$, vertex $(\sigma_{\rm mv}, \mu_{\rm mv})$, and asymptotes

$$\mu = \mu_{\rm mv} \pm \nu_{\rm mv}\,\sigma\,.$$

Frontier Portfolios: Minimum Volatility Portfolio

It is clear that no portfolio has a volatility σ that is less than σ_{mv}. In other words, σ_{mv} is the minimum volatility attainable by diversification.

The unique frontier portfolio corresponding to $(\sigma_{\mathrm{mv}}, \mu_{\mathrm{mv}})$ is called the *minimum volatility portfolio.* Its associated allocation $\mathbf{f}_{\mathrm{mv}}$ is given by

$$\mathbf{f}_{\mathrm{mv}} = \mathbf{f}(\mu_{\mathrm{mv}}) = \mathbf{f}\left(\frac{b}{a}\right) = \frac{1}{a}\mathbf{V}^{-1}\mathbf{1} = \sigma_{\mathrm{mv}}^2\mathbf{V}^{-1}\mathbf{1}\,.$$

This allocation depends only upon $\mathbf{V}$, and is therefore known with greater confidence than any allocation that also depends upon $\mathbf{m}$.

Frontier Portfolios: Allocation Slope

The allocations of the Markowitz frontier portfolios can be expressed as

$$\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}_{\mathrm{mv}} + (\mu - \mu_{\mathrm{mv}})\,\mathbf{g}_{\mathrm{mv}}\,,$$

where the *associated slope* $\mathbf{g}_{\mathrm{mv}}$ is defined by

$$\mathbf{g}_{\mathrm{mv}} = \frac{1}{\nu_{\mathrm{mv}}^2}\,\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1})\,.$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) = b - \mu_{\mathrm{mv}}a = 0$, we see that

$$\mathbf{1}^{\mathrm{T}}\mathbf{g}_{\mathrm{mv}} = \frac{1}{\nu_{\mathrm{mv}}^2}\,\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) = 0\,.$$

Therefore $\mathbf{g}_{\mathrm{mv}} \notin \mathcal{M}$.

Frontier Portfolios: **V**-Scalar Product

Fact 1. The vectors $\mathbf{f}_{\mathrm{mv}}$ and $\mathbf{g}_{\mathrm{mv}}$ are orthogonal with respect to the **V**-scalar product, which is given by $(\mathbf{f}_1 \,|\, \mathbf{f}_2)_{\mathbf{V}} = \mathbf{f}_1^{\mathrm{T}} \mathbf{V} \mathbf{f}_2$.

Proof. Because $\mathbf{1}^{\mathrm{T}} \mathbf{g}_{\mathrm{mv}} = 0$ we see that

$$(\mathbf{f}_{\mathrm{mv}} \,|\, \mathbf{g}_{\mathrm{mv}})_{\mathbf{V}} = \mathbf{f}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{g}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^{2} \, \mathbf{1}^{\mathrm{T}} \mathbf{g}_{\mathrm{mv}} = 0 \,. \qquad \square$$

Therefore

$$\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}_{\mathrm{mv}} + (\mu - \mu_{\mathrm{mv}})\, \mathbf{g}_{\mathrm{mv}}$$

is an orthogonal decomposition of $\mathbf{f}_{\mathrm{mf}}(\mu)$ with respect to the **V**-scalar.

Fact 2. In the associated **V**-norm, which is given by $\|\mathbf{f}\|_{\mathbf{V}}^2 = (\mathbf{f} \,|\, \mathbf{f})_{\mathbf{V}}$, we find that

$$\|\mathbf{f}_{\mathrm{mv}}\|_{\mathbf{V}}^2 = \mathbf{f}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^{4} \mathbf{1}^{\mathrm{T}} \mathbf{V}^{-1} \mathbf{1} = \sigma_{\mathrm{mv}}^{2}$$

$$\|\mathbf{g}_{\mathrm{mv}}\|_{\mathbf{V}}^2 = \mathbf{g}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{g}_{\mathrm{mv}} = \frac{1}{\nu_{\mathrm{mv}}^{4}} (\mathbf{m} - \mu_{\mathrm{mv}} \mathbf{1})^{\mathrm{T}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\mathrm{mv}} \mathbf{1}) = \frac{1}{\nu_{\mathrm{mv}}^{2}} \,.$$

Frontier Portfolios: **V**-Orthogonal Projection

Fact 3. For any $\mathbf{f} \in \mathcal{M}$ we have

$$(\mathbf{f}_{\mathrm{mv}} \,|\, \mathbf{f})_{\mathbf{V}} = \sigma_{\mathrm{mv}}^2 \mathbf{1}^{\mathrm{T}} \mathbf{f} = \sigma_{\mathrm{mv}}^2 \,,$$

$$(\mathbf{g}_{\mathrm{mv}} \,|\, \mathbf{f})_{\mathbf{V}} = \frac{1}{\nu_{\mathrm{mv}}^2} \left(\mathbf{m} - \mu_{\mathrm{mv}} \mathbf{1}\right)^{\mathrm{T}} \mathbf{f} = \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{mv}}^2} \,, \qquad \text{where } \mu = \mathbf{m}^{\mathrm{T}} \mathbf{f} \,.$$

Fact 4. Because $\mathbf{f}_{\mathrm{mv}}$ and $\mathbf{g}_{\mathrm{mv}}$ are **V**-orthogonal, the **V**-orthogonal projection of **f** onto $\mathrm{Span}\{\mathbf{f}_{\mathrm{mv}}, \mathbf{g}_{\mathrm{mv}}\}$ is

$$\begin{aligned} \mathbf{P}\mathbf{f} &= \frac{(\mathbf{f}_{\mathrm{mv}} \,|\, \mathbf{f}\,)_{\mathbf{V}}}{\|\mathbf{f}_{\mathrm{mv}}\|_{\mathbf{V}}^2} \mathbf{f}_{\mathrm{mv}} + \frac{(\mathbf{g}_{\mathrm{mv}} \,|\, \mathbf{f}\,)_{\mathbf{V}}}{\|\mathbf{g}_{\mathrm{mv}}\|_{\mathbf{V}}^2} \mathbf{g}_{\mathrm{mv}} = \frac{\mathbf{f}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{f}}{\mathbf{f}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{mv}}} \mathbf{f}_{\mathrm{mv}} + \frac{\mathbf{g}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{f}}{\mathbf{g}_{\mathrm{mv}}^{\mathrm{T}} \mathbf{V} \mathbf{g}_{\mathrm{mv}}} \mathbf{g}_{\mathrm{mv}} \\ &= \frac{\sigma_{\mathrm{mv}}^2}{\sigma_{\mathrm{mv}}^2} \mathbf{f}_{\mathrm{mv}} + \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{mv}}^2} \frac{\nu_{\mathrm{mv}}^2}{1} \mathbf{g}_{\mathrm{mv}} = \mathbf{f}_{\mathrm{mv}} + (\mu - \mu_{\mathrm{mv}})\, \mathbf{g}_{\mathrm{mv}} = \mathbf{f}_{\mathrm{mf}}(\mu) \,. \end{aligned}$$

Frontier Portfolios: **V**-Orthogonal Decomposition

For any $\mathbf{f} \in \mathcal{M}$ we have the **V**-orthogonal decomposition

$$\mathbf{f} = \mathbf{P}\mathbf{f} + (\mathbf{I} - \mathbf{P})\mathbf{f} = \mathbf{f}_{\mathrm{mf}}(\mu) + \mathbf{g}_{\mathrm{dv}} \,,$$

where $\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}$ and $\mathbf{g}_{\mathrm{dv}} = (\mathbf{I} - \mathbf{P})\mathbf{f}$. This can be decomposed further as

$$\mathbf{f} = \mathbf{f}_{\mathrm{mv}} + (\mu - \mu_{\mathrm{mv}})\,\mathbf{g}_{\mathrm{mv}} + \mathbf{g}_{\mathrm{dv}} \,.$$

Fact 5. Because $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{g}_{\mathrm{mv}}$ and $\mathbf{g}_{\mathrm{dv}}$ are mutually **V**-orthogonal, we have

$$\begin{aligned}\sigma^2 = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} = \|\mathbf{f}\|_{\mathbf{V}}^2 &= \|\mathbf{f}_{\mathrm{mv}}\|_{\mathbf{V}}^2 + (\mu - \mu_{\mathrm{mv}})^2\|\mathbf{g}_{\mathrm{mv}}\|_{\mathbf{V}}^2 + \|\mathbf{g}_{\mathrm{dv}}\|_{\mathbf{V}}^2 \\ &= \sigma_{\mathrm{mv}}^2 + \left(\frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{mv}}}\right)^2 + \sigma_{\mathrm{dv}}^2 \,,\end{aligned}$$

where $\sigma_{\mathrm{dv}} = \|\mathbf{g}_{\mathrm{dv}}\|_{\mathbf{V}}$.

Frontier Portfolios: Interpretation

Remark. This shows that the volatility (risk) associated with any $\mathbf{f} \in \mathcal{M}$ is

$$\sigma(\mathbf{f}) = \sqrt{\sigma_{\mathrm{mv}}^2 + \left(\frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{mv}}}\right)^2 + \sigma_{\mathrm{dv}}^2}\,,$$

where $\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}$ and $\sigma_{\mathrm{dv}} = \|\mathbf{g}_{\mathrm{dv}}\|_{\mathbf{V}}$. Markowitz interpreted the three contributions under the square root respectively as arising from:

- the minimum risk of the market,
- the systemic risk of the market associated with reward μ,
- the diversifiable or specific risk of the portfolio.

Markowitz attributed the first two to "risk of the market" because he was considering the case when N was large enough that $\sigma_{\mathrm{mf}}(\mu)$ would not be significantly reduced by introducing additional assets into the portfolio. *More generally, one should attribute $\sigma_{\mathrm{mf}}(\mu)$ to those risks that are essential to the set of N assets being considered for the portfolio.*

Frontier Portfolios: Two-Fund Property

The fact that $\mathbf{f}_{\mathrm{mf}}(\mu)$ is a linear function of μ leads to the following general property of frontier portfolios that contain two or more risky assets.

Fact 6. Let $\mathbf{f}_{\mathrm{mf}}(\mu)$ be the frontier portfolios associated with $N \geq 2$ risky assets. Let μ_1 and μ_2 be any two return means with $\mu_1 < \mu_2$. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the allocations of the frontier portfolios associated with μ_1 and μ_2. Then $\mathbf{f}_{\mathrm{mf}}(\mu)$ is given by

$$\mathbf{f}_{\mathrm{mf}}(\mu) = \frac{\mu_2 - \mu}{\mu_2 - \mu_1}\,\mathbf{f}_1 + \frac{\mu - \mu_1}{\mu_2 - \mu_1}\,\mathbf{f}_2\,,$$

Proof. Because $\mathbf{f}_{\mathrm{mf}}(\mu)$ is a linear function of μ, it is uniquely determined by $\mathbf{f}_1 = \mathbf{f}_{\mathrm{mf}}(\mu_1)$ and $\mathbf{f}_2 = \mathbf{f}_{\mathrm{mf}}(\mu_2)$. □

Frontier Portfolios: Two Fund Property

Remark. *The foregoing property states that every frontier portfolio can be realized by holding positions in just two funds that have the portfolio allocations* $\mathbf{f}_1$ *and* $\mathbf{f}_2$*.*

- When $\mu \in (\mu_1, \mu_2)$ both funds are held long.
- When $\mu < \mu_1$ the second fund is held short while the first is held long.
- When $\mu > \mu_2$ the first fund is held short while the second is held long.
- When $\mu = \mu_1$ only the first fund is held long.
- When $\mu = \mu_2$ only the second fund is held long.

This property is often called the *Two Mutual Fund Theorem*, which is a label that elevates it to a higher status than it deserves. We will call it simply the *Two Fund Property.*

General Portfolio with Two Risky Assets: Allocations

Consider a portfolio of two risky assets with return mean vector $\mathbf{m}$ and return covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

The constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ give the linear system

$$f_1 + f_2 = 1, \qquad m_1 f_1 + m_2 f_2 = \mu.$$

The vectors $\mathbf{m}$ and $\mathbf{1}$ are not co-linear if and only if $m_1 \neq m_2$. In that case $\mathbf{f}$ is uniquely determine by this linear system to be

$$\mathbf{f} = \mathbf{f}(\mu) = \begin{pmatrix} f_1(\mu) \\ f_2(\mu) \end{pmatrix} = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

General Portfolio with Two Risky Assets: Hyperbola

Because there is just one portfolio for each μ, every portfolio is a frontier portfolio. In other words, $\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}(\mu)$.

These frontier portfolios trace out the hyperbola

$$\begin{aligned}\sigma^2 &= \sigma_{\mathrm{mf}}(\mu)^2 = \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) \\ &= \frac{v_{11}(m_2-\mu)^2 + 2v_{12}(m_2-\mu)(\mu-m_1) + v_{22}(\mu-m_1)^2}{(m_2-m_1)^2}\,.\end{aligned}$$

Remark. Each (σ_i, m_i) lies on the frontier of every two-asset portfolio. Typically each (σ_i, m_i) lies strictly to the right of the frontier of a portfolio that contains more than two risky assets.

General Portfolio with Two Risky Assets: Parameters

The frontier parameters μ_{mv}, σ_{mv}, and ν_{mv} are given by

$$\mu_{\text{mv}} = \frac{(v_{22} - v_{12})m_1 + (v_{11} - v_{12})m_2}{v_{11} + v_{22} - 2v_{12}},$$

$$\sigma_{\text{mv}}^2 = \frac{v_{11}v_{22} - v_{12}^2}{v_{11} + v_{22} - 2v_{12}}, \qquad \nu_{\text{mv}}^2 = \frac{(m_2 - m_1)^2}{v_{11} + v_{22} - 2v_{12}}.$$

Remark. The fact that $\mathbf{V}$ is positive definite implies

$$v_{11}v_{22} - v_{12}^2 = \det(\mathbf{V}) > 0,$$

$$v_{11} + v_{22} - 2v_{12} = \begin{pmatrix} 1 & -1 \end{pmatrix} \mathbf{V} \begin{pmatrix} 1 \\ -1 \end{pmatrix} > 0.$$

General Portfolio with Two Risky Assets: $\mathbf{f}_{\mathrm{mv}}$

The minimum volatility portfolio is

$$\mathbf{f}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^2 \mathbf{V}^{-1}\mathbf{1} = \frac{\sigma_{\mathrm{mv}}^2}{v_{11}v_{22} - v_{12}^2} \begin{pmatrix} v_{22} & -v_{12} \\ -v_{12} & v_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{v_{11} + v_{22} - 2v_{12}} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix} .$$

Remark. Notice that $\mathbf{f}_{\mathrm{mv}}$ holds no short positions if and only if

$$v_{11} - v_{12} \geq 0 \qquad \text{and} \qquad v_{22} - v_{12} \geq 0 \,.$$

This will be the case when $v_{12} \leq 0$, which is when the two assets are not positively correlated.

General Portfolio with Three Risky Assets: Allocations

Consider a portfolio of three risky assets with return mean vector **m** and return covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

The constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ give the linear system

$$f_1 + f_2 + f_3 = 1, \qquad m_1 f_1 + m_2 f_2 + m_3 f_3 = \mu.$$

The vectors **m** and **1** are not co-linear if and only if $m_i \neq m_j$ for some i and j. If we assume that $m_1 \neq m_3$ then a general solution is

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix} + \frac{\phi}{m_3 - m_1} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix},$$

where ϕ is an arbitrary real number.

General Portfolio with Three Risky Assets: Allocations

Therefore every allocation $\mathbf{f}$ that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ can be expressed as the one-parameter family

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\, \mathbf{n} \qquad \text{for some } \phi \in \mathbb{R}\,,$$

where $\mathbf{f}_{13}(\mu)$ and $\mathbf{n}$ are the linearly independent unitless vectors given by

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

It is easily checked that these vectors satisfy

$$\begin{aligned} \mathbf{1}^{\mathrm{T}}\mathbf{f}_{13}(\mu) &= 1\,, & \qquad \mathbf{1}^{\mathrm{T}}\mathbf{n} &= 0\,, \\ \mathbf{m}^{\mathrm{T}}\mathbf{f}_{13}(\mu) &= \mu\,, & \qquad \mathbf{m}^{\mathrm{T}}\mathbf{n} &= 0\,. \end{aligned}$$

In particular, $\mathbf{f}_{13}(\mu)$ is the Markowitz portfolio with return mean μ that is generated by assets 1 and 3.

General Portfolio with Three Risky Assets: Volatility

We can use the family $\mathbf{f} = \mathbf{f}_{13}(\mu) + \phi\,\mathbf{n}$ to find an alternative expression for the frontier. Fix $\mu \in \mathbb{R}$. For Markowitz portfolios we obtain

$$\sigma^2 = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\,\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}\,.$$

Because $\mathbf{V}$ is positive definite we know that $\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0$. By completing the square in the above quadratic function of ϕ, we obtain

$$\sigma^2 = \sigma_{\mathrm{mf}}(\mu)^2 + \left(\phi + \frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\right)^2 \mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}\,, \tag{5.3}$$

where

$$\sigma_{\mathrm{mf}}(\mu)^2 = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) - \frac{\left(\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)\right)^2}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,.$$

General Portfolio with Three Risky Assets: Frontier

The form (5.3) shows that σ has the unique minimizer at

$$\phi = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} ,$$

and minimum of $\sigma = \sigma_{\mathrm{mf}}(\mu)$ where

$$\sigma_{\mathrm{mf}}(\mu)^2 = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) - \frac{\left(\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)\right)^2}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} . \qquad (5.4)$$

Therefore the frontier is given by $\sigma = \sigma_{\mathrm{mf}}(\mu)$.

Remark. The first term on the right-hand side of (5.4) is the unique portfolio with return mean μ that contains only assets 1 and 3. Hence, $\sigma_{\mathrm{mf}}(\mu)$ is less when $\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) \neq 0$.

General Portfolio with Three Risky Assets: Nonuniqueness

The form (5.3) also shows that for every $\sigma > \sigma_{\mathrm{mf}}(\mu)$ there exists two portfolios associated with the point (σ, μ) in the $\sigma\mu$-plane given by

$$\phi = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} \pm \left(\frac{\sigma^2 - \sigma_{\mathrm{mf}}(\mu)^2}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\right)^{\frac{1}{2}} .$$

Remark. This shows that for portfolios with three assets there are two portfolios associated with every point (σ, μ) in the $\sigma\mu$-plane to the right of the frontier hyperbola $\sigma = \sigma_{\mathrm{mf}}(\mu)$.

Remark. For portfolios with more than three assets there is a family of portfolios associated with every point (σ, μ) in the $\sigma\mu$-plane to the right of the frontier hyperbola $\sigma = \sigma_{\mathrm{mf}}(\mu)$.

General Portfolio with Three Risky Assets: Identities

Remark. Whenever $m_1 \neq m_2 \neq m_3 \neq m_1$ the Markowitz portfolios with return mean μ generated by assets 1 and 2 and assets 2 and 3 are

$$\mathbf{f}_{21}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \\ 0 \end{pmatrix}, \qquad \mathbf{f}_{32}(\mu) = \frac{1}{m_3 - m_2} \begin{pmatrix} 0 \\ m_3 - \mu \\ \mu - m_2 \end{pmatrix}.$$

Then we can show that the minimum of σ^2 over all portfolios with return mean μ that contain all three assets can be expressed as

$$\sigma^2 = \mathbf{f}_{13}(\mu)^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_{13}(\mu) = \mathbf{f}_{21}(\mu)^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_{21}(\mu) = \mathbf{f}_{32}(\mu)^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_{32}(\mu),$$

where

$$\widetilde{\mathbf{V}} = \mathbf{V} - \frac{\mathbf{V}\mathbf{n}\mathbf{n}^{\mathrm{T}}\mathbf{V}}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}.$$

General Portfolio with Three Risky Assets: Off Frontier

Remark. The formula on the previous slide shows that for each $i = 1, 2, 3$ the frontier portfolio with $\mu = m_i$ has σ given by

$$\sigma^2 = \mathbf{e}_i^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{e}_i = \mathbf{e}_i^{\mathrm{T}} \mathbf{V} \mathbf{e}_i - \frac{(\mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{e}_i)^2}{\mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{n}} = \sigma_i^2 - \frac{(\mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{e}_i)^2}{\mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{n}} \,.$$

This shows that the point (σ_i, m_i) will lie to the right of the frontier hyperbola in the $\sigma\mu$-plane if and only if

$$\mathbf{n}^{\mathrm{T}} \mathbf{V} \mathbf{e}_i \neq 0 \,.$$

This condition is almost always satisfied by every asset and is always satisfied by at least one asset. We conclude that for portfolios with three or more assets all of the points (σ_i, m_i) will generally lie to the right of the frontier hyperbola in the $\sigma\mu$-plane.

General Portfolio with Three Risky Assets: Off Frontier

Remark. The same formula also shows that for assets i and j we have

$$\sigma^2 = \mathbf{f}_{ij}(\mu)^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}_{ij}(\mu) = \mathbf{f}_{ij}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{ij}(\mu) - \frac{\left(\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{ij}(\mu)\right)^2}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}},$$

This shows that in the $\sigma\mu$-plane the frontier hyperbola associated with only assets i and j lies to the right of the frontier hyperbola associated with all three assets when

$$\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{ij}(\mu) \neq 0 .$$

Because this condition is linear in μ, typically it is satisfied at all but one value of μ. We conclude that for portfolios with three assets the points in the $\sigma\mu$-plane corresponding to portfolios that hold neutral positions in all but two assets will generally lie to the right of the frontier hyperbola at all but one point.

Simple Portfolio with Three Risky Assets: Example

Consider a portfolio of three risky assets with return mean vector $\mathbf{m}$ and return covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$. The last condition is equivalent to the condition that $\mathbf{V}$ is positive definite given $s > 0$. This portfolio has the unrealistic properties that (1) every asset has the same volatility s, (2) every pair of distinct assets has the same correlation r, and (3) the return means are uniformly spaced with difference $d = m_3 - m_2 = m_2 - m_1$. These simplifications will make it easier to follow the ensuing calculations than for the general three-asset portfolio. This simple portfolio can be used to help debug code or to provide illustrative examples.

Simple Portfolio with Three Risky Assets

It is helpful to express $\mathbf{m}$ and $\mathbf{V}$ in terms of $\mathbf{1}$ and $\mathbf{q} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ as

$$\mathbf{m} = m\,\mathbf{1} + d\,\mathbf{q}\,, \qquad \mathbf{V} = s^2(1-r)\left(\mathbf{I} + \frac{r}{1-r}\,\mathbf{1}\,\mathbf{1}^{\mathrm{T}}\right).$$

Notice that $\mathbf{1}^{\mathrm{T}}\mathbf{1} = 3$, $\mathbf{1}^{\mathrm{T}}\mathbf{q} = 0$, $\mathbf{q}^{\mathrm{T}}\mathbf{q} = 2$, $\mathbf{V}\mathbf{1} = s^2(1+2r)\mathbf{1}$, and $\mathbf{V}\mathbf{q} = s^2(1-r)\mathbf{q}$. It can be checked that

$$\mathbf{V}^{-1} = \frac{1}{s^2(1-r)}\left(\mathbf{I} - \frac{r}{1+2r}\,\mathbf{1}\,\mathbf{1}^{\mathrm{T}}\right), \qquad \mathbf{V}^{-1}\mathbf{1} = \frac{1}{s^2(1+2r)}\,\mathbf{1}\,,$$

$$\mathbf{V}^{-1}\mathbf{q} = \frac{1}{s^2(1-r)}\,\mathbf{q}\,, \qquad \mathbf{V}^{-1}\mathbf{m} = \frac{m}{s^2(1+2r)}\,\mathbf{1} + \frac{d}{s^2(1-r)}\,\mathbf{q}\,.$$

Simple Portfolio with Three Risky Assets

The parameters a, b, and c are therefore given by

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} = \frac{3}{s^2(1+2r)}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} = \frac{3m}{s^2(1+2r)}\,,$$

$$c = \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} = \frac{3m^2}{s^2(1+2r)} + \frac{2d^2}{s^2(1-r)}\,.$$

The frontier parameters are then

$$\sigma_{\mathrm{mv}} = \sqrt{\frac{1}{a}} = s\sqrt{\frac{1+2r}{3}}\,, \qquad \mu_{\mathrm{mv}} = \frac{b}{a} = m\,,$$

$$\nu_{\mathrm{mv}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s}\sqrt{\frac{2}{1-r}}\,.$$

The minimum volatility portfolio has allocation

$$\mathbf{f}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^2\mathbf{V}^{-1}\mathbf{1} = \tfrac{1}{3}\,\mathbf{1}\,.$$

Simple Portfolio with Three Risky Assets

The Markowitz frontier is given by

$$\sigma = \sigma_{\mathrm{mf}}(\mu) = s\sqrt{\frac{1+2r}{3} + \frac{1-r}{2}\left(\frac{\mu - m}{d}\right)^2} \qquad \text{for } \mu \in (-\infty, \infty)\,.$$

Notice that each (σ_i, m_i) lies strictly to the right of the frontier because

$$\sigma_{\mathrm{mv}} = \sigma_{\mathrm{mf}}(m) = s\sqrt{\frac{1+2r}{3}} < \sigma_{\mathrm{mf}}(m \pm d) = s\sqrt{\frac{5+r}{6}} < s\,.$$

Notice that as r decreases the frontier moves to the left for $|\mu - m| < \frac{2}{3}\sqrt{3}d$ and to the right for $|\mu - m| > \frac{2}{3}\sqrt{3}d$.

Simple Portfolio with Three Risky Assets

The allocation of the frontier portfolio with return mean μ is

$$\mathbf{f}_{\mathrm{mf}}(\mu) = \mathbf{f}_{\mathrm{mv}} + \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{mv}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) = \begin{pmatrix} \frac{1}{3} - \frac{\mu - m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu - m}{2d} \end{pmatrix} .$$

Notice that the frontier portfolio will hold long postitions in all three assets when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. It will hold a short position in the first asset when $\mu > m + \frac{2}{3}d$, and a short position in the third asset when $\mu < m - \frac{2}{3}d$. In particular, in order to create a portfolio with return mean μ greater than that of any asset contained within it you must short sell the asset with the lowest return mean and invest the proceeds into the asset with the highest return mean. The fact that $\mathbf{f}_{\mathrm{mf}}(\mu)$ is independent of $\mathbf{V}$ is a consequence of the simple forms of both $\mathbf{V}$ and $\mathbf{m}$. This is also why the fraction of the investment in the second asset is a constant.

Simple Portfolio with Three Risky Assets

Remark. The frontier portfolios for this example are independent of all the parameters in **V**. While this is not generally true, it is generally true that they are independent of the overall market volatility. *Said another way, the frontier portfolios depend only upon the correlations c_{ij}, the volatility ratios σ_i/σ_j, and the means m_i. Moreover, the minimum volatility portfolio $\mathbf{f}_{\mathrm{mv}}$ depends only upon the correlations and the volatility ratios.* Because markets can exhibit periods of markedly different volatility, it is natural to ask when correlations and volatility ratios might be relatively stable across such periods.