Portfolios that Contain Risky Assets
10.4. Fortune's Formulas

C. David Levermore

University of Maryland, College Park, MD

Math 420: *Mathematical Modeling*
April 27, 2022 version

Portfolios that Contain Risky Assets
Part II: Probabilistic Models

Portfolios that Contain Risky Assets
Part II: Probabilistic Models

10. Cautious Objectives for Portfolio Models

Fortune's Formulas

Introduction

Given a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ for N risky assets, a choice of positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, and using the *one risk-free rate model* with risk-free rate μ_{rf}, a cautious investor might select a risky asset allocation $\mathbf{f}$ from a set $\Pi \subset \mathcal{M}_+$ that maximizes a cautious objective

$$\widehat{\Gamma}^{\chi}(\mathbf{f}) = \widehat{\gamma}(\mathbf{f}) - \chi\sqrt{\widehat{\theta}(\mathbf{f})}\,, \tag{1.1a}$$

where $\chi \geq 0$ is a caution coefficient chosen by the investor,

$$\begin{aligned} \widehat{\gamma}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \log\Big(1 + \mu_{\mathrm{rf}} + \big(\mathbf{r}(d) - \mu_{\mathrm{rf}}\mathbf{1}\big)^{\mathrm{T}}\mathbf{f}\Big)\,, \\ \widehat{\theta}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \left(\log\Big(1 + \mu_{\mathrm{rf}} + \big(\mathbf{r}(d) - \mu_{\mathrm{rf}}\mathbf{1}\big)^{\mathrm{T}}\mathbf{f}\Big) - \hat{\gamma}(\mathbf{f})\right)^2 . \end{aligned} \tag{1.1b}$$

Introduction

We now consider some settings in which mean-variance approximations to this optimization problem can be solved analytically. These approximations replace the objective (1.1) with estimators that depend only on:

- the risk-free rate μ_{rf},
- the return mean vector $\mathbf{m}$,
- the return covariance matrix $\mathbf{V}$,
- the nonnegative caution coefficient χ,

where $\mathbf{m}$ and $\mathbf{V}$ are obtained from the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ by

$$\mathbf{m} = \sum_{d=1}^{D} w_d \, \mathbf{r}(d) \,, \qquad \mathbf{V} = \sum_{d=1}^{D} w_d \left(\mathbf{r}(d) - \mathbf{m}\right)\left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathrm{T}} . \tag{1.2}$$

Introduction

In the previous section we saw that the maximizer $\mathbf{f}_*$ for such a problem corresponds to a point (σ_*, μ_*) on the efficient frontier. Moreover, we saw that (σ_*, μ_*) is the point in the $\sigma\mu$-plane where the level curves of the objective are tangent to the efficient frontier. While this geometric picture gave insight into how optimal portfolio allocations arise, the form of the approximations that we used made comparing the results messy.

In this section we:

- identify a symmetry in the one risk-free rate model,
- derive some new mean-variance approximations of the family of cautious objectives (1.1) that respect this symmetry,
- solve the maximization problem for these new objectives over their natural domains,
- compare the results and draw some lessons from these comparisons.

Introduction

The explicit formulas for the maximizer $\mathbf{f}_*$ that we derive will confirm the general picture developed in the previous section. Moreover, the symmetry preserving properties of the new approximations facilitate comparisons and allow us to gain insights into the relative merits of the different families of approximate objectives. We will see that

- the maximizers when $\chi = 0$ give different realizations of the Kelly Criterion — so-called *fortune's formulas*;
- the maximizers when $\chi > 0$ give different realizations of fractional Kelly strategies.

We will derive and analyze these formulas after reviewing Markowitz and Tobin frontiers.

Frontiers (Markowitz)

Recall that the Markowitz frontier is the hyperbola in the right-half of the $\sigma\mu$-plane given by

$$\sigma = \sqrt{\sigma_{\rm mv}^2 + \left(\frac{\mu - \mu_{\rm mv}}{\nu_{\rm mv}}\right)^2}\,, \tag{2.3a}$$

where the *frontier parameters* $\sigma_{\rm mv}$, $\mu_{\rm mv}$ and $\nu_{\rm mv}$ are determined by

$$\frac{1}{\sigma_{\rm mv}^2} = \mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}\,, \qquad \mu_{\rm mv} = \frac{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{m}}{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}}\,,$$
$$\nu_{\rm mv}^2 = (\mathbf{m} - \mu_{\rm mv}\mathbf{1})^{\rm T}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm mv}\mathbf{1})\,. \tag{2.3b}$$

The hyperbola given by (2.3a) has vertex $(\sigma_{\rm mv}, \mu_{\rm mv})$ and asymptotes

$$\mu = \mu_{\rm mv} \pm \nu_{\rm mv}\,\sigma \qquad \text{for } \sigma \geq 0\,.$$

Frontiers (Tobin)

If risk-free assets are included using the one risk-free rate model with risk-free rate μ_{rf} then the *Tobin frontier* is the union of the two half-lines given by

$$\mu = \mu_{\mathrm{rf}} \pm \nu_{\mathrm{rf}}\, \sigma \qquad \text{for } \sigma \geq 0\,, \tag{2.4a}$$

where the *frontier parameter* ν_{rf} is determined by

$$\nu_{\mathrm{rf}}^2 = (\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1})^{\mathrm{T}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1})\,. \tag{2.4b}$$

and satisfies the *frontier parameter relation*,

$$\nu_{\mathrm{rf}}^2 = \nu_{\mathrm{mv}}^2 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}} \right)^2 . \tag{2.4c}$$

It is also the *Sharpe ratio* of every portfolio on the *efficient Tobin frontier*.

Frontiers (Tangent Portfolios)

When $\mu_{\rm rf} \neq \mu_{\rm mv}$ the Tobin frontier (2.4a) is tangent to the Markowitz frontier (2.3a) at the point $(\sigma_{\rm tg}, \mu_{\rm tg})$ given by

$$\sigma_{\rm tg} = \sigma_{\rm mv} \sqrt{1 + \left(\frac{\nu_{\rm rf}\, \sigma_{\rm mv}}{\mu_{\rm mv} - \mu_{\rm rf}} \right)^2} \,, \qquad \mu_{\rm tg} = \mu_{\rm mv} + \frac{\nu_{\rm mv}^2\, \sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} \,.$$

The unique *tangency portfolio* associated with this point has allocation

$$\mathbf{f}_{\rm tg} = \frac{\sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} \mathbf{V}^{-1} (\mathbf{m} - \mu_{\rm rf} \mathbf{1}) \,. \tag{2.5}$$

When $\mu_{\rm rf} \neq \mu_{\rm mv}$ every portfolio on the efficient Tobin frontier can be viewed as holding a position in this tangency portfolio and a position in a risk-free asset.

Mean-Variance Approximations

We can select a portfolio on the efficient Tobin frontier by maximizing a mean-variance objective that approximates the cautious objective (1.1). These objectives are contructed by replacing the $\widehat{\gamma}(\mathbf{f})$ and $\widehat{\theta}(\mathbf{f})$ that appear in (1.1a) and that are defined by (1.1b) with mean-variance estimators that depend only on:

- the risk-free rate μ_{rf},
- the return mean vector $\mathbf{m}$,
- the return covariance matrix $\mathbf{V}$.

Here we study three such approximations. Each of these approximations will respect an important symmetry of the cautious objective. This symmetry becomes evident upon rewriting the cautious objective.

Mean-Variance Approximations

The growth rate of the portfolio with allocation $\mathbf{f}$ on day d is

$$\log\Big(1+\mu_{\text{rf}}+(\mathbf{r}(d)-\mu_{\text{rf}}\mathbf{1})^{\text{T}}\mathbf{f}\Big)=\log(1+\mu_{\text{rf}})+\log\Big(1+\tilde{\mathbf{r}}(d)^{\text{T}}\mathbf{f}\Big)\,, \quad (3.6)$$

where *relative return* vector $\tilde{\mathbf{r}}(d)$ is defined by

$$\tilde{\mathbf{r}}(d)=\frac{1}{1+\mu_{\text{rf}}}\,(\mathbf{r}(d)-\mu_{\text{rf}}\mathbf{1})\,. \quad (3.7)$$

The i^{th} entry of $\tilde{\mathbf{r}}(d)$ is the *relative return* of asset i on day d with respect to the risk-free rate μ_{rf}. The so-called *relative growth rate* of the portfolio with allocation $\mathbf{f}$ on day d is

$$\log\Big(1+\tilde{\mathbf{r}}(d)^{\text{T}}\mathbf{f}\Big)\,. \quad (3.8)$$

It is the growth rate of the portfolio relative to that of the safe investment.

Mean-Variance Approximations

It then follows from (1.1b) and (3.6) that

$$\widehat{\gamma}(\mathbf{f}) = \log(1+\mu_{\mathrm{rf}}) + \tilde{\gamma}(\mathbf{f}) \,, \qquad \widehat{\theta}(\mathbf{f}) = \tilde{\theta}(\mathbf{f}) \,, \tag{3.9a}$$

where $\tilde{\gamma}(\mathbf{f})$ and $\tilde{\theta}(\mathbf{f})$ are the *relative growth rate mean* and *variance* that we see from (3.8) are given by

$$\begin{aligned} \tilde{\gamma}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \log\left(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\right) \,, \\ \tilde{\theta}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \left(\log\left(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\right) - \tilde{\gamma}(\mathbf{f})\right)^2 \,. \end{aligned} \tag{3.9b}$$

Notice that these depend only on the relative return history $\{\tilde{\mathbf{r}}(d)\}_{d=1}^{D}$.

Mean-Variance Approximations

It then follows from (1.1a) that the cautious objective can be rewritten as

$$\widehat{\Gamma}^{\chi}(\mathbf{f}) = \log(1+\mu_{\mathrm{rf}}) + \widetilde{\Gamma}^{\chi}(\mathbf{f}), \tag{3.10a}$$

where

$$\widetilde{\Gamma}^{\chi}(\mathbf{f}) = \tilde{\gamma}(\mathbf{f}) - \chi\sqrt{\tilde{\theta}(\mathbf{f})}, \tag{3.10b}$$

where $\tilde{\gamma}(\mathbf{f})$ and $\tilde{\theta}(\mathbf{f})$ are defined by (3.9b).

The natural domain for $\tilde{\gamma}(\mathbf{f})$, $\tilde{\theta}(\mathbf{f})$ and $\widetilde{\Gamma}^{\chi}(\mathbf{f})$ is the same as that for $\widehat{\gamma}(\mathbf{f})$, $\widehat{\theta}(\mathbf{f})$ and $\widehat{\Gamma}^{\chi}(\mathbf{f})$, — namely, the set of solvent Markowitz allocations Ω_+. We see from definition (3.7) of $\tilde{\mathbf{r}}(d)$ that for every $\mathbf{f} \in \mathcal{M}_+$ we have

$$1 + \mu_{\mathrm{rf}} + (\mathbf{r}(d) - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{f} = (1+\mu_{\mathrm{rf}})\left(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\right),$$

which implies that

$$\Omega_+ = \left\{\mathbf{f} \in \mathcal{M}_+ \,:\, 1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f} > 0 \;\; \forall d\right\}.$$

Mean-Variance Approximations

We now collect three observations that will shape how our mean-variance approximations are constructed.

- Because $\tilde{\gamma}(\mathbf{f})$ and $\tilde{\theta}(\mathbf{f})$ defined by (3.9b) depend only on the relative return history $\{\tilde{\mathbf{r}}(d)\}_{d=1}^{D}$, we see that $\widetilde{\Gamma}^{\chi}(\mathbf{f})$ defined by (3.10) depends only on the relative return history $\{\tilde{\mathbf{r}}(d)\}_{d=1}^{D}$ and χ.
- Because by (3.10a) we have

$$\widehat{\Gamma}^{\chi}(\mathbf{f}) = \log(1 + \mu_{\mathrm{rf}}) + \widetilde{\Gamma}^{\chi}(\mathbf{f}) \, ,$$

  we see that maximizers of $\widehat{\Gamma}^{\chi}(\mathbf{f})$ are maximizers of $\widetilde{\Gamma}^{\chi}(\mathbf{f})$.
- Therefore these maximizers can depend only on the relative return history $\{\tilde{\mathbf{r}}(d)\}_{d=1}^{D}$ and χ.

Mean-Variance Approximations

If a mean-variance approximation of $\widehat{\Gamma}^{\chi}(\mathbf{f})$ is going to preserve this symmetry then it should have a maximizer that depends only on:

- the *relative return mean* vector $\tilde{\mathbf{m}}$,
- the *relative return covariance* matrix $\tilde{\mathbf{V}}$,
- the nonnegative caution coefficient χ,

where $\tilde{\mathbf{m}}$ and $\tilde{\mathbf{V}}$ are obtained from the relative return history $\{\tilde{\mathbf{r}}(d)\}_{d=1}^{D}$ by

$$\tilde{\mathbf{m}} = \sum_{d=1}^{D} w_d \, \tilde{\mathbf{r}}(d) \,, \qquad \tilde{\mathbf{V}} = \sum_{d=1}^{D} w_d \left(\tilde{\mathbf{r}}(d) - \tilde{\mathbf{m}}\right) \left(\tilde{\mathbf{r}}(d) - \tilde{\mathbf{m}}\right)^{\mathrm{T}} . \tag{3.11}$$

It then follows from the relation (3.7) between $\tilde{\mathbf{r}}(d)$ and $\mathbf{r}(d)$, and from the definitions (1.2) of $\mathbf{m}$ and $\mathbf{V}$ that

$$\tilde{\mathbf{m}} = \frac{1}{1+\mu_{\mathrm{rf}}} \left(\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1}\right) , \qquad \tilde{\mathbf{V}} = \frac{1}{(1+\mu_{\mathrm{rf}})^2} \mathbf{V} . \tag{3.12}$$

Mean-Variance Approximations

Because by definition (3.10b)

$$\widetilde{\Gamma}^{\chi}(\mathbf{f}) = \tilde{\gamma}(\mathbf{f}) - \chi \sqrt{\tilde{\theta}(\mathbf{f})} \,,$$

we can construct mean-variance approximations of $\widetilde{\Gamma}^{\chi}(\mathbf{f})$ that have a maximizer that depends only on $\widetilde{\mathbf{m}}$, $\widetilde{\mathbf{V}}$ and χ by constucting mean-variance approximations of $\tilde{\gamma}(\mathbf{f})$ and $\tilde{\theta}(\mathbf{f})$ that depend only on $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{V}}$.

We see from definitions (1.1), (3.9b) and (3.10b) that

$$\widehat{\gamma}(\mathbf{f}), \qquad \widehat{\theta}(\mathbf{f}), \qquad \widehat{\Gamma}^{\chi}(\mathbf{f}) \,,$$

with $\mu_{\mathrm{rf}} = 0$ and $\mathbf{r}(d)$ replaced by $\tilde{\mathbf{r}}(d)$ are the same as

$$\tilde{\gamma}(\mathbf{f}), \qquad \tilde{\theta}(\mathbf{f}), \qquad \widetilde{\Gamma}^{\chi}(\mathbf{f}) \,.$$

Therefore mean-variance approximations of $\tilde{\gamma}(\mathbf{f})$ and $\tilde{\theta}(\mathbf{f})$ that depend only on $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{V}}$ can be constructed by adapting mean-variance approximations of $\widehat{\gamma}(\mathbf{f})$ and $\widehat{\theta}(\mathbf{f})$ by setting $\mu_{\mathrm{rf}} = 0$ and replacing $\mathbf{m}$ and $\mathbf{V}$ with $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{V}}$.

Mean-Variance Approximations

We will explicitly solve the maximization problems for the families of parabolic, quadratic and reasonable objectives:

$$\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,, \tag{3.13a}$$

$$\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,, \tag{3.13b}$$

$$\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,. \tag{3.13c}$$

These objectives have the natural domains Ω_{p}, Ω_{q} and Ω_{r} given by

$$\Omega_{\mathrm{p}} = \Pi_{\mathrm{q}} = \mathcal{M}_{+} = \mathbb{R}^N\,, \qquad \Omega_{\mathrm{r}} = \left\{\mathbf{f} \in \mathcal{M}_{+} \,:\, 1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} > 0\right\}. \tag{3.14}$$

Each of these domains is a convex subset of $\mathbb{R}^N$. Each of the objectives given in (3.13) is a strictly concave function over its natural domain. We will find the unique maximizer of each objective over its natural domain.

Mean-Variance Approximations

Before solving the maximization problems, we collect two facts that will be used in the analysis of them.

First, we see from (2.4b) and (3.12) that the Sharpe ratio satisfies

$$\nu_{\mathrm{rf}}^2 = (\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}\,(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{3.15}$$

Second, for every $\mathbf{f} \in \mathbb{R}^N$ we have the *Cauchy inequality*

$$\left|\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right| \leq \nu_{\mathrm{rf}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,. \tag{3.16}$$

Indeed, the Cauchy inequality for the $\widetilde{\mathbf{V}}$-scalar product $(\mathbf{f}\,|\,\mathbf{g})_{\widetilde{\mathbf{V}}} = \mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{g}$ and relation (3.15) imply that

$$\begin{aligned}\left|\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right| &= \left|\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{V}}\,\mathbf{f}\right| = \left|\left(\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,\middle|\,\mathbf{f}\right)_{\widetilde{\mathbf{V}}}\right| \\ &\leq \left\|\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\right\|_{\widetilde{\mathbf{V}}}\|\mathbf{f}\|_{\widetilde{\mathbf{V}}} = \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}\sqrt{\mathbf{f}\widetilde{\mathbf{V}}\,\mathbf{f}} = \nu_{\mathrm{rf}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,.\end{aligned}$$

Parabolic Objectives

First we consider the maximization problem

$$\mathbf{f}_* = \arg\max\left\{ \widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N \right\}, \tag{4.17a}$$

where $\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f})$ is the family of parabolic objectives (3.13a) given by

$$\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,. \tag{4.17b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}} - \widetilde{\mathbf{V}}\,\mathbf{f} - \frac{\chi}{\sigma}\,\widetilde{\mathbf{V}}\,\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} > 0$.

Parabolic Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = \widetilde{\mathbf{m}} - \frac{\sigma_* + \chi}{\sigma_*} \widetilde{\mathbf{V}}\, \mathbf{f}_* \,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_*} > 0$. Upon solving this equation for $\mathbf{f}_*$ we obtain

$$\mathbf{f}_* = \frac{\sigma_*}{\sigma_* + \chi} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \,. \tag{4.18}$$

All that remains is to determine σ_*.

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_* = \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} = \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \nu_{\mathrm{rf}}^2 \,.$$

Parabolic Objectives

We conclude that σ_* satisfies

$$(\sigma_* + \chi)^2 = \nu_{\mathrm{rf}}^2 .$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that χ must satisfy the bounds

$$0 \leq \chi < \nu_{\mathrm{rf}} , \tag{4.19}$$

and that σ_* is determined by

$$\sigma_* + \chi = \nu_{\mathrm{rf}} .$$

Then the maximizer $\mathbf{f}_*$ given by (4.18) becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\mathrm{rf}}}\right) \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} . \tag{4.20}$$

Parabolic Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\mathrm{rf}}$. In that case the positive definiteness of $\widetilde{\mathbf{V}}$, the fact $\chi \geq \nu_{\mathrm{rf}}$ and the *Cauchy inequality* (3.16) imply for every $\mathbf{f} \in \Omega_{\mathrm{p}}$ that

$$
\begin{aligned}
\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) &= \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \nu_{\mathrm{rf}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \leq 0 = \widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{0})\,.
\end{aligned}
$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\mathrm{rf}}$.

Parabolic Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (4.17) is

$$\mathbf{f}_{*\mathrm{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{rf}}}\right) \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathrm{rf}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{rf}} \,. \end{cases} \tag{4.21}$$

This solution is always an efficient Tobin frontier portfolio.
When $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ and $\chi < \nu_{\mathrm{rf}}$ it allocates

- f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ given by (2.5),
- and $(1 - f_{\mathrm{tg}}^{\chi})$ times the portfolio value in a risk-free asset,

where

$$f_{\mathrm{tg}}^{\chi} = \mathbf{1}^{\mathrm{T}} \mathbf{f}_{*\mathrm{p}} = \left(1 - \frac{\chi}{\nu_{\mathrm{rf}}}\right) (1 + \mu_{\mathrm{rf}}) \, \frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}^2} \,.$$

Parabolic Objectives

Remark. Kelly investors take $\chi = 0$, in which case (4.21) reduces to

$$\mathbf{f}_{*\mathrm{p}} = \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{4.22}$$

This is often called *fortune's formula* in the belief that it is a good approximation to the Kelly strategy. In this view formula (4.21) gives a fractional Kelly strategy for every $\chi \in (0, \nu_{\mathrm{rf}})$. However, we will see that formula (4.22) gives an allocation that can be far from the Kelly strategy, and can lead to overbetting.

Quadratic Objectives

Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\left\{\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N\right\}, \tag{5.23a}$$

where $\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f})$ is the family of quadratic objectives (3.13b) given by

$$\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,. \tag{5.23b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}} - \widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \widetilde{\mathbf{V}}\,\mathbf{f} - \frac{\chi}{\sigma}\,\widetilde{\mathbf{V}}\,\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} > 0$.

Quadratic Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = \widetilde{\mathbf{m}} - \widetilde{\mathbf{m}}\, \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* - \frac{\sigma_* + \chi}{\sigma_*} \widetilde{\mathbf{V}}\, \mathbf{f}_* \,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_*} > 0$.

After multiplying this relation by $\widetilde{\mathbf{V}}^{-1}$ and bringing the terms involving $\mathbf{f}_*$ to the left-hand side, we obtain

$$\frac{\sigma_* + \chi}{\sigma_*} \mathbf{f}_* + \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}}\, \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* = \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \,. \tag{5.24}$$

Quadratic Objectives

Now multiply this by $\sigma_* \, \mathbf{m}^{\mathrm{T}}$ and use the *Sharpe ratio* formula (3.15), $\widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} = \nu_{\mathrm{rf}}^2$, to obtain

$$\left(\sigma_* + \chi + \nu_{\mathrm{rf}}^2 \, \sigma_*\right) \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* = \nu_{\mathrm{rf}}^2 \, \sigma_* \, ,$$

which implies that

$$\widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* = \frac{\nu_{\mathrm{rf}}^2 \, \sigma_*}{\sigma_* + \chi + \nu_{\mathrm{rf}}^2 \, \sigma_*} \, .$$

When this expression is placed into (5.24) we can solve for $\mathbf{f}_*$ to find

$$\mathbf{f}_* = \frac{\sigma_*}{\sigma_* + \chi + \nu_{\mathrm{rf}}^2 \, \sigma_*} \, \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \, . \tag{5.25}$$

All that remains is to determine σ_*.

Quadratic Objectives

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_* = \frac{\sigma_*^2}{\left((1+\nu_{\mathrm{rf}}^2)\,\sigma_* + \chi\right)^2}\, \widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}}$$
$$= \frac{\sigma_*^2}{\left((1+\nu_{\mathrm{rf}}^2)\,\sigma_* + \chi\right)^2}\, \nu_{\mathrm{rf}}^2 \,,$$

we conclude that σ_* satisfies

$$\left((1+\nu_{\mathrm{rf}}^2)\,\sigma_* + \chi\right)^2 = \nu_{\mathrm{rf}}^2 \,.$$

Quadratic Objectives

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that χ must satisfy the bounds

$$0 \leq \chi < \nu_{\mathrm{rf}} \,, \tag{5.26}$$

and that σ_* is determined by

$$\left(1 + \nu_{\mathrm{rf}}^2\right) \sigma_* + \chi = \nu_{\mathrm{rf}} \,.$$

Therefore the maximizer $\mathbf{f}_*$ given by (5.25) becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\mathrm{rf}}}\right) \frac{1}{1 + \nu_{\mathrm{rf}}^2} \, \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \,. \tag{5.27}$$

Quadratic Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\mathrm{rf}}$. In that case the positive definiteness of $\widetilde{\mathbf{V}}$, the fact $\chi \geq \nu_{\mathrm{rf}}$ and the *Cauchy inequality* (3.16) imply for every $\mathbf{f} \in \Omega_{\mathrm{q}}$ that

$$\begin{aligned}
\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) &= \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \nu_{\mathrm{rf}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} \leq 0 = \widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{0})\,.
\end{aligned}$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\mathrm{rf}}$.

Quadratic Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (5.23) is

$$\mathbf{f}_{*q} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\rm rf}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{1+\nu_{\rm rf}^2} & \text{if } \chi < \nu_{\rm rf}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\rm rf}\,. \end{cases} \tag{5.28}$$

This solution is always an efficient Tobin frontier portfolio.
When $\mu_{\rm rf} \neq \mu_{\rm mv}$ and $\chi < \nu_{\rm rf}$ it allocates

- $f_{\rm tg}^{\chi}$ times the portfolio value in the tangent portfolio $\mathbf{f}_{\rm tg}$ given by (2.5),
- and $(1 - f_{\rm tg}^{\chi})$ times the portfolio value in a risk-free asset,

where

$$f_{\rm tg}^{\chi} = \mathbf{1}^{\rm T}\mathbf{f}_{*q} = \left(1 - \frac{\chi}{\nu_{\rm rf}}\right) \frac{1+\mu_{\rm rf}}{1+\nu_{\rm rf}^2}\, \frac{\mu_{\rm mv}-\mu_{\rm rf}}{\sigma_{\rm mv}^2}\,.$$

Quadratic Objectives

Remark. Kelly investors take $\chi = 0$, in which case (5.28) reduces to

$$\mathbf{f}_*^{\mathrm{q}} = \frac{1}{1+\nu_{\mathrm{rf}}^2}\,\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{5.29}$$

Formula (5.29) differs significantly from formula (4.22) whenever the Sharpe ratio ν_{rf} is not small. Sharpe ratios are often near 1 and sometimes can be as large as 3. So which of these should be called *fortune's formula*? Certainly not formula (4.22)! To see why, set $\chi = 0$ and $\mathbf{f} = \mathbf{f}_{*\mathrm{p}} = \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}$ into the quadratic objective (5.23b) to obtain

$$\widetilde{\Gamma}_{\mathrm{q}}^{0}(\mathbf{f}_{*\mathrm{p}}) = \tfrac{1}{2}\,\nu_{\mathrm{rf}}^2 - \tfrac{1}{2}\,\nu_{\mathrm{rf}}^4\,,$$

which is negative when $\nu_{\mathrm{rf}} > 1$. So formula (4.22) might overbet!

Reasonable Objectives

Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\left\{ \widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N ,\, 1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} > 0 \right\}, \tag{6.30a}$$

where $\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f})$ is the family of reasonable objectives (3.13c) given by

$$\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}}\,. \tag{6.30b}$$

Because $\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) \to -\infty$ as $\mathbf{f}$ approaches the boundary of the domain being considered in (6.30a), the maximizer must lie in the interior of the domain. If $\mathbf{f} \neq 0$ then the gradient of $\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) = \frac{1}{1+\mu}\,\widetilde{\mathbf{m}} - \widetilde{\mathbf{V}}\mathbf{f} - \frac{\chi}{\sigma}\,\widetilde{\mathbf{V}}\mathbf{f}\,,$$

where $\mu = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}$ and $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} > 0$.

Reasonable Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{f}_* = \frac{1}{1+\mu_*}\,\frac{\sigma_*}{\sigma_*+\chi}\,\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,, \tag{6.31}$$

where $\mu_* = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_*$ and $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}_*} > 0$.

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}_*}$ we have

$$\begin{aligned}\sigma_*^{\,2} = \mathbf{f}_*^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}_* &= \frac{1}{(1+\mu_*)^2}\,\frac{\sigma_*^{\,2}}{(\sigma_*+\chi)^2}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} \\ &= \frac{1}{(1+\mu_*)^2}\,\frac{\sigma_*^{\,2}}{(\sigma_*+\chi)^2}\,\nu_{\mathrm{rf}}^2\,.\end{aligned}$$

Reasonable Objectives

From this we conclude that μ_* and σ_* satisfy

$$(\sigma_* + \chi)^2 = \frac{\nu_{\mathrm{rf}}^2}{(1+\mu_*)^2} .$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < \frac{\nu_{\mathrm{rf}}}{1+\mu_*} , \tag{6.32}$$

and that we can determine σ_* in terms of μ_* from

$$\sigma_* + \chi = \frac{\nu_{\mathrm{rf}}}{1+\mu_*} .$$

Then the maximizer $\mathbf{f}_*$ given by (6.31) becomes

$$\mathbf{f}_* = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathrm{rf}}} \right) \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} , \tag{6.33}$$

Reasonable Objectives

Because $\mu_* = \mathbf{m}^{\mathrm{T}}\mathbf{f}_*$, by the *Sharpe ratio* formula (3.15) we have

$$\mu_* = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathrm{rf}}}\right)\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathrm{rf}}}\right)\nu_{\mathrm{rf}}^2 .$$

This can be reduced to the quadratic equation

$$\left(\frac{\nu_{\mathrm{rf}}}{1+\mu_*}\right)^2 + \left(\frac{1}{\nu_{\mathrm{rf}}} - \chi\right)\frac{\nu_{\mathrm{rf}}}{1+\mu_*} = 1 ,$$

which has the unique positive root

$$\frac{\nu_{\mathrm{rf}}}{1+\mu_*} = -\frac{1}{2}\left(\frac{1}{\nu_{\mathrm{rf}}} - \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1}{\nu_{\mathrm{rf}}} - \chi\right)^2} . \qquad (6.34)$$

Reasonable Objectives

Then condition (6.32) is satisfied if and only if

$$0 < \frac{\nu_{\rm rf}}{1+\mu_*} - \chi = -\frac{1}{2}\left(\frac{1}{\nu_{\rm rf}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1}{\nu_{\rm rf}} - \chi\right)^2}\,.$$

This inequality holds if and only if

$$0 < 1 + \frac{1}{4}\left(\frac{1}{\nu_{\rm rf}} - \chi\right)^2 - \frac{1}{4}\left(\frac{1}{\nu_{\rm rf}} + \chi\right)^2 = 1 - \frac{\chi}{\nu_{\rm rf}}\,.$$

This holds if and only if χ satisfies the bounds

$$0 \leq \chi < \nu_{\rm rf}\,. \tag{6.35}$$

Reasonable Objectives

By using (6.34) to eliminate μ_* from the maximizer $\mathbf{f}_*$ given by (6.33) we find

$$\mathbf{f}_* = \left[-\frac{1}{2}\left(\frac{1}{\nu_{\text{rf}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1}{\nu_{\text{rf}}} - \chi\right)^2}\,\right] \frac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{\nu_{\text{rf}}} \,.$$

This becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\text{rf}}}\right) \frac{1}{D(\chi, \nu_{\text{rf}})} \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} \,, \qquad (6.36a)$$

where

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2} \,. \qquad (6.36b)$$

Reasonable Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\mathrm{rf}}$. In that case the fact that

$$\log(1+r) \leq r \qquad \text{for every } r \in (-1, \infty)\,,$$

the positive definiteness of $\widetilde{\mathbf{V}}$, the fact $\chi \geq \nu_{\mathrm{rf}}$ and the *Cauchy inequality* (3.16) imply for every $\mathbf{f} \in \omega_{\mathrm{r}}$ that

$$\begin{aligned} \widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) &= \log\left(1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} \\ &\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} \\ &\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \nu_{\mathrm{rf}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\,\mathbf{f}} \leq 0 = \widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{0})\,. \end{aligned}$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\mathrm{rf}}$.

Reasonable Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (6.30) is

$$\mathbf{f}_{*\mathrm{r}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{rf}}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{D(\chi, \nu_{\mathrm{rf}})} & \text{if } \chi < \nu_{\mathrm{rf}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{rf}} \,, \end{cases} \tag{6.37}$$

where $D(\chi, y)$ was defined by (6.36b).

This solution is always an efficient Tobin frontier portfolio.
When $\mu_{\mathrm{rf}} \neq \mu_{\mathrm{mv}}$ and $\chi < \nu_{\mathrm{rf}}$ it allocates

- f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ given by (2.5),
- and $(1 - f_{\mathrm{tg}}^{\chi})$ times the portfolio value in a risk-free asset,

where

$$f_{\mathrm{tg}}^{\chi} = \mathbf{1}^{\mathrm{T}}\mathbf{f}_{*\mathrm{r}} = \left(1 - \frac{\chi}{\nu_{\mathrm{rf}}}\right) \frac{1 + \mu_{\mathrm{rf}}}{D(\chi, \nu_{\mathrm{rf}})} \, \frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}^2} \,.$$

Reasonable Objectives

Remark. Kelly investors take $\chi = 0$, in which case (6.37) reduces to

$$\mathbf{f}_{*\mathrm{r}} = \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{rf}}^2}} \, \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} \,. \tag{6.38}$$

This candidate for *fortune's formula* will be compared with the others later.

Remark. Further evidence that for Kelly investors the parabolic maximizer (4.22) can overbet is seen by setting $\chi = 0$ and $\mathbf{f} = \mathbf{f}_{*\mathrm{p}} = \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}$ in the reasonable objective (6.30b) to obtain

$$\widetilde{\Gamma}_{\mathrm{r}}^{0}(\mathbf{f}_{*\mathrm{p}}) = \log\left(1 + \nu_{\mathrm{rf}}^2\right) - \tfrac{1}{2}\,\nu_{\mathrm{rf}}^2 \,,$$

which is negative when $\nu_{\mathrm{rf}} > 1.59$. So formula (4.22) might overbet!

Comparisons

The maximizers for the parabolic, quadratic, and reasonable objectives are given by (4.21), (5.28), and (6.37) respectively. They are

$$\mathbf{f}_{*\mathrm{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{rf}}}\right) \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathrm{rf}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{rf}}\,, \end{cases} \tag{7.39a}$$

$$\mathbf{f}_{*\mathrm{q}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{rf}}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{1 + \nu_{\mathrm{rf}}^2} & \text{if } \chi < \nu_{\mathrm{rf}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{rf}}\,, \end{cases} \tag{7.39b}$$

$$\mathbf{f}_{*\mathrm{r}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{rf}}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{D(\chi, \nu_{\mathrm{rf}})} & \text{if } \chi < \nu_{\mathrm{rf}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{rf}}\,. \end{cases} \tag{7.39c}$$

where $D(\chi, y)$ was defined by (6.36b).

Comparisons

Fact 1. $\mathbf{f}_{*\mathrm{q}}$ is the most conservative and $\mathbf{f}_{*\mathrm{p}}$ is the most agressive.

Proof. Recall from (6.36b) that

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2}\,. \tag{7.40}$$

This is a strictly increasing function of χ because for every $y > 0$ we have

$$\partial_\chi D(\chi, y) = \tfrac{1}{2}y\left(1 - \frac{1 - \chi\, y}{\sqrt{(1 - \chi\, y)^2 + 4y^2}}\right) > 0\,.$$

Hence, for every $\chi \in [0, y)$ we have

$$1 < D(0, y) \leq D(\chi, y) < D(y, y) = 1 + y^2\,. \tag{7.41}$$

Therefore $1 < D(\chi, \nu_{\mathrm{rf}}) < 1 + \nu_{\mathrm{rf}}^2$ when $\chi < \nu_{\mathrm{rf}}$.

Comparisons

We now compare the dependence of $\mathbf{f}_{*\mathrm{q}}$ and $\mathbf{f}_{*\mathrm{r}}$ upon χ and ν_{rf}.

Fact 2. For every $\chi \in [0, \nu_{\mathrm{rf}})$ we have

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{rf}}^2}}{1 + \nu_{\mathrm{rf}}^2} \leq \frac{D(\chi, \nu_{\mathrm{rf}})}{1 + \nu_{\mathrm{rf}}^2} < 1\,, \tag{7.42}$$

where the left-hand side is a strictly decreasing function of ν_{rf}.

Proof. By setting $y = \nu_{\mathrm{rf}}$ in (7.41) we obtain

$$1 + \nu_{\mathrm{rf}}^2 > D(\chi, \nu_{\mathrm{rf}}) \geq D(0, \nu_{\mathrm{rf}}) = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{rf}}^2}\,.$$

The inequalities (7.42) follow. The task of proving the left-hand side of (7.42) is a strictly decreasing function of ν_{rf} is left as an exercise. □

Comparisons

We now use **Fact 2** to show that $\mathbf{f}_{*\mathrm{q}}$ and $\mathbf{f}_{*\mathrm{p}}$ are close when $\nu_{\mathrm{rf}} \leq \frac{2}{3}$.

Fact 3. If $\nu_{\mathrm{rf}} \leq \frac{2}{3}$ then for every $\chi \in [0, \nu_{\mathrm{rf}})$ we have

$$\frac{12}{13} \leq \frac{D(\chi, \nu_{\mathrm{rf}})}{1 + \nu_{\mathrm{rf}}^2} < 1\,. \tag{7.43}$$

Proof. By the monotonicity asserted in **Fact 2** if $\nu_{\mathrm{rf}} \leq \frac{2}{3}$ then

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{rf}}^2}}{1 + \nu_{\mathrm{rf}}^2} \geq \frac{\frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3}}{1 + \frac{4}{9}} = \frac{\frac{4}{3}}{\frac{13}{9}} = \frac{12}{13}\,.$$

Then (7.43) follows from inequality (7.42) of **Fact 2**. □

Comparisons

Remark. We see from (7.39) that when $\chi = 0$

$$\mathbf{f}_{*q} = \frac{1}{1+\nu_{rf}^2}\,\mathbf{f}_{*p}\,, \qquad \mathbf{f}_{*r} = \frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_{rf}^2}}\,\mathbf{f}_{*p}\,.$$

This is the case when the difference between $\mathbf{f}_{*q}$ and $\mathbf{f}_{*r}$ is at its greatest. To get a feel for this difference, when $\nu_{rf} = \sqrt{2}$ these are

$$\mathbf{f}_{*q} = \tfrac{1}{3}\,\mathbf{f}_{*p}\,, \qquad \mathbf{f}_{*r} = \tfrac{1}{2}\,\mathbf{f}_{*p}\,,$$

while when $\nu_{rf} = \sqrt{6}$ these are

$$\mathbf{f}_{*q} = \tfrac{1}{7}\,\mathbf{f}_{*p}\,, \qquad \mathbf{f}_{*r} = \tfrac{1}{3}\,\mathbf{f}_{*p}\,.$$

We see that this difference becomes quite large for Sharpe ratios $\nu_{rf} > 2$.

Comparisons

Finally, we compare the maximizers found here with those found in the previous section. First, observe that the maximizers given by (7.39) each have the form

$$\mathbf{f}_* = \alpha\, \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \qquad \text{for some } \alpha \in [0, 1] \,.$$

By using the *Sharpe formula* (3.15) we see that

$$\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}}\, \mathbf{f}_* = \alpha^2 \nu_{\mathrm{rf}}^2 \,, \qquad \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* = \alpha\, \nu_{\mathrm{rf}}^2 \,.$$

Therefore the maximizer $\mathbf{f}_*$ maps to the point (σ_*, μ_*) in the $\sigma\mu$-plane given by

$$\sigma_* = \alpha\,(1 + \mu_{\mathrm{rf}})\, \nu_{\mathrm{rf}} \,, \qquad \mu_* = \mu_{\mathrm{rf}} + \alpha\,(1 + \mu_{\mathrm{rf}})\, \nu_{\mathrm{rf}}^2 \,. \tag{7.44}$$

The first question to address is whether or not these points lie in the sets Σ_{p}, Σ_{q} or Σ_{r} over which we solved the analogous maximization problems in the previous section.

Comparisons

Recall that

$$
\begin{aligned}
\Sigma_{\rm p} &= \Big\{(\sigma,\mu) \,:\, \sigma \geq 0\Big\}\,,\\
\Sigma_{\rm q} &= \Big\{(\sigma,\mu) \,:\, \sigma \geq 0\,,\ \mu \leq 1\Big\}\,,\\
\Sigma_{\rm r} &= \Big\{(\sigma,\mu) \,:\, \sigma \geq 0\,,\ 1+\mu > 0\Big\}\,.
\end{aligned}
\tag{7.45}
$$

We see from (7.44) that

$$
\begin{aligned}
\sigma_* &= \alpha\,(1+\mu_{\rm rf})\,\nu_{\rm rf} \geq 0\,,\\
1+\mu_* &= (1+\mu_{\rm rf})\,(1+\alpha\,\nu_{\rm rf}^2) > 0\,,
\end{aligned}
$$

whereby it is evident from (7.45) that $(\sigma_*,\mu_*) \in \Sigma_{\rm r} \subset \Sigma_{\rm p}$.

Comparisons

We also see from (7.44) that $\mu_* \leq 1$ if and only if

$$\alpha\,\nu_{\mathrm{rf}}^2 \leq \frac{1-\mu_{\mathrm{rf}}}{1+\mu_{\mathrm{rf}}}\,.$$

For all of our maximizers the α is largest when $\chi = 0$. In that case the above bound becomes

$$\begin{aligned} \nu_{\mathrm{rf}}^2 &\leq \frac{1-\mu_{\mathrm{rf}}}{1+\mu_{\mathrm{rf}}} && \text{for } \mathbf{f}_{*\mathrm{p}}\,, \\ \frac{\nu_{\mathrm{rf}}^2}{1+\nu_{\mathrm{rf}}^2} &\leq \frac{1-\mu_{\mathrm{rf}}}{1+\mu_{\mathrm{rf}}} && \text{for } \mathbf{f}_{*\mathrm{q}}\,, \\ \frac{\nu_{\mathrm{rf}}^2}{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_{\mathrm{rf}}^2}} &\leq \frac{1-\mu_{\mathrm{rf}}}{1+\mu_{\mathrm{rf}}} && \text{for } \mathbf{f}_{*\mathrm{r}}\,. \end{aligned} \tag{7.46}$$

Comparisons

After some algebra it can be shown that the bounds (7.46) are

$$
\begin{aligned}
\nu_{\rm rf}^2 &\leq \frac{1-\mu_{\rm rf}}{1+\mu_{\rm rf}} && \text{for } \mathbf{f}_{*\rm p}\,, \\
\nu_{\rm rf}^2 &\leq \frac{1-\mu_{\rm rf}}{2\,\mu_{\rm rf}} && \text{for } \mathbf{f}_{*\rm q}\,, \\
\nu_{\rm rf}^2 &\leq \frac{2\,(1-\mu_{\rm rf})}{(1+\mu_{\rm rf})^2} && \text{for } \mathbf{f}_{*\rm r}\,.
\end{aligned}
\tag{7.47}
$$

Because $\mu_{\rm rf}$ is usually a small positive number, we see that

- for $\mathbf{f}_{*\rm p}$ the upper bound on $\nu_{\rm rf}$ is just under 1,
- for $\mathbf{f}_{*\rm q}$ the upper bound on $\nu_{\rm rf}$ is huge,
- for $\mathbf{f}_{*\rm r}$ the upper bound on $\nu_{\rm rf}$ is just under $\sqrt{2}$.

Comparisons

Because $\mu_* \leq 1$ for every $\chi \geq 0$ if and only if the Sharpe ratio $\nu_{\rm rf}$ satisfiies the bound (7.47), it is evident from (7.45) that $(\sigma_*, \mu_*) \in \Sigma_{\rm q}$ if and only if the Sharpe ratio $\nu_{\rm rf}$ satisfiies the bound (7.47).

Remark. If $\chi \geq \dfrac{2\,\nu_{\rm rf}\,\mu_{\rm rf}}{1+\mu_{\rm rf}}$ for $\mathbf{f}_{*\rm q}$ then $(\sigma_*, \mu_*) \in \Sigma_{\rm q}$ for any $\nu_{\rm rf} > 0$.

Remark. Because $\alpha \in [0,1]$, it can be shown from (7.44) that

$$1 + \mu_* > \sigma_* \,,$$

which implies that $(\sigma_*, \mu_*) \in \Sigma_{\rm t}$ where

$$\Sigma_{\rm t} = \Big\{ (\sigma, \mu) \,:\, \sigma \geq 0 \,,\, 1 + \mu > \sigma \Big\} .$$

This is the domain over which we maximized $G_{\rm t}^{\chi}(\sigma, \mu)$.

Eight Lessons Learned

Here are eight lessons learned from this study of mean-variance objectives.

1. The return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and risk-free rate μ_{rf} play roles in determining the optimal allocation entirely through $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{V}}$.

2. The Sharpe ratio ν_{rf} and the caution coefficient χ play a huge role in determining the optimal allocation. In particular, when $\chi \geq \nu_{\mathrm{rf}}$ the optimal allocation is entirely in the safe investment.

3. Portfolios with higher Sharpe ratios allow for greater uncertainty.

4. For any choice of χ the maximizer for the quadratic objective is more conservative than the maximizer for the reasonable objective, which is more conservative than the maximizer for the parabolic objective.

Eight Lessons Learned

5. The maximizer for a parabolic objective is agressive and will likely overbet when the Sharpe ratio $\nu_{\rm rf}$ is not small.

6. The maximizers for quadratic and reasonable objectives are close when the Sharpe ratio $\nu_{\rm rf}$ is not large. As χ approaches $\nu_{\rm rf}$, the maximizers for the quadratic and reasonable objectives get closer.

7. We will have greater confidence in the computed Sharpe ratio $\nu_{\rm rf}$ when the tangency portfolio lies towards the "nose" of the Markowitz frontier. This translates into having greater confidence in the maximizers for the quadratic and reasonable objectives.

8. Analyzing the maximizers for both the quadratic and reasonable objectives gave greater insights than analyzing them separately. Together they are fortune's formulas!