Portfolios that Contain Risky Assets
10.2. Mean-Variance Estimators for Cautious Objectives

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Portfolios that Contain Risky Assets
Part II: Probabilistic Models

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10. Cautious Objectives for Portfolio Models

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Mean-Variance Estimators for Cautious Objectives

Introduction (Means and Volatilities)

Consider portfolios built from N risky assets and possibly some risk-free assets. Given a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ of the risky assets and positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, define the return sample mean $\mathbf{m}$ and sample variance $\mathbf{V}$ by

$$\mathbf{m} = \sum_{d=1}^{D} w_d \, \mathbf{r}(d) \,, \qquad \mathbf{V} = \sum_{d=1}^{D} w_d \left(\mathbf{r}(d) - \mathbf{m}\right)\left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathrm{T}} . \tag{1.1}$$

A Markowitz portfolio with a risk-free return r_{rf} and a risky asset allocation $\mathbf{f}$ has the return mean and volatility estimators

$$\hat{\mu} = r_{\mathrm{rf}} + \mathbf{m}^{\mathrm{T}}\mathbf{f} \,, \qquad \hat{\sigma} = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\,\mathbf{f}} \,. \tag{1.2}$$

Remark. The formulas for $\mathbf{m}$ and $\hat{\mu}$ are unbiased IID estimators, while those for $\mathbf{V}$ and $\hat{\sigma}$ are biased IID estimators. These biased estimators are what arise naturally in what follows.

Introduction (Solvency)

A Markowitz portfolio with a risk-free return $r_{\rm rf}$ and a risky asset allocation **f** is said to be solvent if

$$1 + r_{\rm rf} + \mathbf{r}(d)^{\rm T}\mathbf{f} > 0 \quad \forall d\,. \tag{1.3a}$$

Recall that $r_{\rm rf}$ is given in terms of the allocations of any risk-free assets by

$$r_{\rm rf} = \begin{cases} 0 & \text{when } \mathbf{f} \in \mathcal{M}\,, \\ \mu_{\rm rf} f^{\rm rf} & \text{when } (\mathbf{f}, f^{\rm rf}) \in \mathcal{M}_1\,, \\ \mu_{\rm si} f^{\rm si} + \mu_{\rm cl} f^{\rm cl} & \text{when } (\mathbf{f}, f^{\rm si}, f^{\rm cl}) \in \mathcal{M}_2\,. \end{cases} \tag{1.3b}$$

Introduction (Cautious Objectives)

For every solvent Markowitz portfolio a cautious objective has the form

$$\widehat{\Gamma}^{\chi} = \widehat{\gamma} - \chi \sqrt{\widehat{\theta}}\,, \tag{1.4a}$$

where $\chi \geq 0$ is the caution coefficient, $\widehat{\gamma}$ is the growth rate estimator

$$\widehat{\gamma} = \sum_{d=1}^{D} w_d \, \log\Big(1 + r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\Big) \,. \tag{1.4b}$$

and $\widehat{\theta}$ is the growth rate variance estimator

$$\widehat{\theta} = \sum_{d=1}^{D} w_d \, \Big(\log\Big(1 + r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\Big) - \widehat{\gamma}\Big)^2 \,. \tag{1.4c}$$

Introduction (Strategy)

The cautious objective strategy is to maximize $\widehat{\Gamma}^{\chi}$ over a convex subset Π of all solvent Markowitz allocations. This maximizer can be found numerically by convex optimization methods that are typically covered in graduate courses.

Rather than seek the maximizer of $\widehat{\Gamma}^{\chi}$ over Π, our strategy will be to replace the estimator $\widehat{\Gamma}^{\chi}$ with a new estimator for which finding the maximizer is easier. The hope is that the maximizer of $\widehat{\Gamma}^{\chi}$ and that of the new estimator will be close.

This strategy rests upon the fact that $\widehat{\Gamma}^{\chi}$ is itself an approximation. The uncertainties associated with it will translate into uncertainities about its maximizer. The hope is that the difference between the maximizer of $\widehat{\Gamma}^{\chi}$ and that of the new estimator will be within these uncertainties.

Introduction (Mean-Variance Estimators)

We will derive some *mean-variance estimators* for Γ^χ in the form

$$\widehat{\Gamma}^\chi = G(\hat{\sigma}, \hat{\mu}) \,, \tag{1.5a}$$

where $\hat{\sigma}$ and $\hat{\mu}$ are given by (1.2) and $G(\sigma, \mu)$ is a function that is defined over a convex subset Σ of the $\sigma\mu$-plane over which

- $G(\sigma, \mu)$ is a strictly decreasing function of σ,
- $G(\sigma, \mu)$ is a strictly increasing function of μ, (1.5b)
- $G(\sigma, \mu)$ is a concave function of (σ, μ).

The monotonicity properties insure that $\widehat{\Gamma}^\chi$ is larger for more efficient portfolios, which implies that its maximizer over Π, if it exists, will lie on the efficient frontier of Π.

Mean-Variance Estimators (Introduction)

The portfolio with risk-free return r_{rf} and risky asset allocation $\mathbf{f}$ has the return history $\{r(d)\}_{d=1}^{D}$ with

$$r(d) = \hat{\mu}(\mathbf{f}) + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\,,$$

where $\tilde{\mathbf{r}}(d) = \mathbf{r}(d) - \mathbf{m}$. In words, $\tilde{\mathbf{r}}(d)$ is the deviation of $\mathbf{r}(d)$ from its sample mean $\mathbf{m}$. Then we can write

$$\begin{aligned}\log(1 + r(d)) = \log(1 + \hat{\mu}) + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}} \\ - \left(\frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}} - \log\left(1 + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}}\right)\right) .\end{aligned} \tag{2.6}$$

Notice that the last term on the first line has sample mean zero while the concavity of the function $r \mapsto \log(1 + r)$ implies that $r - \log(1 + r) \geq 0$, which implies that the term on the second line is nonpositive.

Mean-Variance Estimators (γ Estimators)

When we studied Kelly objectives this expression was used to derive estimators of $\widehat{\gamma}$. Here it will be used to derive estimators of $\widehat{\theta}$.

More specifically, earlier we used the second-order Taylor approximation $\log(1+z) \approx z - \frac{1}{2}z^2$ in the last term of (2.6) to obtain

$$\log(1 + r(d)) \approx \log\big(1 + \hat{\mu}(\mathbf{f})\big) + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}(\mathbf{f})} - \frac{1}{2}\left(\frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}(\mathbf{f})}\right)^2 .$$

This led to the *Taylor estimator*

$$\widehat{\gamma}_{\mathrm{t}}(\mathbf{f}) = \log\big(1 + \hat{\mu}(\mathbf{f})\big) - \frac{1}{2}\frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}{(1 + \hat{\mu}(\mathbf{f}))^2} .$$

This estimator was not well-behaved, so from it we derived the sensible, reasonable, quadratic, and parabolic estimators, $\widehat{\gamma}_{\mathrm{s}}$, $\widehat{\gamma}_{\mathrm{r}}$, $\widehat{\gamma}_{\mathrm{q}}$, and $\widehat{\gamma}_{\mathrm{p}}$, all of which behave better.

Mean-Variance Estimators (θ Estimators)

Here we use the first-order Taylor approximation $\log(1+z) \approx z$ in the last term of (2.6), which makes it vanish, to obtain

$$\log(1 + r(d)) \approx \log(1 + \hat{\mu}) + \frac{\tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}}{1 + \hat{\mu}} \,. \tag{2.7}$$

When this is placed into definition (1.4c) of $\widehat{\theta}$ we obtain

$$\widehat{\theta} = \sum_{d=1}^{D} w_d \left(\log(1 + r(d)) - \widehat{\gamma} \right)^2 \approx \frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}{(1 + \hat{\mu})^2} \,,$$

which leads to the *Taylor variance estimator*

$$\widehat{\theta}_{\mathrm{t}} = \frac{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}{(1 + \hat{\mu})^2} = \frac{\hat{\sigma}^2}{(1 + \hat{\mu})^2} \,. \tag{2.8}$$

Like the Taylor estimator $\widehat{\gamma}_{\mathrm{t}}$, this is not well-behaved.

Mean-Variance Estimators (Parabolic)

The simplest thing to do is drop the $\hat{\mu}$ term in the denominator of $\widehat{\theta}_{\mathrm{t}}$, which leads to the *quadratic variance estimator*

$$\widehat{\theta}_{\mathrm{q}} = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} = \hat{\sigma}^2 \,. \tag{2.9}$$

When the quadratic variance estimator $\widehat{\theta}_{\mathrm{q}}$ given by (2.9) is combined with the parabolic estimator $\widehat{\gamma}_{\mathrm{p}}$ to estimate the cautious objective $\widehat{\Gamma}^{\chi}$ given by (1.4a), then we obtain the *parabolic estimator*

$$\widehat{\Gamma}^{\chi}_{\mathrm{p}} = \hat{\mu} - \tfrac{1}{2}\,\hat{\sigma}^2 - \chi\,\hat{\sigma} \,. \tag{2.10a}$$

This has the mean-variance form (1.5a) with

$$G^{\chi}_{\mathrm{p}}(\sigma,\mu) = \mu - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma \,, \tag{2.10b}$$

which for every $\chi \geq 0$ has all the properties (1.5b) over the set

$$\Sigma_{\mathrm{p}} = \Big\{(\sigma,\mu) \,:\, \sigma \geq 0\Big\} \,. \tag{2.10c}$$

Mean-Variance Estimators (Quadratic)

When the quadratic variance estimator $\widehat{\theta}_{\rm q}$ given by (2.9) is combined with the quadratic estimator $\widehat{\gamma}_{\rm q}$ to estimate the cautious objective $\widehat{\Gamma}^{\chi}$ given by (1.4a), then we obtain the *quadratic estimator*

$$\widehat{\Gamma}^{\chi}_{\rm q} = \hat{\mu} - \tfrac{1}{2}\,\hat{\mu}^2 - \tfrac{1}{2}\,\hat{\sigma}^2 - \chi\,\hat{\sigma}\,. \tag{2.11a}$$

This has the mean-variance form (1.5a) with

$$G^{\chi}_{\rm q}(\sigma,\mu) = \mu - \tfrac{1}{2}\,\mu^2 - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,, \tag{2.11b}$$

which for every $\chi \geq 0$ has all the properties (1.5b) over the set

$$\Sigma_{\rm q} = \Big\{(\sigma,\mu)\,:\,\sigma \geq 0\,,\,\mu \leq 1\Big\}\,. \tag{2.11c}$$

Mean-Variance Estimators (Reasonable)

When the quadratic variance estimator $\widehat{\theta}_{\mathrm{q}}$ given by (2.9) is combined with the reasonable estimator $\widehat{\gamma}_{\mathrm{r}}$ to estimate the cautious objective $\widehat{\Gamma}^{\chi}$ given by (1.4a), then we obtain the *reasonable estimator*

$$\widehat{\Gamma}_{\mathrm{r}}^{\chi} = \log(1+\hat{\mu}) - \tfrac{1}{2}\,\hat{\sigma}^2 - \chi\,\hat{\sigma}\,. \tag{2.12a}$$

This has the mean-variance form (1.5a) with

$$G_{\mathrm{r}}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,, \tag{2.12b}$$

which for every $\chi \geq 0$ has all the properties (1.5b) over the set

$$\Sigma_{\mathrm{r}} = \Big\{(\sigma,\mu)\,:\,\sigma \geq 0\,,\ 1+\mu > 0\Big\}\,. \tag{2.12c}$$

Mean-Variance Estimators (Sensible)

When the quadratic variance estimator $\widehat{\theta}_{\mathrm{q}}$ given by (2.9) is combined with the sensible estimator $\widehat{\gamma}_{\mathrm{s}}$ to estimate the cautious objective $\widehat{\Gamma}^{\chi}$ given by (1.4a), then we obtain the *sensible estimator*

$$\widehat{\Gamma}_{\mathrm{s}}^{\chi} = \log(1+\hat{\mu}) - \tfrac{1}{2}\,\frac{\hat{\sigma}^2}{1+\hat{\mu}} - \chi\,\hat{\sigma}\,. \tag{2.13a}$$

This has the mean-variance form (1.5a) with

$$G_{\mathrm{s}}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+\mu} - \chi\,\sigma\,, \tag{2.13b}$$

which for every $\chi \geq 0$ has all the properties (1.5b) over the set

$$\Sigma_{\mathrm{s}} = \Big\{(\sigma,\mu)\,:\,\sigma \geq 0\,,\,1+\mu > 0\Big\}\,. \tag{2.13c}$$

Mean-Variance Estimators (Taylor)

When the quadratic variance estimator $\widehat{\theta}_{\mathrm{q}}$ given by (2.9) is combined with the Taylor estimator $\widehat{\gamma}_{\mathrm{t}}$ to estimate the cautious objective $\widehat{\Gamma}^{\chi}$ given by (1.4a), then we obtain the *Taylor estimator*

$$\widehat{\Gamma}_{\mathrm{t}}^{\chi} = \log(1+\hat{\mu}) - \tfrac{1}{2}\,\frac{\hat{\sigma}^2}{(1+\hat{\mu})^2} - \chi\,\hat{\sigma}\,. \tag{2.14a}$$

This has the mean-variance form (1.5a) with

$$G_{\mathrm{t}}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,, \tag{2.14b}$$

which for every $\chi \geq 0$ has all the properties (1.5b) over the set

$$\Sigma_{\mathrm{t}} = \Big\{(\sigma,\mu)\,:\,1+\mu \geq \sigma \geq 0\,,\,1+\mu > 0\Big\}\,. \tag{2.14c}$$

Mean-Variance Estimators (Ultimate)

Finally, when the Taylor variance estimator $\widehat{\theta}_t$ given by (2.8) is combined with the Taylor estimator $\widehat{\gamma}_t$ to estimate the cautious objective $\widehat{\Gamma}^\chi$ given by (1.4a), then we obtain the *ultimate estimator*

$$\widehat{\Gamma}^\chi_u = \log(1+\hat{\mu}) - \tfrac{1}{2}\,\frac{\hat{\sigma}^2}{(1+\hat{\mu})^2} - \chi\,\frac{\hat{\sigma}}{1+\hat{\mu}}\,. \tag{2.15a}$$

This has the mean-variance form (1.5a) with

$$G^\chi_u(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\frac{\sigma}{1+\mu}\,, \tag{2.15b}$$

which for every $\chi \in [0,1)$ has all the properties (1.5b) over the set

$$\Sigma^\chi_u = \left\{(\sigma,\mu)\,:\,1+\mu \geq \frac{\sigma}{1-\chi} \geq 0\,,\ 1+\mu > 0\right\}\,. \tag{2.15c}$$

Remark. Here "ultimate" means "last" rather than "best"!

Properties of the Estimators (Introduction)

The *mean-variance estimators* that we have derived all have the form

$$\widehat{\Gamma}^{\chi} = G(\hat{\sigma}, \hat{\mu}) \,, \tag{3.16}$$

where $\hat{\sigma}$ and $\hat{\mu}$ are given by (1.2). Here we show that each $G(\sigma, \mu)$ is a function that is defined over a convex subset Σ of the $\sigma\mu$-plane over which

- $G(\sigma, \mu)$ is a strictly decreasing function of σ,
- $G(\sigma, \mu)$ is a strictly increasing function of μ, (3.17)
- $G(\sigma, \mu)$ is a concave function of (σ, μ).

Specifically, we verify these properties for

$$\widehat{\Gamma}^{\chi}_{\mathrm{p}} \,, \qquad \widehat{\Gamma}^{\chi}_{\mathrm{q}} \,, \qquad \widehat{\Gamma}^{\chi}_{\mathrm{r}} \,, \qquad \widehat{\Gamma}^{\chi}_{\mathrm{s}} \,, \qquad \widehat{\Gamma}^{\chi}_{\mathrm{t}} \,, \qquad \widehat{\Gamma}^{\chi}_{\mathrm{u}} \,,$$

that are the parabolic, quadratic, reasonable, sensible, Taylor and ultimate estimators given by (2.10a), (2.11a), (2.12a), (2.13a), (2.14a) and (2.15a) respectively.

Properties of the Estimators (Functions $G(\sigma, \mu)$)

If $\widehat{\Gamma}$ is $\widehat{\Gamma}_{\mathrm{p}}^{\chi}$, $\widehat{\Gamma}_{\mathrm{q}}^{\chi}$, $\widehat{\Gamma}_{\mathrm{r}}^{\chi}$, $\widehat{\Gamma}_{\mathrm{s}}^{\chi}$, $\widehat{\Gamma}_{\mathrm{t}}^{\chi}$, or $\widehat{\Gamma}_{\mathrm{u}}^{\chi}$ for some $\chi \geq 0$ then

$$G_{\mathrm{p}}^{\chi}(\sigma, \mu) = \mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,, \tag{3.18a}$$

$$G_{\mathrm{q}}^{\chi}(\sigma, \mu) = \mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,, \tag{3.18b}$$

$$G_{\mathrm{r}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,, \tag{3.18c}$$

$$G_{\mathrm{s}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+\mu} - \chi\,\sigma\,, \tag{3.18d}$$

$$G_{\mathrm{t}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,, \tag{3.18e}$$

$$G_{\mathrm{u}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\frac{\sigma}{1+\mu} \quad \text{if } \chi \in [0,1)\,. \tag{3.18f}$$

Properties of the Estimators (Sets Σ)

These are the parabolic, quadratic, reasonable, sensible, Taylor, and ultimate estimators respectively. They are considered over the sets

$$\Sigma_{\mathrm{p}} = \Big\{ (\sigma, \mu) \in \mathbb{R}^2 \;:\; \sigma \geq 0 \Big\}\,, \tag{3.19a}$$

$$\Sigma_{\mathrm{q}} = \Big\{ (\sigma, \mu) \in \Sigma_{\mathrm{p}} \;:\; \mu \leq 1 \Big\}\,, \tag{3.19b}$$

$$\Sigma_{\mathrm{r}} = \Big\{ (\sigma, \mu) \in \Sigma_{\mathrm{p}} \;:\; 1 + \mu > 0 \Big\}\,, \tag{3.19c}$$

$$\Sigma_{\mathrm{s}} = \Big\{ (\sigma, \mu) \in \Sigma_{\mathrm{p}} \;:\; 1 + \mu > 0 \Big\}\,, \tag{3.19d}$$

$$\Sigma_{\mathrm{t}} = \Big\{ (\sigma, \mu) \in \Sigma_{\mathrm{r}} \;:\; 1 + \mu \geq \sigma \Big\}\,, \tag{3.19e}$$

$$\Sigma_{\mathrm{u}}^{\chi} = \Big\{ (\sigma, \mu) \in \Sigma_{\mathrm{r}} \;:\; 1 + \mu \geq \tfrac{\sigma}{1-\chi} \Big\} \quad \text{if } \chi \in [0, 1)\,. \tag{3.19f}$$

These are convex subsets of $\mathbb{R}^2$ that satisfy $\Sigma_{\mathrm{u}}^{\chi} \subset \Sigma_{\mathrm{t}} \subset \Sigma_{\mathrm{s}} = \Sigma_{\mathrm{r}} \subset \Sigma_{\mathrm{p}}$ and $\Sigma_{\mathrm{q}} \subset \Sigma_{\mathrm{p}}$ with $\Sigma_{\mathrm{u}}^{\chi} = \Sigma_{\mathrm{t}}$ when $\chi = 0$.

Properties of the Estimators (Derivatives)

It is evident that each $G(\sigma, \mu)$ given in (3.18) is infinitely differentiable over the convex set Σ that is respectively given in (3.19). We will examine the following properties of $G(\sigma, \mu)$ over Σ:

- $G(\sigma, \mu)$ is a strictly decreasing function of σ over Σ , (3.20a)
- $G(\sigma, \mu)$ is a strictly increasing function of μ over Σ , (3.20b)
- $G(\sigma, \mu)$ is concave over Σ . (3.20c)
- $G(\sigma, \mu)$ is strictly concave over the interior of Σ . (3.20d)

Recall that

- property (3.20a) holds when $G_\sigma < 0$ over the interior of Σ,
- property (3.20b) holds when $G_\mu > 0$ over the interior of Σ,
- property (3.20c) holds where the Hessian is nonpositive definite,
- property (3.20d) holds where the Hessian is negative definite.

Properties of the Estimators (Parabolic)

We now check properties (3.20) for the parabolic, quadratic, reasonable, sensible, Taylor, and ultimate estimators.

For the *parabolic estimator* we see from (3.18a) that

$$G(\sigma, \mu) = \mu - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\sigma - \chi\,, \qquad G_\mu = 1\,,$$
$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Because $\chi \geq 0$, properties (3.20a-c) hold over the set Σ_{p} given by (3.19a). Here $G(\sigma, \mu)$ is not strictly concave anywhere in Σ_{p}, so property (3.20d) does not hold.

Properties of the Estimators (Quadratic)

For the *quadratic estimator* we see from (3.18b) that

$$G(\sigma,\mu) = \mu - \tfrac{1}{2}\,\mu^2 - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\sigma - \chi\,, \qquad G_\mu = 1 - \mu\,,$$
$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Because $\chi \geq 0$, properties (3.20) hold over the set Σ_{q} given by (3.19b).

Properties of the Estimators (Reasonable)

For the *reasonable estimator* we see from (3.18c) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\sigma - \chi\,, \qquad G_\mu = \frac{1}{1+\mu}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -\dfrac{1}{(1+\mu)^2} \end{pmatrix}.$$

Because $\chi \geq 0$, properties (3.20) hold over the set Σ_{r} given by (3.19c).

Properties of the Estimators (Sensible)

For the *sensible estimator* we see from (3.18d) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+\mu} - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\frac{\sigma}{1+\mu} - \chi\,, \qquad G_\mu = \frac{1}{1+\mu} + \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{1+\mu} & \dfrac{\sigma}{(1+\mu)^2} \\ \dfrac{\sigma}{(1+\mu)^2} & -\dfrac{1}{(1+\mu)^2} - \dfrac{\sigma^2}{(1+\mu)^3} \end{pmatrix}\,,$$

$$\det\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \frac{1}{(1+\mu)^3}\,.$$

Because $\chi \geq 0$, properties (3.20) hold over the set Σ_s given by (3.19d).

Properties of the Estimators (Taylor)

For the *Taylor estimator* we see from (3.18e) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\frac{\sigma}{(1+\mu)^2} - \chi\,, \qquad G_\mu = \frac{1}{1+\mu} + \frac{\sigma^2}{(1+\mu)^3}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{(1+\mu)^2} & \dfrac{2\,\sigma}{(1+\mu)^3} \\ \dfrac{2\,\sigma}{(1+\mu)^3} & -\dfrac{1}{(1+\mu)^2} - \dfrac{3\,\sigma^2}{(1+\mu)^4} \end{pmatrix}.$$

$$\det\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \frac{1}{(1+\mu)^4}\left(1 - \frac{\sigma^2}{(1+\mu)^2}\right).$$

Because $\chi \geq 0$, properties (3.20) hold over the set Σ_t given by (3.19e).

Properties of the Estimators (Ultimate)

If $\chi < 1$ then for the *ultimate estimator* we see from (3.18f) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \frac{\chi\,\sigma}{1+\mu}\,,$$

whereby

$$G_\sigma = -\tfrac{\sigma}{(1+\mu)^2} - \tfrac{\chi}{1+\mu}\,, \qquad G_\mu = \tfrac{1}{1+\mu} + \tfrac{\sigma^2}{(1+\mu)^3} + \tfrac{\chi\,\sigma}{(1+\mu)^2}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -\tfrac{1}{(1+\mu)^2} & \tfrac{2\sigma}{(1+\mu)^3} + \tfrac{\chi}{(1+\mu)^2} \\ \tfrac{2\sigma}{(1+\mu)^3} + \tfrac{\chi}{(1+\mu)^2} & -\tfrac{1}{(1+\mu)^2} - \tfrac{3\sigma^2}{(1+\mu)^4} - \tfrac{2\chi\,\sigma}{(1+\mu)^3} \end{pmatrix}.$$

$$\det\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \frac{1}{(1+\mu)^4}\left(1 - \left(\frac{\sigma}{(1+\mu)} + \chi\right)^2\right).$$

Because $\chi \in [0,1)$, properties (3.20) hold over the set $\Sigma_{\mathrm{u}}^{\chi}$ given by (3.19f).