Portfolios that Contain Risky Assets
10.1. Cautious Objectives for Markowitz Portfolios

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Portfolios that Contain Risky Assets
Part II: Probabilistic Models

6. Independent, Identically-Distributed Models for Assets
7. Assessing Independent, Identically-Distributed Models
8. Independent, Identically-Distributed Models for Portfolios
9. Kelly Objectives for Portfolio Models
10. Cautious Objectives for Portfolio Models

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Part II: Probabilistic Models

10. Cautious Objectives for Portfolio Models

10.1. Cautious Objectives for Markowitz Portfolios

10.2. Mean-Variance Estimators for Cautious Objectives

10.3. Optimization of Mean-Variance Objectives

10.4. Fortune's Formulas

Cautious Objectives for Markowitz Portfolios

1. Introduction
2. Cautious Objectives
3. Central Limit Theorem
4. Downside Uncertainties
5. Estimator Uncertainty

Introduction

The Kelly criterion says that investors whose objective is to maximize the value of their portfolio over an extended period should maximize its growth rate mean. More precisely, it suggests that such investors should select an allocation that maximizes the sample mean estimator $\hat{\gamma}$. This suggestion rests upon:

- the validity of an IID model,
- the Law of Large Numbers,
- the accuracy of the estimator $\hat{\gamma}$.

If these assumptions hold then the Kelly criterion would be suitable for many young investors, but not for those older investors who depend upon their portfolios for their income. Of course, the first assumption is questionable while the last is foolhardy, so even young investors should be more cautious.

Introduction

The Kelly criterion exposes some investors to potential downside events from which it might be hard to recover.

- Older investors who depend upon their portfolios for their income might be drawing down on their portfolio at a rate of 4% per year, hoping that this income stream will last at least 20 years.
- But if the value of their portfolio is reduced by 40% in a market downturn or by an inflationary period then the value of their future income stream will be similarly reduced.
- This puts them in a tough spot if their income stream no longer covers their fixed expenses. They might feel forced to draw down at 6.5% per year, which would rapidly erode the value of their portfolio.

Here we will develop objective functions that are better suited for more cautious investors. We will do so within the framework of IID models.

Cautious Objectives

Given

- a set of assets with a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$,
- a choice of positive weights $\{w_d\}_{d=1}^{D}$ that sum to one,

the Kelly Criterion selects the portfolio allocation that maximizes $\hat{\gamma}$ over a class Π of Markowitz portfolio allocations, where the objective $\hat{\gamma}$ is the sample mean estimator of γ given by

$$\hat{\gamma} = \sum_{d=1}^{D} w_d \log\left(1 + r(d)\right), \tag{2.1}$$

where $\{r(d)\}_{d=1}^{D}$ is the return history for a Markowitz portfolio with a risk-free return r_{rf} and a risky asset allocation $\mathbf{f}$, which is given by

$$r(d) = r_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}. \tag{2.2}$$

Cautious Objectives

Here we present the family of *cautious objectives* that have a form like

$$\widehat{\Gamma}^{\chi} = \hat{\gamma} - \chi \sqrt{\hat{\theta}}\,, \tag{2.3}$$

where the nonnegative parameter χ is the so-called *caution coefficient* and $\hat{\theta}$ is the sample variance estimator of θ given by

$$\hat{\theta} = \frac{1}{1 - \bar{w}_D} \sum_{d=1}^{D} w_d \left(\log\left(1 + r(d)\right) - \hat{\gamma} \right)^2 , \tag{2.4}$$

with $\bar{w}_D$ defined by

$$\bar{w}_D = \sum_{d=1}^{D} w_d^2 \,. \tag{2.5}$$

Cautious Objectives

When $\hat{\gamma}$ is a strictly concave function over Π, it is clear that $\widehat{\Gamma}^{\chi}$ given by (2.3) will be is a strictly concave function over any bounded subset of Π provided that the caution coefficient χ is small enough.

If $\sqrt{\hat{\theta}}$ is convex over relevant sets of Π then for every $\chi > 0$ the additional term in $\widehat{\Gamma}^{\chi}$ will enhance the strict concavity of $\hat{\gamma}$, which helps guard against overbetting.

The choice of a value for the caution coefficient χ is up to each individual investor. It characterizes how cautious the investor wishes to be. Caution can arise from many sources, each of which has to be quantified in order to guide the choice of χ.

Cautious Objectives

Here we consider contributions to the caution coefficient from two sources:

- the desire to reduce the impact of downside events;
- our uncertainty in the estimator $\hat{\gamma}$ given by (2.1).

The analysis of these contributions will require more information than the Law of Large Numbers provides. However, this additional information can be estimated with the aid of the *Central Limit Theorem.*

Other potential contributions to the caution coefficient will be explored in the projects. These might include our confidence in the IID model or our assesment of economic factors.

Central Limit Theorem (Introduction)

Because the *Central Limit Theorem* will play a major role in our analysis, we now review it. Let $\{X_d\}_{d=1}^{\infty}$ be any sequence of IID random variables drawn from a probability density $p(X)$ with mean γ and variance $\theta > 0$. Let $\{Y_d\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y_d = \frac{1}{d} \sum_{d'=1}^{d} X_{d'} \qquad \text{for every } d = 1, \cdots, \infty\,. \tag{3.6}$$

Recall that

$$\mathrm{Ex}(Y_d) = \gamma\,, \qquad \mathrm{Vr}(Y_d) = \frac{\theta}{d}\,.$$

The Law of Large Numbers says that Y_d is will approach γ as $d \to \infty$. However, it does not say how the Y_d are distibuted around γ for fixed d. The Central Limit Theorem gives such information.

Central Limit Theorem (Standardization)

Let $\{Z_d\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Z_d = \frac{Y_d - \gamma}{\sqrt{\theta/d}} \qquad \text{for every } d = 1, \cdots, \infty\,.$$

These random variables have been *standardized* so that

$$\mathrm{Ex}(Z_d) = 0\,, \qquad \mathrm{Vr}(Z_d) = 1\,.$$

The Central Limit Theorem says that as $d \to \infty$ the limiting distribution of Z_d will be the mean-zero, variance-one normal distribution.

Remark. In general a random variable Z is said to be *centralized* if $\mathrm{Ex}(Z) = 0$, and is said to be *standardized* if $\mathrm{Ex}(Z) = 0$ and $\mathrm{Vr}(Z) = 1$.

Central Limit Theorem (Statement)

More precisely, it says that for every $\zeta \in \mathbb{R}$ we have

$$\lim_{d\to\infty} \Pr\Big\{ Z_d \geq -\zeta \Big\} = \mathrm{N}(\zeta)\,, \tag{3.7}$$

where $\mathrm{N}(\zeta)$ is the normal cummulative distribution function defined by

$$\mathrm{N}(\zeta) \equiv \int_{-\infty}^{\zeta} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\,\mathrm{d}Z = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\,\mathrm{d}Z\,. \tag{3.8}$$

We can express the limit (3.7) in terms of Y_d as

$$\lim_{d\to\infty} \Pr\left\{ Y_d \geq \gamma - \zeta\sqrt{\theta/d} \right\} = \mathrm{N}(\zeta)\,. \tag{3.9}$$

Central Limit Theorem (Remarks)

Remark. The normal cummulative distribution function N is a strictly increasing, continous function that maps $\mathbb{R}$ onto $(0,1)$. It thereby has an inverse N^{-1} that is a strictly increasing, continuous function that maps $(0,1)$ onto $\mathbb{R}$. Both of these functions are infinitely differentiable.

Remark. The power of the Central Limit Theorem is that it assumes so little about the underlying probability density $p(X)$. Specifically, all that it assumes is

$$\int_{-\infty}^{\infty} X^2 p(X)\,\mathrm{d}X < \infty \qquad \text{and} \qquad \theta > 0\,,$$

where

$$\theta = \int_{-\infty}^{\infty} (X-\gamma)^2 p(X)\,\mathrm{d}X\,, \quad \text{and} \quad \gamma = \int_{-\infty}^{\infty} X\,p(X)\,\mathrm{d}X\,.$$

Central Limit Theorem (Berry-Esseen Theorem)

The Central Limit Theorem does not estimate how fast the limit (3.9) is approached. Such estimates require additional assumptions about the underlying probability density $p(X)$. The simplest such theorem is the Berry-Esseen Theorem, which is not covered in most undergraduate probability courses.

The Berry-Esseen Theorem says that there exists $C_{\mathrm{BE}} \in \mathbb{R}$ such that if

$$\rho = \int_{-\infty}^{\infty} |X - \gamma|^3 p(X)\, \mathrm{d}X < \infty \,, \qquad (3.10)$$

then for every $\zeta \in \mathbb{R}$ we have

$$\left| \mathrm{Pr}\left\{ Y_d \geq \gamma - \zeta\sqrt{\theta/d} \right\} - \mathrm{N}(\zeta) \right| \leq C_{\mathrm{BE}} \, \frac{\rho}{\sqrt{\theta^3}} \, \frac{1}{\sqrt{d}} \,. \qquad (3.11)$$

Central Limit Theorem (Berry-Esseen Bound)

Remark. The Berry-Esseen Theorem shows that the rate of convergence of the limit (3.9) in the Central Limit Theorem is $d^{-\frac{1}{2}}$ as $d \to \infty$.

Remark. The constant C_{BE} is *universal* because it does not depend upon the probability density $p(X)$. Its best value is known to lie within the interval $(0.4, 0.5)$. Bounding its value is the subject of current research.

The error bound (3.11) depends upon $p(X)$ through the ratio

$$\frac{\rho}{\sqrt{\theta^3}},$$

where θ is the variance of $p(X)$ and ρ is the quantity defined in (3.10). The Hölder inequality bounds this ratio below by

$$1 \leq \frac{\rho}{\sqrt{\theta^3}}.$$

Central Limit Theorem (Berry-Esseen Examples)

When $p(X)$ is the normal density given by

$$p(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right),$$

then it can be shown (exercise!) that

$$\frac{\rho}{\sqrt{\theta^3}} = \frac{4}{\sqrt{2\pi}} \quad (\approx 1.59577).$$

The ratio is larger for densities with wider tails. For example, when

$$p(X) = \frac{1}{2\eta} \exp\left(-\frac{|X-\gamma|}{\eta}\right) \qquad \text{for some } \eta > 0,$$

then it can be shown (another exercise!) that $\theta = 2\eta^2$ and

$$\frac{\rho}{\sqrt{\theta^3}} = \frac{3}{\sqrt{2}} \quad (\approx 2.12132).$$

Downside Uncertainties (Introduction)

The IID model for a Markowitz portfolio has the growth rate mean γ and variance θ that are estimated by $\hat{\gamma}$ and $\hat{\theta}$ given by (2.1) and (2.4).

Let $\{X_d\}_{d=1}^{\infty}$ be an IID growth rate history drawn from this model and let $\{Y_d\}_{d=1}^{\infty}$ be defined by (3.6). The Law of Large Numbers says that as $d \to \infty$ the values of Y_d become strongly peak around γ. This behavior seems to be consistent with the idea that it is reasonable to pick the allocation that maximizes the estimator $\hat{\gamma}$. However, by taking $\zeta = 0$ in (3.9) we see that the Central Limit Theorem implies

$$\lim_{d \to \infty} \mathrm{Pr}\{ Y_d \geq \gamma \} = \mathrm{N}(0) = \tfrac{1}{2} \,.$$

This shows that in the long run the growth rate of a portfolio will exceed γ with a probability of only $\frac{1}{2}$. Cautious investors might want the portfolio to exceed a smaller objective with a higher probability.

Downside Uncertainties (Central Limit Approximation)

The Central Limit Theorem says that if T is large enough then we can use the approximation

$$\Pr\left\{ Y_T \geq \gamma - \zeta\sqrt{\theta/T} \right\} \approx \mathrm{N}(\zeta)\,. \tag{4.12}$$

Let $\lambda_{\mathrm{d}} \in (\frac{1}{2}, 1)$ be the probability that we do not want to experience a downside event. Set

$$\zeta_{\mathrm{d}} = \mathrm{N}^{-1}(\lambda_{\mathrm{d}})\,.$$

Then approximation (4.12) becomes

$$\Pr\left\{ Y_T \geq \gamma - \frac{\zeta_{\mathrm{d}}}{\sqrt{T}}\sqrt{\theta} \right\} \approx \lambda_{\mathrm{d}}\,. \tag{4.13}$$

Downside Uncertainties (Caution Coefficient)

This suggests that if downside tail events were our only concern then we could pick the caution coefficient

$$\chi_{\rm d} = \frac{\mathrm{N}^{-1}(\lambda_{\rm d})}{\sqrt{T}} = \frac{\zeta_{\rm d}}{\sqrt{T}}\,, \tag{4.14}$$

whereby

$$\widehat{\Gamma}^{\chi} = \hat{\gamma} - \frac{\zeta_{\rm d}}{\sqrt{T}}\sqrt{\hat{\theta}}\,. \tag{4.15}$$

Remark. When $\hat{\gamma}$ is a strictly concave function over Π, it is clear that $\widehat{\Gamma}^{\chi}$ will be is a strictly concave function over any bounded subset of Π provided that $\chi_{\rm d}$ is small enough.

Downside Uncertainties (Remarks)

Remark. The only assumption that we made beyond the validity of the IID model in order to construct this objective is that T is large enough for the Central Limit Theorem to yield a good approximation of the distribution of growth rates.

Remark. Investors often choose T to be the interval at which the portfolio will be rebalanced, regardless of whether T is large enough for the Central Limit Theorem approximation to be valid. If an investor plans to rebalance once a year then $T = 252$, twice a year then $T = 126$, four times a year then $T = 63$, and twelve times a year then $T = 21$. The smaller T, the less likely it is that the Central Limit Theorem approximation is valid.

Downside Uncertainties (Picking $\chi_{\rm d}$)

The idea is now to select the allocation in Π that maximizes $\widehat{\Gamma}^{\chi}$ for a given $\chi_{\rm d}$, where Π and $\chi_{\rm d}$ are chosen by the investor. When $\chi_{\rm d}$ is given by (4.14) this objective is to maximize the growth rate that will be exceeded by the portfolio with probability $\lambda_{\rm d}$ when it is held for T trading days. Because $1 - \lambda_{\rm d}$ is the fraction of times the investor is willing to experience a downside tail event, the choice of $\lambda_{\rm d}$ reflects the caution of the investor. More cautious investors will select a higher $\lambda_{\rm d}$.

Remark. The caution of an investor can increase with age. Retirees whose portfolio provide an income that covers much of their living expenses will often be extremely cautious. Investors within ten years of retirement may be fairly cautious because they have less time for their portfolio to recover from an economic downturn. In constrast, young investors can be less cautious because they have more time to experience economic upturns and because they are typically far from their peak earning capacity.

Downside Uncertainties (Picking λ_d)

Remark. The caution of an investor should also depend on a careful reading of economic factors or an analysis of the historical data. For example, if the historical data shows evidence of a bubble then any investor should be more cautious.

An investor should select λ_d to be a probability that reflects his or her caution. For example, an investor can select $\zeta_\text{d} = \text{N}^{-1}(\lambda_\text{d})$ based on the following tabulations

$$\begin{aligned}
&\text{N}\left(\tfrac{1}{4}\right) \approx .5987\,, \quad \text{N}\left(\tfrac{1}{2}\right) \approx .6915\,, \quad \text{N}\left(\tfrac{3}{4}\right) \approx .7734\,, \quad \text{N}(1) \approx .8413\,, \\
&\text{N}\left(\tfrac{5}{4}\right) \approx .8944\,, \quad \text{N}\left(\tfrac{3}{2}\right) \approx .9332\,, \quad \text{N}\left(\tfrac{7}{4}\right) \approx .9599\,, \quad \text{N}(2) \approx .9772\,, \\
&\text{N}\left(\tfrac{9}{4}\right) \approx .9878\,, \quad \text{N}\left(\tfrac{5}{2}\right) \approx .9938\,, \quad \text{N}\left(\tfrac{11}{4}\right) \approx .9970\,, \quad \text{N}(3) \approx .9987\,.
\end{aligned}$$

Downside Uncertainties (Picking ζ_{d})

An investor who is willing to experience a downside tail event roughly

$$
\begin{aligned}
\text{once every two years might select } \zeta_{\mathrm{d}} &= 0\,,\\
\text{twice every five years might select } \zeta_{\mathrm{d}} &= \tfrac{1}{4}\,,\\
\text{thrice every ten years might select } \zeta_{\mathrm{d}} &= \tfrac{1}{2}\,,\\
\text{twice every nine years might select } \zeta_{\mathrm{d}} &= \tfrac{3}{4}\,,\\
\text{once every six years might select } \zeta_{\mathrm{d}} &= 1\,,\\
\text{once every ten years might select } \zeta_{\mathrm{d}} &= \tfrac{5}{4}\,,\\
\text{once every fifteen years might select } \zeta_{\mathrm{d}} &= \tfrac{3}{2}\,,\\
\text{once every twenty five years might select } \zeta_{\mathrm{d}} &= \tfrac{7}{4}\,,\\
\text{once every forty four years might select } \zeta_{\mathrm{d}} &= 2\,.
\end{aligned}
$$

Downside Uncertainties (Remarks)

Remark. We see that $\zeta_{\mathrm{d}} = 2$ corresponds to a fairly conservative investor.

Remark. We should pick a larger value of ζ_{d} whenever our analysis of the historical data gives us less confidence in either

- the health of the economy,
- the calibration of $\mathbf{m}$ and $\mathbf{V}$, or
- the validity of an IID model.

These are the questions that are addressed in the projects.

Remark. This approach is similar to something in financal management called *value at risk*. The finance problem is much harder because the time horizon T considered there is much shorter, typically no longer than days. The Central Limit Theorem approximation is likely invalid in that case.

Estimator Uncertainty (Introduction)

The IID model for a Markowitz portfolio has the growth rate mean γ and variance θ that are estimated by $\hat{\gamma}$ and $\hat{\theta}$ given by (2.1) and (2.4).

Let $\{X_d\}_{d=1}^{\infty}$ be an IID growth rate sample drawn from this model and let $\{Y_D\}_{D=1}^{\infty}$ be defined as in (3.6) by

$$Y_D = \frac{1}{D}\sum_{d=1}^{D} X_d \qquad \text{for every } D = 1, \cdots, \infty\,. \tag{5.16}$$

Notice that Y_D is precisely $\hat{\gamma}$ computed with uniform weights for D days. The Law of Large Numbers says that the values of Y_D become strongly peak around γ as $D \to \infty$. The Central Limit Theorem makes the more precise statement that for every $\zeta \in \mathbb{R}$ we have

$$\lim_{D\to\infty} \mathrm{Pr}\Big\{ Y_D \leq \gamma + \zeta\sqrt{\theta/D} \Big\} = \mathrm{N}(\zeta)\,. \tag{5.17}$$

Estimator Uncertainty (Idea)

The result (5.17) suggests that if D is large enough then

$$\Pr\left\{\gamma \geq \hat{\gamma} - \zeta \sqrt{\hat{\theta}/D}\right\} \approx \mathrm{N}(\zeta)\,. \tag{5.18}$$

Here we have replaced Y_D in (5.17) with the estimator $\hat{\gamma}$ and the unknown variance θ with its estimator $\hat{\theta}$ given by (2.4).

As we saw in the setting of the simple betting game, overbetting arises when $\hat{\gamma}$ *overestimates* γ.

- The Central Limit Theorem tell us that this happen half of the time.
- We want to be fairly certain that when this happens, γ will not be too much lower than $\hat{\gamma}$.
- This suggests that we should pick $\zeta = \zeta_{\mathrm{e}}$ large enough that we are confident in the lower bound on γ given in (5.18).

Estimator Uncertainty (Picking $\zeta_{\rm e}$)

The table that we showed earlier suggests that good values for $\zeta_{\rm e}$ lie in the interval $[1, 2]$.

- For $\zeta_{\rm e} = 1$ the lower bound will be wrong about one year in six.
- For $\zeta_{\rm e} = 1.5$ the lower bound will be wrong about one year in fifteen.
- For $\zeta_{\rm e} = 2$ the lower bound will be wrong about one year in forty four.

Once $\zeta_{\rm e}$ is seclected, the caution coefficient becomes

$$\chi_{\rm e} = \frac{\zeta_{\rm e}}{\sqrt{D}}\,, \tag{5.19}$$

whereby

$$\widehat{\Gamma}^{\chi} = \hat{\gamma} - \frac{\zeta_{\rm e}}{\sqrt{D}}\sqrt{\hat{\theta}}\,. \tag{5.20}$$

Estimator Uncertainty (Caution Coefficient)

We can then combine the caution coefficients from (4.14) and (5.19) into a single caution coefficient as

$$\chi = \chi_{\mathrm{d}} + \chi_{\mathrm{e}} = \frac{\zeta_{\mathrm{d}}}{\sqrt{T}} + \frac{\zeta_{\mathrm{e}}}{\sqrt{D}} \,. \tag{5.21}$$

In the common case where $T = D$ this becomes

$$\chi = \frac{\zeta}{\sqrt{D}} \,, \qquad \text{where } \zeta = \zeta_{\mathrm{d}} + \zeta_{\mathrm{e}} \,. \tag{5.22}$$

Based on our earlier considerations, we might pick ζ in the interval $[2, 4]$. When $D = 252$ the values for χ given by (5.22) then range from about $\frac{1}{8}$ to about $\frac{1}{2}$.