Math 420, Spring 2021
Project Two: Choosing a Caution Coefficient

presentation dates Tuesday, 4 May to Tuesday 11 May
report due Friday, 14 May, 2021

This project explores how economic and modeling metrics might guide the choice of the caution coefficient χ. Specifically, based on what was found in Project One, we examine the proximity and leverage metrics, and based on what was found in the homework for Project Two, we examine the signal-to-noise and an indentical distribution metric.

Consider the risky assets in groups (A) (B) and (C) of your project. Use adjusted closing prices to compute the return of each asset for each trading day over the last fifteen calendar years — namely the years ending December 31 of 2006-2020.

Task 1. Describe each of your nine risky assets.

There are 60 quarters within this fifteen year period. There are 57 one-year periods within these fifteen years — the first consisting of quarters 1-4, the second consisting of quarters 2-5, and so on until the last consisting of quarters 57-60. We call the return histories over these 57 one-year periods *rolling histories* and label each by its first quarter.

For each of the 57 one-year histories compute $\mathbf{m}$ and $\mathbf{V}$ using uniform weights for the assests in groups (A), (B), and (C) combined. Use the U.S. T-Bill (13 week) rate available at the beginning of each history as the safe investment rate for the data from that period. Assume that the credit line rate for each period is three points higher than the U.S. T-Bill rate. For each of these one-year histories:

Task 2. For each of the 57 one-year histories do the following.

- Compute the return mean and volatility estimators

  $$\hat{\mu} = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \hat{\sigma} = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,,$$

  and the signal-to-noise metric

  $$\omega^{\mathrm{SN}} = \frac{\hat{\mu}^2}{\bar{w}\hat{\sigma}^2 + \hat{\mu}^2}\,,$$

  for the safe tangent allocation $\mathbf{f}_{\mathrm{st}}$ and for the long tangent allocation $\mathbf{f}_{\mathrm{lt}}$. (Save for later the portfolios along each of the 57 efficient long frontiers that you computed to do this.) On a single graph plot these two metrics as a function of the first quarter of each history. Based on this graph decide which of these metrics is most informative.

- Compute the identically-distributed metrics

  $$\omega^{\mathrm{m}}\,, \qquad \omega^{\mathrm{v}}\,, \qquad \omega^{\mathrm{KS}}\,,$$

  for the safe tangent allocation $\mathbf{f}_{\mathrm{st}}$ and for the long tangent allocation $\mathbf{f}_{\mathrm{lt}}$, where each quarter is compared with every other quarter in its history. Because there are six comparisons for each history, one for each pair of quarters. On a single graph plot these six metrics as a function of the first quarter of each history. Based on this graph decide which of these metrics is most informative.

- Compute the proximity metric ω^{prx} for for the long tangent allocation $\mathbf{f}_{\text{lt}}$ and for the S&P 500 index fund. On a single graph plot these two metrics as a function of the first quarter of each history. Based on this graph decide which of these metrics is most informative.
- Compute the leverage metric ω^{lev} for the safe tangent allocation $\mathbf{f}_{\text{st}}$ and for credit tangent allocation $\mathbf{f}_{\text{ct}}$. On a single graph plot these two metrics as a function of the first quarter of each history. Based on this graph decide which of these metrics is most informative.
- On a single graph plot the four metrics that you have selected as a function of the first quarter of each history.

Remark. A tangent portfolio on the frontier hyperbola will exist for any risk-free rate μ_{rf} provided that $\mu_{\text{rf}} \neq \mu_{\text{mv}}$. Its allocation is given by

$$\mathbf{f}_{\text{tg}} = \frac{\mu_{\text{tg}} - \mu_{\text{rf}}}{\nu_{\text{rf}}^2} \mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\text{rf}}\mathbf{1}\big) \,,$$

where

$$\nu_{\text{rf}}^2 = \big(\mathbf{m} - \mu_{\text{rf}}\mathbf{1}\big)^{\text{T}}\mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\text{rf}}\mathbf{1}\big) \,, \qquad \mu_{\text{tg}} - \mu_{\text{rf}} = \frac{\nu_{\text{rf}}^2\,\sigma_{\text{mv}}^2}{\mu_{\text{mv}} - \mu_{\text{rf}}} \,.$$

When $\mu_{\text{rf}} = \mu_{\text{si}}$ these formulas yield $\mathbf{f}_{\text{st}}$, ν_{si}^2, and μ_{st} associated with the safe tangent portfolio. When $\mu_{\text{rf}} = \mu_{\text{cl}}$ they yield $\mathbf{f}_{\text{ct}}$, ν_{cl}^2, and μ_{ct} associated with the credit tangent portfolio.

Task 3. For long portfolios with a safe investment the reasonable estimator of the cautious objective is

$$\widehat{\Gamma}_{\text{r}}^{\chi}(\mathbf{f}) = \log\big(1 + \hat{\mu}(\mathbf{f})\big) - \tfrac{1}{2}\hat{\sigma}(\mathbf{f})^2 - \chi\,\hat{\sigma}(\mathbf{f}) \,,$$

where χ is the caution coefficient and

$$\hat{\mu}(\mathbf{f}) = \mu_{\text{si}} + \big(\mathbf{m} - \mu_{\text{si}}\mathbf{1}\big)^{\text{T}}\mathbf{f} \,, \qquad \hat{\sigma}(\mathbf{f}) = \sqrt{\mathbf{f}^{\text{T}}\mathbf{V}\mathbf{f}} \,.$$

Let $\nu_{\text{si}} > 0$ be given by

$$\nu_{\text{si}}^2 = \big(\mathbf{m} - \mu_{\text{si}}\mathbf{1}\big)^{\text{T}}\mathbf{V}^{-1}\big(\mathbf{m} - \mu_{\text{si}}\mathbf{1}\big) \,.$$

Let ρ_{lt} be given by

$$\rho_{\text{lt}} = \begin{cases} 0 & \text{if } \mu_{\text{si}} \geq \mu_{\text{mx}} \,, \\ \dfrac{\mu_{\text{lt}} - \mu_{\text{si}}}{\sigma_{\text{lt}}} & \text{if } \mu_{\text{si}} < \mu_{\text{mx}} \,. \end{cases}$$

- Plot $\rho_{\text{lt}}/\nu_{\text{si}}$ as a function of the first quarter of each history. How does this graph compare with the ones for the proximity metrics? Explain any similarities seen.
- For each one-year history with $\rho_{\text{lt}} > 0$ find the allocation $\mathbf{f}_{\text{mx}}^{\chi}$ that maximizes $\widehat{\Gamma}_{\text{r}}^{\chi}(\mathbf{f})$ over the set of long portfolios with a safe investment when $\chi = \zeta\rho_{\text{lt}}$ with

  $$\zeta = 0.000\,,\ 0.125\,,\ 0.250\,,\ 0.375\,,\ 0.500\,,\ 0.625\,,\ 0.750\,,\ 0.875\,,\ 1.000\,.$$

  For $\zeta = 1.000$ we have $\mathbf{f}_{\text{mx}}^{\chi} = \mathbf{0}$. Why? (This leaves as many as of $57 \times 8 = 456$ optimization problems. This is where you use the efficient long frontier portfolios that you saved earlier!) For each calendar year plot the efficient long frontier and the points $(\hat{\sigma}(\mathbf{f}), \hat{\mu}(\mathbf{f}))$ for $\mathbf{f}_{\text{lt}}$ and for each of the nine $\mathbf{f}_{\text{mx}}^{\chi}$ computed above. There should be fifteen such graphs, one for each calender year.

- For each of the first 53 one-year histories with $\rho_{\mathrm{lt}} > 0$ determine which of the nine allocations $\mathbf{f}_{\mathrm{mx}}^{\chi}$ computed above yields the maximum value of

$$\sum_{d=1}^{D} \log\bigl(1 + r(d,\mathbf{f})\bigr)\,, \qquad \text{where} \quad r(d,\mathbf{f}) = \mu_{\mathrm{si}} + \bigl(\mathbf{r}(d) - \mu_{\mathrm{si}}\mathbf{1}\bigr)^{\mathrm{T}}\mathbf{f}\,,$$

  and the data μ_{si} and $\{\mathbf{r}(d)\}_{d=1}^{D}$ are from the one-year history *subsequent* to the one used to compute the nine allocations. Let χ_{opt} be the assiciated value of χ. Plot χ_{opt} as a function of the first quarter of each history.

Task 4. Index each of the first 53 one-year histories with $\rho_{\mathrm{lt}} > 0$ by h. For each of these histories we have the optimal caution coefficient $\chi_{\mathrm{opt}}(h)$ found in Task 3, and the four metrics chosen in Task 2, let's call them

$$\omega^{\mathrm{SN}}(h)\,, \qquad \omega^{\mathrm{ID}}(h)\,, \qquad \omega^{\mathrm{prx}}(h)\,, \qquad \omega^{\mathrm{lev}}(h)\,,$$

Use linear least squares to find the coefficients α, β^{SN}, β^{ID}, β_0^{prx}, β_1^{prx}, β_0^{lev} and β_1^{lev} that minimize the sum of the squares of the residuals $\epsilon(h)$, where

$$\begin{aligned}\chi_{\mathrm{opt}}(h) = \alpha &- \beta^{\mathrm{SN}}\omega^{\mathrm{SN}}(h) - \beta^{\mathrm{ID}}\omega^{\mathrm{ID}}(h) - \beta_0^{\mathrm{prx}}\omega^{\mathrm{prx}}(h) \\ &- \beta_1^{\mathrm{prx}}\omega^{\mathrm{prx}}(h-1) - \beta_0^{\mathrm{lev}}\omega_0^{\mathrm{lev}}(h) - \beta_1^{\mathrm{lev}}\omega^{\mathrm{lev}}(h-1) + \epsilon(h)\,.\end{aligned}$$

More specifically, find α, β^{SN}, β^{ID}, β_0^{prx}, β_1^{prx}, β_0^{lev}, and β_1^{lev} that minimize

$$\begin{aligned}\sum_h \epsilon(h)^2 = \sum_h \Bigl(\alpha &- \beta^{\mathrm{SN}}\omega^{\mathrm{SN}}(h) - \beta^{\mathrm{ID}}\omega^{\mathrm{ID}}(h) - \beta_0^{\mathrm{prx}}\omega^{\mathrm{prx}}(h) \\ &- \beta_1^{\mathrm{prx}}\omega^{\mathrm{prx}}(h-1) - \beta_0^{\mathrm{lev}}\omega^{\mathrm{lev}}(h) - \beta_1^{\mathrm{lev}}\omega^{\mathrm{lev}}(h-1)\Bigr)^2\,,\end{aligned}$$

where the sum begins at the second one-year history with $\rho_{\mathrm{lt}} > 0$. Comment on the meaning of the coefficient values that you found.