Portfolios that Contain Risky Assets 16: Optimization of Mean-Variance Objectives

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Math 420: *Mathematical Modeling*
April 27, 2021 version

Portfolios that Contain Risky Assets
Part II: Stochastic Models

Optimization of Mean-Variance Objectives

Mean-Variance Objectives (Introduction)

We consider portfolios that contains N risky assets along with a risk-free safe investment and possibly a risk-free credit line. Given the return mean vector $\mathbf{m}$, return covariance matrix $\mathbf{V}$, and risk-free rates $\mu_{\rm si}$ and $\mu_{\rm cl}$, a mean-variance objective for a portfolio allocation $(f_{\rm si}, f_{\rm cl}, \mathbf{f})$ has the form

$$\widehat{\Gamma}(f_{\rm si}, f_{\rm cl}, \mathbf{f}) = G(\hat{\sigma}(\mathbf{f}), \hat{\mu}(f_{\rm si}, f_{\rm cl}, \mathbf{f}))\,, \tag{1.1a}$$

where the return variance and mean estimators are given by

$$\hat{\sigma}(\mathbf{f}) = \sqrt{\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}}\,, \qquad \hat{\mu}(f_{\rm si}, f_{\rm cl}, \mathbf{f}) = \mu_{\rm si} f_{\rm si} + \mu_{\rm cl} f_{\rm cl} + \mathbf{m}^{\rm T}\mathbf{f}\,, \tag{1.1b}$$

and the allocation $(f_{\rm si}, f_{\rm cl}, \mathbf{f})$ satisfies the constraint

$$f_{\rm si} + f_{\rm cl} + \mathbf{1}^{\rm T}\mathbf{f} = 1\,. \tag{1.1c}$$

Mean-Variance Objectives (One Risk-Free Rate Case)

For the One Risk-Free Rate model with risk-free rate μ_{rf} these mean-variance objectives take the form

$$\widehat{\Gamma}(\mathbf{f}) = G(\hat{\sigma}(\mathbf{f}), \hat{\mu}(\mathbf{f})) , \tag{1.2a}$$

where the return variance and mean estimators are given by

$$\hat{\sigma}(\mathbf{f}) = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} , \qquad \hat{\mu}(\mathbf{f}) = \mu_{\mathrm{rf}}(1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}) + \mathbf{m}^{\mathrm{T}}\mathbf{f} . \tag{1.2b}$$

For portfolios without risk-free assets we also add the constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$.

Markowitz Portfolio Theory says that a rational investor should choose a portfolio allocation on the efficient frontier corresponding to a class Π of Markowitz portfolio allocations. Here we show how to use a mean-variance objective to select an optimal portfolio on the efficient frontier.

Mean-Variance Objectives (Some Examples)

If $\widehat{\Gamma}$ is $\widehat{\Gamma}_{\rm p}^{\chi}$, $\widehat{\Gamma}_{\rm q}^{\chi}$, $\widehat{\Gamma}_{\rm r}^{\chi}$, $\widehat{\Gamma}_{\rm s}^{\chi}$, $\widehat{\Gamma}_{\rm t}^{\chi}$, or $\widehat{\Gamma}_{\rm u}^{\chi}$ for some $\chi \geq 0$ then

$$G_{\rm p}^{\chi}(\sigma,\mu) = \mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,, \tag{1.3a}$$

$$G_{\rm q}^{\chi}(\sigma,\mu) = \mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,, \tag{1.3b}$$

$$G_{\rm r}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,, \tag{1.3c}$$

$$G_{\rm s}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+2\mu} - \chi\,\sigma\,, \tag{1.3d}$$

$$G_{\rm t}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,, \tag{1.3e}$$

$$G_{\rm u}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\frac{\sigma}{1+\mu} \quad \text{if } \chi \in [0,1)\,. \tag{1.3f}$$

Mean-Variance Objectives (Their Domains)

These are the parabolic, quadratic, reasonable, sensible, Taylor, and ultimate estimators respectively. Their respective domains are

$$\Sigma_{\rm p} = \left\{(\sigma,\mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\right\}, \tag{1.4a}$$

$$\Sigma_{\rm q} = \left\{(\sigma,\mu) \in \Sigma_{\rm p} \,:\, \mu \leq 1\right\}, \tag{1.4b}$$

$$\Sigma_{\rm r} = \left\{(\sigma,\mu) \in \Sigma_{\rm p} \,:\, 1+\mu > 0\right\}, \tag{1.4c}$$

$$\Sigma_{\rm s} = \left\{(\sigma,\mu) \in \Sigma_{\rm r} \,:\, 1+\mu > \tfrac{1}{2}\right\}, \tag{1.4d}$$

$$\Sigma_{\rm t} = \left\{(\sigma,\mu) \in \Sigma_{\rm r} \,:\, 1+\mu \geq \sigma\right\}, \tag{1.4e}$$

$$\Sigma_{\rm u}^{\chi} = \left\{(\sigma,\mu) \in \Sigma_{\rm r} \,:\, 1+\mu \geq \tfrac{\sigma}{1-\chi}\right\} \quad \text{if } \chi \in [0,1)\,. \tag{1.4f}$$

The domains given in (1.4) are convex subsets of $\mathbb{R}^2$ that satisfy

$$\Sigma_{\rm q} \subset \Sigma_{\rm p}\,, \qquad \Sigma_{\rm s} \subset \Sigma_{\rm r} \subset \Sigma_{\rm p}\,, \qquad \Sigma_{\rm u}^{\chi} \subset \Sigma_{\rm t} \subset \Sigma_{\rm r} \subset \Sigma_{\rm p}\,. \tag{1.5}$$

When $\chi \in (0,1)$ all of these containments are proper. When $\chi = 0$ we have $\Sigma_{\rm u}^0 = \Sigma_{\rm t}$ and all of the remaining containments are proper.

Mean-Variance Objectives (General Properties)

It is evident that each $G(\sigma, \mu)$ given in (1.3) is infinitely differentiable over the convex domain Σ that is respectively given in (1.4). Moreover, we will show that:

- each $G(\sigma, \mu)$ is a strictly decreasing function of σ over its Σ,
- each $G(\sigma, \mu)$ is a strictly increasing function of μ over its Σ,
- each $G(\sigma, \mu)$ is a concave function over its Σ,
- each $G(\sigma, \mu)$ except $G_{\mathrm{p}}^{\chi}(\sigma, \mu)$ is a strictly concave function over its Σ.

All of these properties except the last (strict concavity) are properties shared by all mean-variance objectives. This last property will be replaced with a weaker one that is also shared by all mean-variance objectives. These shared properties will allow us to use such a mean-variance objective to select an optimal portfolio on the efficient frontier.

Explicit Level Sets of Some Objectives (Introduction)

Recall that the efficient frontier associated with any given set of portfolios is a strictly increasing, concave curve $\mu = \mu_{\text{ef}}(\sigma)$ in $\mathbb{R}^2$. A mean-variance objective (1.1) given by $G(\sigma, \mu)$ that is defined over a convex set $\Sigma \subset \mathbb{R}^2$ will select an optimal portfolio on the efficient frontier provided:

- the efficient frontier lies within Σ,
- the function $G(\sigma, \mu)$ has a unique maximum on the efficient frontier.

Because the efficient frontier is a strictly increasing, concave curve, the second property will hold if the level sets of $G(\sigma, \mu)$ are suitable strictly increasing, strictly convex curves in Σ, where the level set of $G(\sigma, \mu)$ associated with a possible value $\Gamma \in \mathbb{R}$ is defined by

$$\Sigma(\Gamma) = \left\{ (\sigma, \mu) \in \Sigma \, : \, G(\sigma, \mu) = \Gamma \right\} . \tag{2.6}$$

This set will be empty when no point $(\sigma, \mu) \in \Sigma$ satisfies $G(\sigma, \mu) = \Gamma$. The next three sections show that these sets lie on strictly increasing, strictly convex curves.

Explicit Level Sets of Some Objectives (Parabolic)

For the parabolic estimator (1.3a) the points (σ, μ) in $\Sigma_{\rm p}(\Gamma)$ satisfy

$$\mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma = \Gamma\,.$$

Upon solving this for μ and completing the square we obtain

$$\begin{aligned}\mu &= \tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma \\ &= \tfrac{1}{2}(\sigma + \chi)^2 + \Gamma - \tfrac{1}{2}\chi^2\,.\end{aligned}$$

This equation yields a parabola with

vertex (minimum)	$\left(-\chi\,,\,\Gamma - \tfrac{1}{2}\chi^2\right)$,
focus	$\left(-\chi\,,\,\Gamma - \tfrac{1}{2}\chi^2 + \tfrac{1}{2}\right)$,
focal length	$\tfrac{1}{2}$.

Explicit Level Sets of Some Objectives (Parabolic)

The level set $\Sigma_{\mathrm{p}}(\Gamma)$ is the restriction of this parabola to Σ_{p}. Because by (1.4a)

$$\Sigma_{\mathrm{p}} = \left\{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\right\},$$

we have

$$\Sigma_{\mathrm{p}}(\Gamma) = \left\{\left(\sigma\,,\, \mu_{\mathrm{p}}^{\chi}(\sigma, \Gamma)\right) \,:\, \sigma \geq 0\right\}. \tag{2.7a}$$

where $\mu = \mu_{\mathrm{p}}^{\chi}(\sigma, \Gamma)$ is given by

$$\mu_{\mathrm{p}}^{\chi}(\sigma, \Gamma) = \tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma\,. \tag{2.7b}$$

We thereby see that Σ_{p} is foilated by strictly increasing, strictly convex segments of the parabolas in the family given by $\mu = \mu_{\mathrm{p}}^{\chi}(\sigma, \Gamma)$. These parabolas shift upward with increasing Γ.

Explicit Level Sets of Some Objectives (Quadratic)

For the quadratic estimator (1.3b) the points (σ, μ) in $\Sigma_{\mathrm{q}}(\Gamma)$ satisfy

$$\mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma = \Gamma\,.$$

By completing squares we see that this equation has the form

$$\tfrac{1}{2}(\sigma + \chi)^2 + \tfrac{1}{2}(\mu - 1)^2 = \tfrac{1}{2}\chi^2 + \tfrac{1}{2} - \Gamma\,.$$

This equation clearly has no solution unless $\chi^2 + 1 \geq 2\Gamma$. When $\chi^2 + 1 \geq 2\Gamma$ it yields a circle in the $\sigma\mu$-plane with

$$\text{center} \quad \big(-\chi\,,\, 1 \big)\,,$$

$$\text{radius} \quad \sqrt{\chi^2 + 1 - 2\Gamma}\,.$$

Explicit Level Sets of Some Objectives (Quadratic)

The level set $\Sigma_{\rm q}(\Gamma)$ is the restriction of this circle to $\Sigma_{\rm q}$. Because by (1.4b)

$$\Sigma_{\rm q} = \left\{(\sigma,\mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\,,\ \mu \leq 1\right\},$$

we can show that $\Sigma_{\rm q}(\Gamma)$ is empty when $\Gamma > \frac{1}{2}$, and when $\Gamma \leq \frac{1}{2}$ we have

$$\Sigma_{\rm q}(\Gamma) = \left\{\left(\sigma\,,\ \mu_{\rm q}^{\chi}(\sigma,\Gamma)\right) \,:\, 0 \leq \sigma \leq \sqrt{\chi^2 + 1 - 2\Gamma} - \chi\right\}, \tag{2.8a}$$

where $\mu = \mu_{\rm q}^{\chi}(\sigma,\Gamma)$ is given by

$$\mu_{\rm q}^{\chi}(\sigma,\Gamma) = 1 - \sqrt{\chi^2 + 1 - 2\Gamma - (\sigma+\chi)^2}\,. \tag{2.8b}$$

We thereby see that $\Sigma_{\rm q}$ is foilated by strictly increasing, strictly convex arcs of the circles centered at $(-\chi, 1)$ in the family given by $\mu = \mu_{\rm q}^{\chi}(\sigma,\Gamma)$ for every $\Gamma \leq \frac{1}{2}$. The radius of these circles is χ when $\Gamma = \frac{1}{2}$ and increases with decreasing Γ.

Explicit Level Sets of Some Objectives (Reasonable)

For the reasonable estimator (1.3c) the points (σ, μ) in $\Sigma_{\mathrm{r}}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\sigma^2 - \chi\,\sigma = \Gamma\,.$$

Upon solving this for μ and completing the square we obtain

$$\begin{aligned}\mu &= \exp\left(\tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma\right) - 1\\ &= \exp\left(\tfrac{1}{2}(\sigma+\chi)^2 + \Gamma - \tfrac{1}{2}\chi^2\right) - 1\,.\end{aligned}$$

The graph of this function is strictly convex with a minimum at

$$\Big(-\chi\,,\, \exp\left(\Gamma - \tfrac{1}{2}\chi^2\right) - 1\Big)\,.$$

Because $e^z - 1 > z$ for every $z \neq 0$, we see that this curve lies above the corresponding parabola associated with the parabolic estimator.

Explicit Level Sets of Some Objectives (Reasonable)

The level set $\Sigma_{\mathrm{r}}(\Gamma)$ is the restriction of this curve to Σ_{r}. Because by (1.4c)

$$\Sigma_{\mathrm{r}} = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0 \,,\, 1 + \mu > 0 \right\},$$

we have

$$\Sigma_{\mathrm{r}}(\Gamma) = \left\{ \left(\sigma \,,\, \mu_{\mathrm{r}}^{\chi}(\sigma, \Gamma)\right) \,:\, \sigma \geq 0 \right\}, \tag{2.9a}$$

where $\mu = \mu_{\mathrm{r}}^{\chi}(\sigma, \Gamma)$ is given by

$$\mu_{\mathrm{r}}^{\chi}(\sigma, \Gamma) = \exp\left(\tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma\right) - 1\,. \tag{2.9b}$$

We thereby see that Σ_{r} is foilated by strictly increasing, strictly convex segments of the curves in the family given by $\mu = \mu_{\mathrm{r}}^{\chi}(\sigma, \Gamma)$. These curves move upward with increasing Γ.

Implicit Level Sets of Objectives (Introduction)

At this point the explicit approach that we have been taking breaks down. For the sensible estimator the points (σ, μ) in the level set $\Sigma_{\mathrm{s}}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{1+2\mu} - \chi\,\sigma = \Gamma\,.$$

For the Taylor estimator the points (σ, μ) in the level set $\Sigma_{\mathrm{t}}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma = \Gamma\,.$$

For the ultimate estimator the points (σ, μ) in the level set $\Sigma_{\mathrm{u}}^{\chi}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\frac{\sigma}{1+\mu} = \Gamma\,.$$

These equations cannot be solved for μ explicitly. Of course, they can be solved for σ explicitly. However, it is easier to analyze the level sets that they define implicitly because this avoids messy formulas.

Implicit Level Sets of Objectives (Level Curves)

We will carry out this implicit analysis in the general setting of an equation in the form

$$G(\sigma, \mu) = \Gamma\,,$$

where we assume that $G_\mu(\sigma, \mu) > 0$ over the interior of the convex set $\Sigma \subset \mathbb{R}^2$. (Here G_μ denotes the partial derivative of G with respect to μ.)

By the Implicit Function Theorem this assumption implies that there exists a unique function $\mu(\sigma, \Gamma)$ such that

$$G(\sigma, \mu(\sigma, \Gamma)) = \Gamma\,. \tag{3.10}$$

Moreover, the function $\mu(\sigma, \Gamma)$ is infinitely differentiable. Then for each Γ in the range of G, the graph of the function $\sigma \mapsto \mu(\sigma, \Gamma)$ is a level curve.

Implicit Level Sets of Objectives (Strictly Increasing in Γ)

By taking the partial derivative of (3.10) with respect to Γ we find that

$$G_\mu(\sigma, \mu) \, \frac{\partial \mu}{\partial \Gamma} = 1 \, .$$

Because $G_\mu(\sigma, \mu) > 0$, this can be solved to obtain

$$\frac{\partial \mu}{\partial \Gamma} = \frac{1}{G_\mu(\sigma, \mu)} > 0 \, .$$

Therefore $\mu(\sigma, \Gamma)$ is a strictly increasing function of Γ.

Implicit Level Sets of Objectives (Strictly Increasing in σ)

By taking the partial derivative of (3.10) with respect to σ we find that

$$G_\sigma(\sigma,\mu) + G_\mu(\sigma,\mu)\,\frac{\partial\mu}{\partial\sigma} = 0\,,$$

Because $G_\mu(\sigma,\mu) > 0$, this can be solved to obtain

$$\frac{\partial\mu}{\partial\sigma} = -\frac{G_\sigma(\sigma,\mu)}{G_\mu(\sigma,\mu)}\,.$$

(Here G_σ denotes the partial derivative of G with respect to σ.)

Therefore, if we assume that $G_\sigma(\sigma,\mu) < 0$ over the interior of the convex set $\Sigma \subset \mathbb{R}^2$ then $\mu(\sigma,\Gamma)$ is a strictly increasing function of σ.

Implicit Level Sets of Objectives (Strictly Convex in σ)

Finally, by taking the second partial derivative of (3.10) with respect to σ, using the foregoing result, and again using the fact that $G_\mu(\sigma,\mu) > 0$, we find after some calculation that

$$\frac{\partial^2 \mu}{\partial \sigma^2} = -\frac{1}{G_\mu^3} \begin{pmatrix} G_\mu & -G_\sigma \end{pmatrix} \begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\sigma\mu} & G_{\mu\mu} \end{pmatrix} \begin{pmatrix} G_\mu \\ -G_\sigma \end{pmatrix} ,$$

where the (σ,μ) arguments of all the functions have been suppressed. (Here $G_{\sigma\sigma}$, $G_{\sigma\mu}$, $G_{\mu\mu}$, denote the various second-order partial derivatives of G with respect to σ and μ.)

Therefore if we assume that the right-hand side is positive over the interior of the convex set $\Sigma \subset \mathbb{R}^2$ then $\mu(\sigma,\Gamma)$ is a strictly convex function of σ.

Implicit Level Sets of Objectives (Summary)

In summary, if $G(\sigma, \mu)$ considered over the interior of the convex set Σ has the properties

$$G_\sigma < 0\,, \qquad G_\mu > 0\,, \tag{3.11a}$$

$$\begin{pmatrix} G_\mu & -G_\sigma \end{pmatrix} \begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} \begin{pmatrix} G_\mu \\ -G_\sigma \end{pmatrix} < 0\,, \tag{3.11b}$$

then the level sets of $G(\sigma, \mu)$ that lie within the convex set Σ are curves given by $\mu = \mu(\sigma, \Gamma)$ where $\mu(\sigma, \Gamma)$ is:

- a strictly increasing, strictly convex function of σ,
- a strictly increasing function of Γ.

Indeed, the functions $\mu_{\mathrm{p}}^{\chi}(\sigma, \Gamma)$, $\mu_{\mathrm{q}}^{\chi}(\sigma, \Gamma)$, and $\mu_{\mathrm{r}}^{\chi}(\sigma, \Gamma)$ that are given explicitly by (2.7b), (2.8b), and (2.9b), have these properties.

Implicit Level Sets of Objectives (Remark)

Remark. Properties (3.11) are implied when $G(\sigma, \mu)$ considered over the interior of the convex set Σ has the properties

$$G_\sigma < 0\,, \qquad G_\mu > 0\,, \tag{3.12a}$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} \quad \text{is negative definite}\,. \tag{3.12b}$$

Verifying the negative definiteness property (3.12b) is often the fastest way to verify property (3.11b). As we will see in the next section, all of the functions $G(\sigma, \mu)$ given by (1.3) except $G^{\chi}_{\mathrm{p}}(\sigma, \mu)$ satisfy the negative definiteness property (3.12b).

Examples (Parabolic Estimator)

For completeness, we now verify properties (3.11) for the parabolic, quadratic, reasonable, sensible, Taylor, and ultimate estimators.

For the parabolic estimator we see from (1.3a) that

$$G(\sigma,\mu) = \mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,,$$

whereby

$$\begin{aligned} &G_\sigma = -\sigma - \chi\,, \qquad G_\mu = 1\,, \\ &\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.13}$$

Hence, because $\chi \geq 0$, properties (3.11) hold for every (σ,μ) in the interior of Σ_{p} given by (1.4a). However it is clear from (4.13) that the parabolic estimator does not have the negative definiteness property (3.12b).

Examples (Quadratic Estimator)

For the quadratic estimator we see from (1.3b) that

$$G(\sigma,\mu) = \mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\sigma - \chi\,, \qquad G_\mu = 1 - \mu\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, because $\chi \geq 0$, properties (3.12) (and thereby properties (3.11)) hold for every (σ,μ) in the interior of Σ_{q} given by (1.4b).

Examples (Reasonable Estimator)

For the reasonable estimator we see from (1.3c) that

$$G(\sigma, \mu) = \log(1 + \mu) - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\sigma - \chi\,, \qquad G_\mu = \tfrac{1}{1+\mu}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -\frac{1}{(1+\mu)^2} \end{pmatrix}.$$

Hence, because $\chi \geq 0$, properties (3.12) (and thereby properties (3.11)) hold for every (σ, μ) in the interior of Σ_{r} given by (1.4c).

Examples (Sensible Estimator)

For the sensible estimator we see from (1.3d) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{1+2\mu} - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\tfrac{\sigma}{1+2\mu} - \chi\,, \qquad G_\mu = \tfrac{1}{1+\mu} + \tfrac{\sigma^2}{(1+2\mu)^2}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -\frac{1}{1+2\mu} & \frac{2\sigma}{(1+2\mu)^2} \\ \frac{2\sigma}{(1+2\mu)^2} & -\frac{1}{(1+\mu)^2} - \frac{4\sigma^2}{(1+2\mu)^3} \end{pmatrix}.$$

Hence, because $\chi \geq 0$, properties (3.12) (and thereby properties (3.11)) hold for every (σ,μ) in the interior of Σ_{s} given by (1.4d).

Examples (Sensible Estimator)

Because

$$G_{\rm s}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{1+2\mu} - \chi\,\sigma\,,$$

for every $\sigma > 0$ the mapping $\mu \mapsto G_{\rm s}^{\chi}(\sigma,\mu)$ is strictly increasing and maps the interval $(\frac{1}{2},\infty)$ onto $\mathbb{R}$. Therefore, for every $\sigma > 0$ and every $\Gamma \in \mathbb{R}$ there exists a unique $\mu_{\rm s}^{\chi}(\sigma,\Gamma) > -\frac{1}{2}$ such that

$$G_{\rm s}^{\chi}(\sigma,\mu_{\rm s}^{\chi}(\sigma,\Gamma)) = \Gamma\,.$$

Moreover, for $\sigma = 0$ and every $\Gamma > \log(\frac{1}{2})$ we have

$$\mu_{\rm s}^{\chi}(0,\Gamma) = \exp(\Gamma) - 1\,.$$

We thereby see that $\Sigma_{\rm s}$ is foilated by strictly increasing, strictly convex segments of the curves in the family given by $\mu = \mu_{\rm s}^{\chi}(\sigma,\Gamma)$. These curves move upward with increasing Γ.

Examples (Taylor Estimator)

For the Taylor estimator we see from (1.3e) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,,$$

whereby

$$G_\sigma = -\tfrac{\sigma}{(1+\mu)^2} - \chi\,, \qquad G_\mu = \tfrac{1}{1+\mu} + \tfrac{\sigma^2}{(1+\mu)^3}\,,$$
$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -\tfrac{1}{(1+\mu)^2} & \tfrac{2\sigma}{(1+\mu)^3} \\ \tfrac{2\sigma}{(1+\mu)^3} & -\tfrac{1}{(1+\mu)^2} - \tfrac{3\sigma^2}{(1+\mu)^4} \end{pmatrix}.$$

Hence, because $\chi \geq 0$, properties (3.12) (and thereby properties (3.11)) hold for every (σ,μ) in the interior of Σ_{t} given by (1.4e).

Examples (Taylor Estimator)

Because

$$G_t^\chi(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,,$$

for every $\sigma > 0$ the mapping $\mu \mapsto G_t^\chi(\sigma,\mu)$ is strictly increasing and maps the interval $(\sigma - 1, \infty)$ onto the interval $(\log(\sigma) - \frac{1}{2} - \chi\,\sigma, \infty)$. Therefore, for every $\sigma > 0$ and every $\Gamma \in (\log(\sigma) - \frac{1}{2} - \chi\,\sigma, \infty)$ there exists a unique $\mu_t^\chi(\sigma,\Gamma) > \sigma - 1$ such that

$$G_t^\chi(\sigma, \mu_t^\chi(\sigma,\Gamma)) = \Gamma\,.$$

Moreover, for $\sigma = 0$ and every $\Gamma \in \mathbb{R}$ we have

$$\mu_t^\chi(0,\Gamma) = \exp(\Gamma) - 1\,.$$

We thereby see that Σ_t is foilated by strictly increasing, strictly convex segments of the curves in the family given by $\mu = \mu_t^\chi(\sigma,\Gamma)$. These curves move upward with increasing Γ.

Example (Ultimate Estimator)

If $\chi < 1$ then for the ultimate estimator we see from (1.3f) that

$$G(\sigma, \mu) = \log(1+\mu) - \frac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \frac{\chi\,\sigma}{1+\mu}\,,$$

whereby

$$G_\sigma = -\frac{\sigma}{(1+\mu)^2} - \frac{\chi}{1+\mu}\,, \qquad G_\mu = \frac{1}{1+\mu} + \frac{\sigma^2}{(1+\mu)^3} + \frac{\chi\,\sigma}{(1+\mu)^2}\,,$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} = \begin{pmatrix} -\frac{1}{(1+\mu)^2} & \frac{2\sigma}{(1+\mu)^3} + \frac{\chi}{(1+\mu)^2} \\ \frac{2\sigma}{(1+\mu)^3} + \frac{\chi}{(1+\mu)^2} & -\frac{1}{(1+\mu)^2} - \frac{3\sigma^2}{(1+\mu)^4} - \frac{2\chi\,\sigma}{(1+\mu)^3} \end{pmatrix}.$$

With some effort it can be checked that, because $\chi \in [0, 1)$, properties (3.12) (and thereby properties (3.11)) hold for every (σ, μ) in the interior of $\Sigma_{\mathrm{u}}^{\chi}$ given by (1.4f).

Examples (Ultimate Estimator)

Because

$$G_{\mathrm{u}}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \frac{\chi\,\sigma}{1+\mu}\,,$$

for every $\sigma > 0$ the mapping $\mu \mapsto G_{\mathrm{u}}^{\chi}(\sigma,\mu)$ is strictly increasing and maps the interval $(\sigma/(1-\chi)-1,\infty)$ onto the interval $(\Gamma_L^{\chi}(\sigma),\infty)$ where

$$\Gamma_L^{\chi}(\sigma) = \log\left(\frac{\sigma}{1-\chi}\right) - \tfrac{1}{2}(1-\chi)^2 - \chi\,(1-\chi)\,.$$

Therefore, for every $\sigma > 0$ and every $\Gamma \in (\Gamma_L^{\chi}(\sigma),\infty)$ there exists a unique $\mu_{\mathrm{u}}^{\chi}(\sigma,\Gamma) > \sigma/(1-\chi)-1$ such that $G_{\mathrm{u}}^{\chi}(\sigma,\mu_{\mathrm{u}}^{\chi}(\sigma,\Gamma)) = \Gamma$. Moreover, for $\sigma = 0$ and every $\Gamma \in \mathbb{R}$ we have $\mu_{\mathrm{u}}^{\chi}(0,\Gamma) = \exp(\Gamma) - 1$. We thereby see that $\Sigma_{\mathrm{u}}^{\chi}$ is foilated by strictly increasing, strictly convex segments of the curves in the family given by $\mu = \mu_{\mathrm{u}}^{\chi}(\sigma,\Gamma)$. These curves move upward with increasing Γ.

Reduced Maximization Problem

Mean-variance objectives have the feature that they can be optimized by simply maximizing $G(\sigma, \mu)$ over the efficient frontier of Π in the $\sigma\mu$-plane. Recall that given any choice of Markowitz portfolio allocations Π its efficient frontier is a curve $\mu = \mu_{\text{ef}}(\sigma)$ in the $\sigma\mu$-plane given by a strictly increasing, concave, continuous, piecewise differentiable function $\mu_{\text{ef}}(\sigma)$. The function $\mu_{\text{ef}}(\sigma)$

- is defined over the interval $[0, \infty)$ for the unlimited leverage, One Risk-Free Rate and Two Risk-Free Rates models,
- is defined over some bounded interval $[0, \sigma_{\text{mx}}]$ for every portfolio model with limited leverage that includes a safe investment.

Reduced Maximization Problem

A mean-variance objective given by $G(\sigma, \mu)$ that is defined over a convex set $\Sigma \subset \mathbb{R}^2$ will select an optimal portfolio on the efficient frontier provided:

- the efficient frontier lies within Σ,
- the function $G(\sigma, \mu)$ has a unique maximum on the efficient frontier.

Assume for the moment that the first condition is met. Here we show how to meet the second when $G(\sigma, \mu)$ has properties (3.11). We define the function $\Gamma_{\text{ef}}(\sigma)$ over this interval by

$$\Gamma_{\text{ef}}(\sigma) = G\left(\sigma, \mu_{\text{ef}}(\sigma)\right).$$

Reduced Maximization Problem

Fact. If $G(\sigma,\mu)$ is a strictly decreasing function of σ and a strictly increasing function of μ over Σ then we have

$$\max\{\, G(\hat{\sigma}(\mathbf{f}),\hat{\mu}(\mathbf{f})) \;:\; \mathbf{f}\in\Pi\} = \max\{\Gamma_{\rm ef}(\sigma) \;:\; \sigma\in[0,\sigma_{\rm mx}]\}\;.$$

Reason. Because frontier portfolios minimize $\hat{\sigma}$ for a given value of $\hat{\mu}$, and because $G(\hat{\mu},\hat{\sigma})$ is a strictly decreasing function of $\hat{\sigma}$, the optimal $\mathbf{f}_*$ clearly must be a frontier portfolio. Because the optimal portfolio must also be more efficient than every other portfolio with the same volatility, because $G(\hat{\mu},\hat{\sigma})$ is a strictly increasing function of $\hat{\mu}$, the optimal portfolio must lie on the efficient frontier.

Reduced Maximization Problem

This reduced maximization problem can be visualized by considering the family of level set curves in the $\sigma\mu$-plane parameterized by Γ as

$$G(\sigma, \mu) = \Gamma \,.$$

When $G(\sigma, \mu)$ has properties (3.11) then these curves are strictly increasing, strictly convex functions of σ. As Γ increases the curve shifts upward in the $\sigma\mu$-plane.

For some values of Γ the corresponding curve will intersect the efficient frontier, which is given by $\mu = \mu_{\text{ef}}(\sigma)$. There is clearly a maximum such Γ. As the level set curve is strictly convex while the efficient frontier is concave, for this maximum Γ the intersection will consist of a single point $(\sigma_{\text{opt}}, \mu_{\text{opt}})$. Then $\sigma = \sigma_{\text{opt}}$ is the maximizer of $\Gamma_{\text{ef}}(\sigma)$.

Reduced Maximization Problem

Remark. This reduction is appealing because the efficient frontier only depends on general information about an investor, like whether he or she will take short positions. Once it is computed, the problem of maximizing any given $\hat{\Gamma}(\mathbf{f})$ over all allocations $\mathbf{f}$ reduces to the problem of maximizing the associated $\Gamma_{\mathrm{ef}}(\sigma)$ over all admissible σ — a problem over one variable.

Remark. The maximum problem

$$\max\{\Gamma_{\mathrm{ef}}(\sigma) \,:\, \sigma \in [0, \sigma_{\mathrm{mx}}]\} \,.$$

is easy to solve numerically. We simply evaluate $G(\sigma, \mu)$ at the points (σ_k, μ_k) that were computed to find the efficient frontier numerically. The maximizer is the point (σ_k, μ_k) at which $G(\sigma_k, \mu_k)$ is largest.

Reduced Maximization Problem

Let us consider what might happen. Because $\mu_{\text{ef}}(\sigma)$ has a piecewise derivative, the function $\Gamma_{\text{ef}}(\sigma)$ has the piecewise derivative

$$\Gamma_{\text{ef}}'(\sigma) = \partial_\mu G(\sigma, \mu_{\text{ef}}(\sigma))\, \mu_{\text{ef}}'(\sigma) + \partial_\sigma G(\sigma, \mu_{\text{ef}}(\sigma)) \,.$$

Because $\mu_{\text{ef}}(\sigma)$ is concave, $\Gamma_{\text{ef}}'(\sigma)$ is strictly decreasing.

Reduced Maximization Problem

Because $\Gamma'_{\rm ef}(\sigma)$ is strictly decreasing, there are three possibilities.

- $\Gamma_{\rm ef}(\sigma)$ takes its maximum at $\sigma = 0$, the left endpoint of its interval of definition. This case arises whenever $\Gamma'_{\rm ef}(0) \leq 0$.
- $\Gamma_{\rm ef}(\sigma)$ takes its maximum in the interior of its interval of definition at the unique point $\sigma = \sigma_{\rm opt}$ where $\Gamma'_{\rm ef}(\sigma)$ changes sign. This case arises for the unlimited leverage models whenever $\Gamma'_{\rm ef}(0) > 0$, and for a limited leverage portfolio model whenever $\Gamma'_{\rm ef}(\sigma_{\rm mx}) < 0 < \Gamma'_{\rm ef}(0)$.
- $\Gamma_{\rm ef}(\sigma)$ takes its maximum at $\sigma = \sigma_{\rm mx}$, the right endpoint of its interval of definition. This case arises only for limited leverage portfolio models whenever $\Gamma'_{\rm ef}(\sigma_{\rm mx}) \geq 0$.

Reduced Maximization Problem

In summary, our approach to portfolio selection has six steps:

1. Choose a return rate history for some set of risky assets.
2. Calibrate its mean vector **m** and covariance matrix **V**.
3. Given **m**, **V**, μ_{si}, μ_{cl}, and any portfolio constraints, compute $\mu_{\text{ef}}(\sigma)$.
4. Choose a mean-variance objective specificed by some $G(\sigma, \mu)$.
5. Find the maximizer σ_{opt} of the function $\Gamma_{\text{ef}}(\sigma) = G(\sigma, \mu_{\text{ef}}(\sigma))$.
6. Evaluate the unique efficient frontier portfolio allocation $\mathbf{f}_{\text{ef}}(\sigma_{\text{opt}})$.

The third step is the most computationally intensive for most choices of portfolio constraints. This step is simplest for unlimited leverage portfolios with a single risk-free rate model. In that case $\mu_{\text{ef}}(\sigma) = \mu_{\text{rf}} + \nu_{\text{tg}}\sigma$, where ν_{tg} is the Sharpe ratio.

Reduced Maximization Problem

We will consider mean-variance objectives (1.1) or given by a function $G(\sigma, \mu)$ that is defined over a convex set $\Sigma \subset \mathbb{R}^2$ which is consistent with the class of Markowitz portfolio allocations Π in the sense that

$$\Sigma(\Pi) = \left\{ \left(\hat{\sigma}(\mathbf{f}) \,,\, \hat{\mu}(\mathbf{f}) \right) \,:\, \mathbf{f} \in \Pi \right\} \subset \Sigma \,. \tag{5.14}$$

For example, it can be shown that

$$\Sigma(\Omega_{(0,2)}) \subset \Sigma_{\mathrm{q}} \,,$$

where $\Omega_{(0,2)}$ is the set of all portfolio allocations with value-ratios in $(0, 2)$;

$$\Sigma(\Omega) \subset \Sigma_{\mathrm{r}} \,,$$

where Ω is the set of all solvent portfolio allocations; and

$$\Sigma(\Omega_{\frac{1}{2}}) \subset \Sigma_{\mathrm{s}} \,,$$

where $\Omega_{\frac{1}{2}}$ is the set of all portfolio allocations with value-ratios in $(\frac{1}{2}, \infty)$.

Reduced Maximization Problem

The set $\Omega_{(\frac{1}{2},2)}$ of all portfolio allocations with value-ratios in $(\frac{1}{2},2)$ satisfies

$$\Sigma\Big(\Omega_{(\frac{1}{2},2)}\Big) \subset \Big\{(\sigma,\mu)\in\mathbb{R}^2 \,:\, \sigma\geq 0\,,\,(1-\mu)(\mu+\tfrac{1}{2})>\sigma^2\Big\}\,,$$

whereby

$$\Sigma\Big(\Omega_{(\frac{1}{2},2)}\Big)\subset\Sigma_{\mathrm{q}}\,,\qquad \Sigma\Big(\Omega_{(\frac{1}{2},2)}\Big)\subset\Sigma_{\mathrm{r}}\,,\qquad \Sigma\Big(\Omega_{(\frac{1}{2},2)}\Big)\subset\Sigma_{\mathrm{s}}\,,$$
$$\Sigma\Big(\Omega_{(\frac{1}{2},2)}\Big)\subset\Sigma_{\mathrm{t}}\,,\qquad \Sigma\Big(\Omega_{(\frac{1}{2},2)}\Big)\subset\Sigma_{\mathrm{u}}^{\chi}\quad \text{for every } \chi\in[0,\tfrac{1}{4}]\,.$$

This means if we choose Π such that $\Pi\subset\Omega_{(\frac{1}{2},2)}$ then the consistency condition (5.14) will hold for each of the estimators given in (5.14) provided our caution coefficient satifies $\chi\leq\frac{1}{4}$. In practice $\chi<\frac{1}{4}$ is always satisfied while $\Pi\subset\Omega_{(\frac{1}{2},2)}$ is satisfied for portfolios with a sufficient leverge limit.

Reduced Maximization Problem

The consistency condition (5.14) implies that

$$\left\{ \left(\hat{\sigma}(\mathbf{f}) \,,\, \hat{\mu}(\mathbf{f}) \right) \,:\, \mathbf{f} \in \Pi(\Gamma) \right\} \subset \Sigma(\Gamma) \,, \tag{5.15a}$$

where $\Pi(\Gamma)$ is defined by

$$\Pi(\Gamma) = \left\{ \mathbf{f} \in \Pi \,:\, \hat{\Gamma}(\mathbf{f}) = \Gamma \right\} . \tag{5.15b}$$

Remark. In economics and finance the objective is commonly called the utility because it is choosen by the investor to score the usefulness of portfolios. In that setting level sets are commonly called indifference sets, and level curves are commonly called indifference curves.

Remark. Property (3.12b) implies that $G(\sigma, \mu)$ is a strictly concave function over the convex set Σ, which combines with property (3.12a) to imply that the mean-variance objective $\widehat{\Gamma}(\mathbf{f})$ given by (1.1) is strictly concave over any class Π of portfolio allocations that is consistent with Σ in the sense (5.14).