Portfolios that Contain Risky Assets 3: Markowitz Portfolios

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

Markowitz Portfolios

Portfolios: Positions

We will consider portfolios in which an investor can hold one of three positions with respect to any risky asset. The investor can:

- hold a *long position* by owning shares of the asset;
- hold a *short position* by selling borrowed shares of the asset;
- hold a *neutral position* by doing neither of the above.

In order to keep things simple, we will not consider derivative assets.

Remark. The potential downside of a long position is limited to the amount that you invest in the asset. This happens when the asset becomes worthless.

Portfolios: Short Positions

Remark. We hold a short position by borrowing shares of an asset from a lender (usually our broker) and selling them immediately.

- If the share price subsequently goes down then we can buy the same number of shares and give them to the lender, thereby paying off our loan and profiting by the price difference minus transaction costs.
- If the price goes up then our lender can force us either to increase our collateral or to pay off the loan by buying shares at this higher price, thereby taking a loss that might be larger than the original value of your entire portfolio.

Because the potential downside of a short position can bankrupt you, *short positions should be monitored constantly.*

Portfolios: Specification

We will consider portfolios that hold positions in N assets indexed by $i = 1, \cdots, N$. For each of these assets we have its closing share price history $\{s_i(d)\}_{d=0}^{D}$ over a common period of $D + 1$ trading days. These portfolios will hold $n_i(d)$ shares of asset i throughout trading day d for each $d = 1, \cdots, D$, whereby each portfolio is specified by $\{n_i(d)\}_{d=1,\, i=1}^{D,\ \ N}$.

- If we hold a long position in asset i on day d then $n_i(d) > 0$.
- If we hold a short position in asset i on day d then $n_i(d) < 0$.
- If we hold a neutral position in asset i on day d then $n_i(d) = 0$.

A portfolio that never holds a short position is called a *long portfolio.* Otherwise it is called a *leveraged portfolio.* Many investors have long portfolios because portfolios that hold short positions can be very risky.

Portfolios: Solvent

The value of such a portfolio at the end of trading day d is

$$\pi(d) = \sum_{i=1}^{N} n_i(d)\, s_i(d)\,. \tag{1.1}$$

If $\pi(d) > 0$ for every trading day d then we say that it is a *solvent portfolio.* The return of such a portfolio for trading day d is

$$r(d) = \frac{\pi(d)}{\pi(d-1)} - 1 = \frac{\pi(d) - \pi(d-1)}{\pi(d-1)}\,. \tag{1.2}$$

Notice that if $\pi(d-1) = 0$ for some d then the return is undefined. Worse, if $\pi(d-1) < 0$ for some d then a postive return implies that $\pi(d) < \pi(d-1)$, which means that the portfolio is losing value! *Therefore, if return is to be a good proxy for reward then we should restrict our considerations to solvent portfolios.*

Portfolios: Solvent

Fact 1. A portfolio will be solvent over a given return history if and only if

$$\pi(0) > 0 \qquad \text{and} \qquad 1 + r(d) > 0 \quad \text{for every } d = 1, \cdots, D\,. \tag{1.3}$$

Proof. By the definition of a solvent portfolio we know that $\pi(0) > 0$ and that $\pi(d) > 0$ and $\pi(d-1) > 0$ for every $d = 1, \cdots, D$. It follows by (1.2) that $1 + r(d) > 0$ for every $d = 1, \cdots, D$. Therefore (1.3) holds.

Conversely, suppose (1.3) holds. It follows from (1.2) by induction that

$$\pi(d) = \pi(0) \prod_{d'=1}^{d} \big(1 + r(d')\big) \qquad \text{for every } d = 1, \cdots, D\,.$$

Then (1.3) implies that $\pi(d) > 0$ for every $d = 1, \cdots, D$ as well as $\pi(0) > 0$. Therefore the portfolio is solvent. $\square$

Portfolios: Self-Financing

A portfolio is called *self-financing* if $\{n_i(d)\}_{i=1}^N$ satisfies

$$\pi(d-1) = \sum_{i=1}^{N} n_i(d)\, s_i(d-1) \quad \text{for every } d = 1, \cdots, D\,. \tag{1.4}$$

This is an idealization that neglects trading costs and assumes that the opening price for a share of asset i on day d is $s_i(d-1)$.

If a portfolio is self-financing then by using (1.1) and (1.4) in (1.2) we see that its return for trading day d is

$$r(d) = \sum_{i=1}^{N} \frac{n_i(d)}{\pi(d-1)} \left(s_i(d) - s_i(d-1)\right). \tag{1.5}$$

Markowitz Portfolios and Allocations: Introduction

A 1952 paper by Harry Markowitz had enormous influence on the theory and practice of portfolio management and financial engineering ever since.

- It presented his doctoral dissertation work at the Unversity of Chicago, for which he was awarded the Nobel Prize in Economics in 1990.
- It was the first work to quantify how diversifying a portfolio can reduce its risk without changing its potential reward. It did this because it was the first work to use the covariance between different assets in an essential way.

The key to carrying out this work was modeling. The first modeling step was to develop a class of idealized portfolios that is simple enough to analyze, yet is rich enough to yield useful results.

Markowitz Portfolios and Allocations: Idealization

Markowitz carried out his analysis on a class of idealized portfolios that are each characterized by a set of real numbers $\{f_i\}_{i=1}^N$ that satisfy

$$\sum_{i=1}^{N} f_i = 1\,. \tag{2.6}$$

The portfolio picks $n_i(d)$ at the beginning of each trading day d so that

$$\frac{n_i(d)s_i(d-1)}{\pi(d-1)} = f_i\,, \tag{2.7}$$

where $n_i(d)$ need not be an integer. We call these *Markowitz portfolios.*

The number f_i is called the *allocation of asset i.* It uniquely determines $n_i(d)$ at the start of each trading day.

Markowitz Portfolios and Allocations: Idealization

Specifically, at the start of trading day d we determine $\{n_i(d)\}_{i=1}^N$ from (2.7) as

$$n_i(d) = \frac{f_i\,\pi(d-1)}{s_i(d-1)}\,. \tag{2.8}$$

We see that for so long as $\pi(d-1) > 0$ and $s_i(d-1) > 0$ the portfolio

- holds a long position in asset i if $f_i > 0$,
- holds a short position in asset i if $f_i < 0$,
- holds a neutral position in asset i if $f_i = 0$.

Remark. If every f_i is nonnegative then f_i is the fraction of the portfolio's value held in asset i at the start of each day.

Markowitz Portfolios and Allocations: Vector Notation

Therefore a Markowitz portfolio is determined by the vector **f** given by

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} .$$

We call **f** the *allocation vector* or simply the *allocation* of the portfolio. The constraint (2.6) can be expressed in the compact form

$$\mathbf{1}^{\mathrm{T}}\mathbf{f} = \sum_{i=1}^{N} f_i = 1 \,, \tag{2.9}$$

where **1** denotes the N-vector with each entry equal to 1.

Markowitz Portfolios and Allocations: All

Therefore the set of allocations for all Markowitz portfolios is

$$\mathcal{M} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \right\}. \tag{2.10}$$

Let $\mathbf{e}_i$ denote the vector whose i^{th} entry is 1 while every other entry is 0. Because $\mathbf{1}^{\mathrm{T}}\mathbf{e}_i = 1$, we see that $\mathbf{e}_i \in \mathcal{M}$ for every $i = 1, \cdots, N$. For every $\mathbf{f} \in \mathcal{M}$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where

$$\mathbf{1}^{\mathrm{T}}\mathbf{f} = \sum_{i=1}^{N} f_i = 1 \,.$$

This shows that $\mathcal{M}$ is simply the $N-1$ dimensional plane in $\mathbb{R}^N$ passing through the unit vectors $\mathbf{e}_1, \cdots, \mathbf{e}_N$. We can visualize $\mathcal{M}$ when N is small.

Markowitz Portfolios and Allocations: All

When $N = 2$ it is the line that passes through the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is orthogonal to the vector $\mathbf{1}$.
When $N = 3$ it is the plane that passes through the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

It is orthogonal to the vector $\mathbf{1}$.

Markowitz Portfolios and Allocations: Long

A Markowitz portfolio is long if and only if

$$f_i \geq 0 \quad \text{for every } i = 1, \cdots, N\,.$$

We express this as

$$\mathbf{f} \geq \mathbf{0}\,,$$

where $\mathbf{0}$ denotes the N-vector with each entry equal to 0.
The set of allocations for long Markowitz portfolios is thereby

$$\Lambda = \left\{ \mathbf{f} \in \mathcal{M} \;:\; \mathbf{f} \geq \mathbf{0} \right\}. \tag{2.11}$$

This set will play an important role in what follows.

Markowitz Portfolios and Allocations: Long

For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where

$$f_i \geq 0 \quad \text{for } i = 1, \cdots, N \,,$$

$$\mathbf{1}^{\mathrm{T}} \mathbf{f} = \sum_{i=1}^{N} f_i = 1 \,.$$

This shows that Λ is just the $N - 1$ dimensional simplex in $\mathbb{R}^N$ whose vertices are the unit vectors $\mathbf{e}_1, \cdots, \mathbf{e}_N$. Because the sides of this simplex are identical, it is a regular simplex. We can visualize Λ when N is small.

Markowitz Portfolios and Allocations: Long

When $N = 2$ it is the convex combination of the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

This is just a line segment, which is the regular 1-dimensional simplex. When $N = 3$ it is the convex combination of the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} .$$

This is an equilateral triangle, which is the regular 2-dimensional simplex.

Markowitz Portfolio Returns: Self-Financing

We now derive a simple expression for the returns of Markowitz portfolios. We start by showing that every Markowitz portfolio is *self-financing.* Recall that for the Markowitz portfolio with allocations $\{f_i\}_{i=1}^N$ if $\pi(d-1)$ is the value of the portfolio at the end of trading day $d-1$ then the number of shares of asset i held on trading day d is given by formula (2.8) as

$$n_i(d) = \frac{f_i\,\pi(d-1)}{s_i(d-1)}\,.$$

Hence, the value of the portfolio at the beginning of trading day d is

$$\sum_{i=1}^N n_i(d)\,s_i(d-1) = \sum_{i=1}^N \frac{f_i\,\pi(d-1)}{s_i(d-1)}\,s_i(d-1) = \pi(d-1)\sum_{i=1}^N f_i = \pi(d-1)\,.$$

Here we have used (2.6), the fact that the allocations $\{f_i\}_{i=1}^N$ sum to 1. Because this portfolio satifies (1.4), it is self-financing.

Markowitz Portfolio Returns: Formula

Because Markowitz portfolios are self-financing, it follows from (1.5) and relationship (2.7) between $n_i(d)$ and f_i that the return $r(d)$ of a Markowitz portfolio for trading day d is

$$r(d) = \sum_{i=1}^{N} \frac{n_i(d)s_i(d-1)}{\pi(d-1)} \, \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \sum_{i=1}^{N} f_i \, r_i(d) \,. \qquad (3.12)$$

The return $r(d)$ for the Markowitz portfolio characterized by $\{f_i\}_{i=1}^{N}$ is thereby simply the linear combination of the $r_i(d)$ with the coefficients f_i. *This relationship makes the class of Markowitz portfolios easy to analyze. It is the reason we will use Markowitz portfolios to model real portfolios.*

Markowitz Portfolio Returns: Vector Notation

Relationship (3.12) can be expressed in the compact form

$$r(d) = \mathbf{r}(d)^{\mathrm{T}} \mathbf{f} \,, \tag{3.13}$$

where $\mathbf{f} \in \mathcal{M}$ and $\mathbf{r}(d)$ are the N-vector defined by

$$\mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix} .$$

We call $\mathbf{r}(d)$ the *return vector* or simply the *returns* for day d.

Markowitz Portfolio Returns: Solvent

Because by (3.13) we have $r(d) = \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}$ for the Markowitz portfolio with allocation $\mathbf{f}$, it follows from charactertization (1.3) of solvent portfolios given by **Fact 1** that this portfolio is solvent if and only if

$$1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} > 0 \qquad \text{for every } d = 1, \cdots, D\,. \tag{3.14}$$

The set of allocations for solvent Markowitz portfolios is thereby

$$\Omega = \left\{ \mathbf{f} \in \mathcal{M} \;:\; 1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} > 0 \;\; \forall d \right\}. \tag{3.15}$$

Unlike $\mathcal{M}$ and Λ, this set depends upon the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$, with each day giving an inequality constraint. Because of this complication, we will not try to visualize Ω now.

Markowitz Portfolio Returns: Long are Solvent

Rather, we now show that *every long Markowitz portfolio is solvent*. In other words, we show that $\Lambda \subset \Omega$. This shows that many solvent portfolios exist. The proof uses the fact that $\mathbf{1} + \mathbf{r}(d) > \mathbf{0}$, which states that every entry of $\mathbf{1} + \mathbf{r}(d)$ is positive.

Fact 2. We have $\Lambda \subset \Omega$.

Proof. Let $\mathbf{f} \in \Lambda$. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{f} \geq \mathbf{0}$, at least one entry of $\mathbf{f}$ must be positive. Because $\mathbf{1} + \mathbf{r}(d) > \mathbf{0}$, $\mathbf{f} \geq \mathbf{0}$, and at least one entry of $\mathbf{f}$ is positive, we have

$$1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} = \left(\mathbf{1} + \mathbf{r}(d)\right)^{\mathrm{T}}\mathbf{f} > 0 \qquad \text{for every } d = 1, \cdots, D\,.$$

We conclude from (3.15) that $\mathbf{f} \in \Omega$. Therefore $\Lambda \subset \Omega$. □

Markowitz Portfolio Returns: Containments

Therefore the sets Λ, Ω, and $\mathcal{M}$ of allocations for long, solvent, and all Markowitz portfolios are related by

$$\Lambda \subset \Omega \subset \mathcal{M}\,. \tag{3.16}$$

We will start by doing analysis in $\mathcal{M}$. It is the easiest set of allocations to study because it is defined by the single equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$. However, when working in $\mathcal{M}$ we must be mindful that results that fall outside of Ω are nonsense. We will then do analysis in Λ. It is defined by the additional the N inequality constraints $\mathbf{f} \geq \mathbf{0}$. We will try to avoid analysis in Ω because it is defined by the additional D inequality constraints $1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} > 0$, which can lead numerical difficulties.

Markowitz Portfolio Statistics: m and V

Recall that if we assign weights $\{w(d)\}_{d=1}^{D}$ to the trading days of a daily return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ then the N-vector of return means $\mathbf{m}$ and the $N \times N$-matrix of return covariances $\mathbf{V}$ can be expressed in terms of $\mathbf{r}(d)$ as

$$\mathbf{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix} = \sum_{d=1}^{D} w(d)\, \mathbf{r}(d)\,,$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{pmatrix} = \sum_{d=1}^{D} w(d)\, \big(\mathbf{r}(d) - \mathbf{m}\big)\, \big(\mathbf{r}(d) - \mathbf{m}\big)^{\mathrm{T}}\,.$$

The choices of the daily return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and weights $\{w(d)\}_{d=1}^{D}$ specify the *calibration* of our models. *Ideally* $\mathbf{m}$ *and* $\mathbf{V}$ *should not be overly sensitive to these choices.*

Markowitz Portfolio Statistics: Return Mean

The return mean μ and return variance v for the Markowitz portfolio with allocation $\mathbf{f}$ can be expressed simply in terms of the return mean vector $\mathbf{m}$ and the covariance matrix $\mathbf{V}$.

Because $r(d) = \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$, the portfolio return mean μ for the Markowitz portfolio with allocation $\mathbf{f}$ is given by

$$\begin{aligned} \mu &= \sum_{d=1}^{D} w(d)\, r(d) = \sum_{d=1}^{D} w(d)\, \mathbf{r}(d)^{\mathrm{T}} \mathbf{f} \\ &= \left(\sum_{d=1}^{D} w(d)\, \mathbf{r}(d)^{\mathrm{T}} \right) \mathbf{f} = \mathbf{m}^{\mathrm{T}} \mathbf{f} \,. \end{aligned}$$

Hence, $\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}$.

Markowitz Portfolio Statistics: Return Variance

Because $r(d) = \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$, the portfolio return variance v for the Markowitz portfolio with allocation $\mathbf{f}$ is given by

$$
\begin{aligned}
v &= \sum_{d=1}^{D} w(d)\left(r(d) - \mu\right)^2 = \sum_{d=1}^{D} w(d)\left(\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} - \mathbf{m}^{\mathrm{T}}\mathbf{f}\right)^2 \\
&= \sum_{d=1}^{D} w(d)\left(\mathbf{f}^{\mathrm{T}}\mathbf{r}(d) - \mathbf{f}^{\mathrm{T}}\mathbf{m}\right)\left(\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} - \mathbf{m}^{\mathrm{T}}\mathbf{f}\right) \\
&= \mathbf{f}^{\mathrm{T}}\left(\sum_{d=1}^{D} w(d)\left(\mathbf{r}(d) - \mathbf{m}\right)\left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathrm{T}}\right)\mathbf{f} = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,.
\end{aligned}
$$

Hence, $v = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$. Because $\mathbf{V}$ is positive definite, $v > 0$.

Markowitz Portfolio Statistics: Volatility

Summarizing, the return mean μ and return variance v for the Markowitz portfolio with allocation $\mathbf{f}$ are given by

$$\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad v = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,. \tag{4.17}$$

The return volatility σ for the portfolio is thereby

$$\sigma = \sqrt{v} = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,. \tag{4.18}$$

These simple formulas are a reason to prefer returns over growth rates when compiling statistics of markets. This simplicity arises because $\mathbf{f}$ is independent of d and because the return $r(d)$ for the Markowitz portfolio with allocation $\mathbf{f}$ depends linearly upon the vector $\mathbf{r}(d)$ of returns as

$$r(d) = \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,.$$

Markowitz Portfolio Statistics: Growth Rate Contrast

Remark. In contrast, the growth rates $x(d)$ of any solvent Markowitz portfolio with allocation $\mathbf{f}$ are given by

$$
\begin{aligned}
x(d) &= \log\left(\frac{\pi(d)}{\pi(d-1)}\right) = \log(1 + r(d)) \\
&= \log\left(1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\right) = \log\left(1 + \sum_{i=1}^{N} r_i(d)\, f_i\right) \\
&= \log\left(1 + \sum_{i=1}^{N} \left(e^{x_i(d)} - 1\right)\, f_i\right) = \log\left(\sum_{i=1}^{N} e^{x_i(d)}\, f_i\right) .
\end{aligned}
$$

Because the $x(d)$ are not linear functions of the $x_i(d)$, averages of $x(d)$ over d are not simply expressed in terms of averages of $x_i(d)$ over d. Moreover, they are not defined for allocations outside Ω.

Critique: Unrealistic Aspects

Aspects of Markowitz portfolios are unrealistic. These include:

- the fact portfolios can contain fractional shares of any asset;
- the fact portfolios are rebalanced every trading day;
- the fact transaction costs and taxes are neglected;
- the fact dividends are neglected.

By making these simplifications the subsequent analysis becomes easier. The idea is to find the Markowitz portfolio that is best for a given investor. The expectation is that any real portfolio with an allocation close to that for the optimal Markowitz portfolio will perform similarly. In practice, most investors rebalance at most a few times per year, and not every asset is involved each time. Transaction costs and taxes are thereby limited. Similarly, borrowing costs are kept to a minimum by not borrowing often. The model accounts for dividends by using adjusted closing prices.

Critique: Limitations

Remark. Portfolios of risky assets can be designed for *trading or investing.*

Traders often take positions that require constant attention. They might buy and sell assets on short time scales in an attempt to profit from market fluctuations. They might also take highly leveraged positions that expose them to enormous gains or loses depending how the market moves. They must be ready to handle margin calls. Trading is often a full time job.

Investors operate on longer time scales. They buy or sell an asset based on their assessment of its fundamental value over time. Investing does not have to be a full time job. Indeed, most people who hold risky assets are investors who are saving for retirement. Lured by the promise of high returns, sometimes investors will buy shares in funds designed for traders. At that point they have become gamblers, whether they realize it or not.

The ideas presented in this course are designed to balance investment portfolios, not trading portfolios.