Portfolios that Contain Risky Assets 8: Limited Portfolios and Their Frontiers

C. David Levermore

University of Maryland, College Park, MD

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

Limited Portfolios and Their Frontiers

Leveraged Markowitz Portfolios

Leveraged portfolios are ones that take short positions. Short positions can offer the promise of great reward, but come with the potential for greater losses. They are favored by quantitative hedge funds, notable examples being the Medallion, RIEF, and RIDA funds run by Renaissance Technologies. They were also favored by investment banks during the first decade of the 21st century, and played a major role in bringing about the subsequent recession. They had a similar role in bringing about the great depression seventy eight years earlier. In fact, they have played a role in every major market crash, such as the dot-com crash of 2000.

Leveraged portfolios contribute to a bubble crash because there are limits on their leverage. When their leverage exceeds their limit then their margins are called and they have to liquidate positions. Because leveraged portfolios can create systemic risk, they are something about which every investor should have some knowledge.

Leveraged Markowitz Portfolios

In order to quantify the leverage of a general Markowitz, we decompose its allocation $\mathbf{f}$ into its long and short positions as

$$\mathbf{f} = \mathbf{f}_+ - \mathbf{f}_- \,, \tag{1.1}$$

where $f_i^{\pm}$, the i^{th} entry of $\mathbf{f}_{\pm}$, is given by

$$f_i^+ = \max\{f_i\,, 0\}\,, \qquad f_i^- = \max\{-f_i\,, 0\}\,.$$

This is the so-called *long-short decomposition* of $\mathbf{f}$. The vectors $\mathbf{f}_+$ and $\mathbf{f}_-$ in this decomposition are characterized by

$$\mathbf{f}_+ \geq \mathbf{0}\,, \qquad \mathbf{f}_- \geq \mathbf{0}\,, \qquad \mathbf{f}_+^{\mathrm{T}}\mathbf{f}_- = 0\,.$$

The multiples of the portfolio value that are held in long and short positions respectively are

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_+\,, \quad \text{and} \quad \mathbf{1}^{\mathrm{T}}\mathbf{f}_-\,. \tag{1.2}$$

Leveraged Markowitz Portfolios

Recall that the set of allocations for long Markowitz portfolios is

$$\Lambda = \{ \mathbf{f} \in \mathcal{M} \,:\, \mathbf{f} \geq \mathbf{0} \} \,. \tag{1.3}$$

Fact 1. We have

$$\Lambda = \{ \mathbf{f} \in \mathcal{M} \,:\, \mathbf{1}^{\mathrm{T}} \mathbf{f}_{-} = 0 \} \,. \tag{1.4}$$

Proof. Let $\mathbf{f} \in \Lambda$. Because $\mathbf{f} \geq \mathbf{0}$, we have $\mathbf{f}_{+} = \mathbf{f}$ and $\mathbf{f}_{-} = \mathbf{0}$. Therefore $\mathbf{1}^{\mathrm{T}} \mathbf{f}_{-} = 0$.

Conversely, let $\mathbf{f} \in \mathcal{M}$ such that $\mathbf{1}^{\mathrm{T}} \mathbf{f}_{-} = 0$. Let $\mathbf{f} = \mathbf{f}_{+} - \mathbf{f}_{-}$ be the long-short decomposition of $\mathbf{f}$ given by (1.1). Because $\mathbf{f}_{-} \geq \mathbf{0}$ while $\mathbf{1}^{\mathrm{T}} \mathbf{f}_{-} = 0$, we conclude that $\mathbf{f}_{-} = \mathbf{0}$. Therefore $\mathbf{f} = \mathbf{f}_{+} \geq \mathbf{0}$. Hence, $\mathbf{f} \in \Lambda$.
$\square$

Leveraged Markowitz Portfolios

Recall that leverged portfolios are ones that are not long — i.e. are ones that hold short positions. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-}$ is the multiple of the portfolio value that is held in short positions, we call $\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-}$ the *leverage* of the portfolio. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \geq 0$, the leverage is always nonnegative. By **Fact 1** we see that:

- long portfolios are those with zero leverage;
- leveraged portfolios are those with positive leverge.

The leverage of the Markowitz portfolio with allocation $\mathbf{f}$ can be computed without computing the long-short decomposition of $\mathbf{f}$.

Leveraged Markowitz Portfolios

The constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and decompostion (1.1) imply that

$$1 = \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mathbf{1}^{\mathrm{T}}\mathbf{f}_{+} - \mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \,.$$

We also have

$$\|\mathbf{f}\|_1 = \mathbf{1}^{\mathrm{T}}\mathbf{f}_{+} + \mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \,,$$

where $\|\mathbf{f}\|_1$ denotes the ℓ^1-norm of $\mathbf{f}$, which is defined by

$$\|\mathbf{f}\|_1 = \sum_{i=1}^{N} |f_i| \,.$$

Notice that $1 = |\mathbf{1}^{\mathrm{T}}\mathbf{f}| \leq \|\mathbf{f}\|_1$. By first adding and subtracting the top relation above from the second, and then multiplying by $\frac{1}{2}$, we obtain

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{+} = \tfrac{1}{2}\big(\|\mathbf{f}\|_1 + 1\big) \,, \qquad \mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} = \tfrac{1}{2}\big(\|\mathbf{f}\|_1 - 1\big) \,. \tag{1.5}$$

Notice that $\mathbf{1}^{\mathrm{T}}\mathbf{f}_{+} \geq 1$ and that $\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \geq 0$.

Leveraged Markowitz Portfolios

The second formula of (1.5) gives a simple way to compute the leverage of a portfolio with allocation **f** — namely,

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} = \tfrac{1}{2}\left(\|\mathbf{f}\|_1 - 1\right).$$

This formula does not require the long-short decomposition of **f**. It is very easy to program.

By **Fact 1** and the second formula of (1.5) we have

$$\Lambda = \left\{\mathbf{f} \in \mathcal{M} \;:\; \|\mathbf{f}\|_1 = 1\right\}. \tag{1.6}$$

This fact gives a simple way to determine if a portfolio with allocation **f** is long — namely, check if $\|\mathbf{f}\|_1 = 1$.

Leveraged Markowitz Portfolios

Remark. The Efficient Market Hypothesis asserts that index funds such as VFINX and VBMFX should lie near the efficient frontier. Therefore a long portfolio containing such funds should lie even closer to the efficient frontier. The Efficient Market Hypothesis can be checked by computing the leverage of portfolios that lie on the frontier. The most important such portfolios are $\mathbf{f}_{\mathrm{mv}}$, $\mathbf{f}_{\mathrm{st}}$, and $\mathbf{f}_{\mathrm{ct}}$. We define the metrics

$$\omega_{\mathrm{mv}} = \frac{1}{\|\mathbf{f}_{\mathrm{mv}}\|_1},$$

$$\omega_{\mathrm{st}} = \begin{cases} \dfrac{1}{\|\mathbf{f}_{\mathrm{st}}\|_1} & \text{if } \mathbf{f}_{\mathrm{st}} \text{ exists}, \\ 0 & \text{otherwise}, \end{cases} \qquad \omega_{\mathrm{ct}} = \begin{cases} \dfrac{1}{\|\mathbf{f}_{\mathrm{ct}}\|_1} & \text{if } \mathbf{f}_{\mathrm{ct}} \text{ exists}, \\ 0 & \text{otherwise}. \end{cases}$$

These take values in $[0, 1]$. The efficient market hypothesis says these should take values close to 1.

Limited-Leverage Portfolios

The class $\mathcal{M}$ of Markowitz portfolios is unrealistic because it allows an investor to take short positions without much collateral. In practice short positions are restricted by *credit limits*.

If we assume that in each case the lender is the broker and the collateral is part of the portfolio then a simple model for credit limits is to constrain the total short position of the portfolio to be at most a positive multiple ℓ of the portfolio value. The value of ℓ is called the *leverage limit* of the portfolio and will depend upon market conditions, but brokers will often allow $\ell > 1$ and seldom allow $\ell > 5$.

Remark. Just because a broker allows a particular value of ℓ does not mean it is in the best interest of an investor to build a portfolio with that value of ℓ. We will use this model to understand what values of ℓ might not be prudent. This understanding will give us a measure of when markets are stressed.

Limited-Leverage Portfolios

The constraint that the multiple of the portfolio value held in short positions is bounded by a leverage limit ℓ can be expressed as

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \leq \ell \,. \tag{2.7}$$

By (1.3) this is equivalent to

$$\tfrac{1}{2}\big(\|\mathbf{f}\|_1 - 1\big) = \mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \leq \ell \,.$$

Therefore the set of allocations for Markowitz portfolios with a leverage limit $\ell \in [0, \infty)$ is

$$\Pi_\ell = \left\{\mathbf{f} \in \mathcal{M} \,:\, \|\mathbf{f}\|_1 \leq 1 + 2\ell\right\}. \tag{2.8}$$

It is clear that if $\ell, \ell' \in [0, \infty)$ then

$$\ell \leq \ell' \qquad \Longrightarrow \qquad \Pi_\ell \subset \Pi_{\ell'} \,.$$

Limited-Leverage Portfolios

By **Fact 1** and (1.6) we see that

$$\Pi_0 = \{\mathbf{f} \in \mathcal{M} \,:\, \|\mathbf{f}\|_1 \leq 1\} = \Lambda \,. \tag{2.9}$$

Hence, the limited leverage Markowitz portfolios with leverage limit $\ell = 0$ are exactly the long Markowitz portfolios.

It is clear from (2.8) that if $\ell \in [0, \infty)$ then $\Pi_\ell \subset \mathcal{M}$. Moreover, it is also clear that

$$\bigcup_{\ell \in [0,\infty)} \Pi_\ell = \mathcal{M} \,. \tag{2.10}$$

In words, the union of the sets Π_ℓ over $\ell \in [0, \infty)$ is $\mathcal{M}$.

Limited-Leverage Portfolios

Because Π_ℓ is a bounded set, its return means are bounded. Recall that

$$\mu_{\mathrm{mn}} = \min\{m_1\,,\, m_2\,,\, \cdots\,,\, m_N\}\,, \qquad \mu_{\mathrm{mx}} = \max\{m_1\,,\, m_2\,,\, \cdots\,,\, m_N\}\,,$$

For every $\ell \in [0, \infty)$ define

$$\mu^\ell_{\mathrm{mn}} = \mu_{\mathrm{mn}} - \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,, \qquad \mu^\ell_{\mathrm{mx}} = \mu_{\mathrm{mx}} + \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,.$$

Then for every $\mathbf{f} \in \Pi^\ell$ the return mean μ satisfies the bounds

$$\mu^\ell_{\mathrm{mn}} \leq \mu \leq \mu^\ell_{\mathrm{mx}}\,.$$

These bounds are sharp:

- $\mu = \mu^\ell_{\mathrm{mn}}$ for any portfolio with allocation $-\ell$ in an asset with return mean μ_{mx} and allocation $1+\ell$ in an asset with return mean μ_{mn}.
- $\mu = \mu^\ell_{\mathrm{mx}}$ for any portfolio with allocation $-\ell$ in an asset with return mean μ_{mn} and allocation $1+\ell$ in an asset with return mean μ_{mx}.

Limited-Leverage Portfolios

Because Π_ℓ is a bounded set, its return variances are bounded. Recall that

$$v_{\mathrm{mx}} = \max\{v_{11}\,,\ v_{22}\,,\ \cdots\,,\ v_{NN}\}\,,$$

Then for every $\ell \in [0, \infty)$ and every $\mathbf{f} \in \Pi^\ell$ the return variance v satisfies the bounds

$$0 < v_{\mathrm{mv}} \leq v \leq v_{\mathrm{mx}}(1 + 2\ell)^2\,.$$

These bounds are not sharp. We will prove better bounds later.

Limited-Leverage Constraints

Recall that Π_ℓ is the set of all limited-leverage portfolio allocations with leverage limit $\ell \geq 0$ and that $\Pi_\ell(\mu)$ is the set of all such allocations with return mean μ. These sets are given by

$$\Pi_\ell = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \|\mathbf{f}\|_1 \leq 1 + 2\ell\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \right\},$$
$$\Pi_\ell(\mu) = \left\{ \mathbf{f} \in \Pi_\ell \,:\, \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}.$$

Clearly $\Pi_\ell(\mu) \subset \Pi_\ell$ for every $\mu \in \mathbb{R}$.

The set Π_ℓ is a convex polytope of dimension $N-1$ that is contained in the hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\}$. The set $\Pi_\ell(\mu)$ is the intersection of Π_ℓ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. Because we have assumed that $\mathbf{m}$ and $\mathbf{1}$ are not proportional, the intersection of these hyperplanes is a set of dimension $N-2$. Therefore the set $\Pi_\ell(\mu)$ is a convex polytope of dimension at most $N-2$, but it might be empty.

Limited-Leverage Constraints

We start by charactering those μ for which $\Pi_\ell(\mu)$ is nonempty. Recall that

$$\mu_{\mathrm{mn}} = \min\{m_i \,:\, i = 1, \cdots, N\}\,, \quad \mu_{\mathrm{mx}} = \max\{m_i \,:\, i = 1, \cdots, N\}\,.$$

We expect that the ℓ-limited leverage portfolio with the highest mean return would have a long allocation of $1+\ell$ in an asset with mean return μ_{mx} and short allocation of $-\ell$ in an asset with mean return μ_{mn}. The mean return of such a portfolio is

$$\mu^\ell_{\mathrm{mx}} = (1+\ell)\mu_{\mathrm{mx}} - \ell\mu_{\mathrm{mn}} = \mu_{\mathrm{mx}} + \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,.$$

Similarly, we expect that the ℓ-limited leverage portfolio with the lowest mean return would have a long allocation of $1+\ell$ in an asset with mean return μ_{mn} and short allocation of $-\ell$ in an asset with mean return μ_{mx}. The mean return of such a portfolio is

$$\mu^\ell_{\mathrm{mn}} = (1+\ell)\mu_{\mathrm{mn}} - \ell\mu_{\mathrm{mx}} = \mu_{\mathrm{mn}} - \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,.$$

Limited-Leverage Constraints

Indeed, we will prove the following.

Fact. *For every* $\ell \geq 0$ *the set* $\Pi_\ell(\mu)$ *is nonempty if and only if* $\mu \in [\mu_{\mathrm{mn}}^\ell, \mu_{\mathrm{mx}}^\ell]$*, where*

$$\mu_{\mathrm{mn}}^\ell = \mu_{\mathrm{mn}} - \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}) \,, \qquad \mu_{\mathrm{mx}}^\ell = \mu_{\mathrm{mx}} + \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}) \,.$$

Remark. Because we have assumed that $\mathbf{m}$ and $\mathbf{1}$ are not proportional, the mean returns $\{m_i\}_{i=1}^N$ are not identical. This implies that $\mu_{\mathrm{mn}} < \mu_{\mathrm{mx}}$, which implies that the interval $[\mu_{\mathrm{mn}}^\ell, \mu_{\mathrm{mx}}^\ell]$ does not reduce to a point. Indeed, when $\ell_2 > \ell_1 \geq 0$ we have

$$\mu_{\mathrm{mn}}^{\ell_2} < \mu_{\mathrm{mn}}^{\ell_1} < \mu_{\mathrm{mx}}^{\ell_1} < \mu_{\mathrm{mx}}^{\ell_2} \,.$$

Limited-Leverage Constraints

Proof. Let $\Pi_\ell(\mu)$ be nonempty for some $\mu \in \mathbb{R}$. Let $\mathbf{f} \in \Pi_\ell(\mu)$ and let $\mathbf{f} = \mathbf{f}_+ - \mathbf{f}_-$ be the long-short decomposistion of $\mathbf{f}$. Because $\mu_{\mathrm{mn}}\mathbf{1} \leq \mathbf{m} \leq \mu_{\mathrm{mx}}\mathbf{1}$, and because $\mathbf{f}_\pm \geq \mathbf{0}$, we have

$$\mu_{\mathrm{mn}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_\pm \leq \mathbf{m}^{\mathrm{T}}\mathbf{f}_\pm \leq \mu_{\mathrm{mx}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_\pm \,.$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ we have

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_+ = \mathbf{1}^{\mathrm{T}}(\mathbf{f} + \mathbf{f}_-) = \mathbf{1}^{\mathrm{T}}\mathbf{f} + \mathbf{1}^{\mathrm{T}}\mathbf{f}_- = 1 + \mathbf{1}^{\mathrm{T}}\mathbf{f}_- \,.$$

Because $\mathbf{f} \in \Pi_\ell$ we have $\mathbf{1}^{\mathrm{T}}\mathbf{f}_- \leq \ell$. These facts combine to give

$$\begin{aligned}\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mathbf{m}^{\mathrm{T}}\mathbf{f}_+ - \mathbf{m}^{\mathrm{T}}\mathbf{f}_- &\leq \mu_{\mathrm{mx}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_+ - \mu_{\mathrm{mn}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_- \\ &= \mu_{\mathrm{mx}} + (\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\mathbf{1}^{\mathrm{T}}\mathbf{f}_- \\ &\leq \mu_{\mathrm{mx}} + \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}) = \mu_{\mathrm{mx}}^{\ell} \,.\end{aligned}$$

Limited-Leverage Constraints

Similarly,

$$\begin{aligned}\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mathbf{m}^{\mathrm{T}}\mathbf{f}_{+} - \mathbf{m}^{\mathrm{T}}\mathbf{f}_{-} &\geq \mu_{\mathrm{mn}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_{+} - \mu_{\mathrm{mx}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \\ &= \mu_{\mathrm{mn}} - (\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \\ &\geq \mu_{\mathrm{mn}} - \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}) = \mu^{\ell}_{\mathrm{mn}} \,.\end{aligned}$$

Therefore $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$. Because $\mathbf{f} \in \Pi_{\ell}(\mu)$ was arbitrary, we conclude that *if $\Pi_{\ell}(\mu)$ is nonempty then $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$.*

Conversely, first choose $\mathbf{e}_{\mathrm{mn}}$ and $\mathbf{e}_{\mathrm{mx}}$ so that

$$\begin{aligned}\mathbf{e}_{\mathrm{mn}} &= \mathbf{e}_{i} \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\mathrm{mn}} \,, \\ \mathbf{e}_{\mathrm{mx}} &= \mathbf{e}_{j} \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\mathrm{mx}} \,.\end{aligned}$$

Limited-Leverage Constraints

Now let $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ and set

$$\mathbf{f} = \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mathbf{e}_{\mathrm{mx}} \, .$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} = \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} = 1$, $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} = \mu_{\mathrm{mn}}$, and $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} = \mu_{\mathrm{mx}}$, we see

$$\begin{aligned}
\mathbf{1}^{\mathrm{T}}\mathbf{f} &= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} \\
&= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} = 1 \, , \\
\mathbf{m}^{\mathrm{T}}\mathbf{f} &= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} \\
&= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mu_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} \, \mu_{\mathrm{mx}} = \mu \, .
\end{aligned}$$

Hence, $\mathbf{f} \in \Pi_{\infty}(\mu)$. We still need to show that $\mathbf{f} \in \Pi_{\ell}(\mu)$.

Limited-Leverage Constraints

Because $\mu \in [\mu_{\text{mn}}^{\ell}, \mu_{\text{mx}}^{\ell}]$ the allocations of $\mathbf{f}$ are bounded by

$$-\ell = \frac{\mu_{\text{mx}} - \mu_{\text{mx}}^{\ell}}{\mu_{\text{mx}} - \mu_{\text{mn}}} \leq \frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}} \leq \frac{\mu_{\text{mx}} - \mu_{\text{mn}}^{\ell}}{\mu_{\text{mx}} - \mu_{\text{mn}}} = 1 + \ell \,,$$

$$-\ell = \frac{\mu_{\text{mn}}^{\ell} - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}} \leq \frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}} \leq \frac{\mu_{\text{mx}}^{\ell} - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}} = 1 + \ell \,.$$

Because they sum to 1, at most one of them is negative. Hence,

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \leq \max\left\{ -\frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}} \,,\, -\frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}} \right\} \leq \ell \,.$$

Hence, $\mathbf{f} \in \Pi_{\ell}(\mu)$. *Therefore if $\mu \in [\mu_{\text{mn}}^{\ell}, \mu_{\text{mx}}^{\ell}]$ then $\Pi_{\ell}(\mu)$ is nonempty.* □

Limited-Leverage Constraints

Remark. For every $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$ the set $\Pi_{\ell}(\mu)$ *is a nonempty, closed, bounded, convex polytope of dimension at most* $N - 2$*.*

- When $N = 2$ it is a point.
- When $N = 3$ it is either a point or a line segment.
- When $N = 4$ it is either a point, a line segment, or a convex polygon.

Limited-Leverage Frontiers

The set Π_ℓ in $\mathbb{R}^N$ of all portfolio allocations with leverage limit ℓ is associated with the set $\Sigma(\Pi_\ell)$ in the $\sigma\mu$-plane of volatilities and return means given by

$$\Sigma(\Pi_\ell) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \ : \ \sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \ \mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \ \mathbf{f} \in \Pi_\ell \right\} .$$

The set $\Sigma(\Pi_\ell)$ is the image in $\mathbb{R}^2$ of the polytope Π_ℓ in $\mathbb{R}^N$ under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set Π_ℓ is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set $\Sigma(\Pi_\ell)$ is compact.

Limited-Leverage Frontiers

We have seen that the set $\Pi_\ell(\mu)$ of all ℓ-limited portfolio allocations with return mean μ is nonempty if and only if $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$. Hence, $\Sigma(\Pi_\ell)$ can be expressed as

$$\Sigma(\Pi_\ell) = \left\{ \left(\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,,\, \mu \right) \,:\, \mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]\,,\ \mathbf{f} \in \Pi_\ell(\mu) \right\}.$$

The points on the boundary of $\Sigma(\Pi_\ell)$ that correspond to those ℓ-limited portfolios that have less volatility than every other ℓ-limited portfolio with the same return mean is called the *ℓ-limited frontier.*

Limited-Leverage Frontiers

The ℓ-limited frontier is the curve in the $\sigma\mu$-plane given by the equation

$$\sigma = \sigma_{\mathrm{f}}^{\ell}(\mu) \quad \text{over} \quad \mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}] \,,$$

where the value of $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is obtained for each $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ by solving the constrained minimization problem

$$\sigma_{\mathrm{f}}^{\ell}(\mu)^2 = \min\left\{ \sigma^2 \;:\; (\sigma,\mu) \in \Sigma(\Pi_\ell) \right\} = \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Pi_\ell(\mu) \right\} .$$

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is continuous over the compact set $\Pi_\ell(\mu)$, *a minimizer exists.*

Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Pi_\ell(\mu)$, whereby *the minimizer is unique.*

Limited-Leverage Frontiers

If we denote this unique minimizer by $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ then for every $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ the function $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is given by

$$\sigma_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{f}}^{\ell}(\mu)}\,,$$

where $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\mathbf{f}_{\mathrm{f}}^{\ell}(\mu) = \arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \ : \ \mathbf{f} \in \Pi_{\ell}(\mu) \right\}.$$

Here $\arg\min$ is read *"the argument that minimizes".* It means that $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ subject to the given constraints.
Remark. This problem cannot be solved by Lagrange multipliers because the set $\Pi_{\ell}(\mu)$ is defined by inequality constraints. It is harder to solve analytically than the analogous minimization problem for portfolios with unlimited leverage. Therefore we will first present a numerical approach that can generally be applied.

Quadratic Programming

Because the function being minimized is quadratic in $\mathbf{f}$ while the constraints are linear in $\mathbf{f}$, this is called a *quadratic programming problem*. It can be solved for a particular $\mathbf{V}$, $\mathbf{m}$, and μ by using either the Matlab command "quadprog" or an equivalent command in some other language.

The Matlab command quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\mathrm{eq}}, \mathbf{d}_{\mathrm{eq}})$ returns the solution of a quadratic programming problem in the standard form

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} \ : \ \mathbf{x} \in \mathbb{R}^M ,\ \mathbf{C}\mathbf{x} \leq \mathbf{d} ,\ \mathbf{C}_{\mathrm{eq}}\mathbf{x} = \mathbf{d}_{\mathrm{eq}} \right\} ,$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^M$, $\mathbf{C} \in \mathbb{R}^{K\times M}$, $\mathbf{d} \in \mathbb{R}^K$, $\mathbf{C}_{\mathrm{eq}} \in \mathbb{R}^{K_{\mathrm{eq}}\times M}$, and $\mathbf{d}_{\mathrm{eq}} \in \mathbb{R}^{K_{\mathrm{eq}}}$. Here K and K_{eq} are the number of inequality and equality constraints respectively.

Quadratic Programming

Given $\mathbf{V}$, $\mathbf{m}$, and $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$, the problem that we want to solve to obtain $\mathbf{f}^{\ell}_{\mathrm{f}}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N ,\; \|\mathbf{f}\|_1 \leq 1 + 2\ell ,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 ,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\} .$$

By comparing this with the standard quadratic programming problem on the previous slide we see that if we set $\mathbf{x} = \mathbf{f}$ then $M = N$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \mathbf{V} , \quad \mathbf{b} = \mathbf{0} , \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix} , \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} .$$

However, it is not as clear how to express the inequality constraint $\|\mathbf{f}\|_1 \leq 1 + 2\ell$ in the standard form $\mathbf{C}\mathbf{f} \leq \mathbf{d}$.

Quadratic Programming

The inequality $\|\mathbf{f}\|_1 \leq 1 + 2\ell$ can be expressed as the inequality constraints

$$\pm f_1 \pm f_2 \pm \cdots \pm f_{N-1} \pm f_N \leq 1 + 2\ell \,,$$

where there are 2^N choices of $\pm$ signs. When the $\pm$ are chosen to be the same sign then the inequality constraint is always satisfied because of the the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$. That leaves $2^N - 2$ inequality constraints that still need to be imposed.

The number $2^N - 2$ grows too fast with N for this approach to be useful for all but small values of N. For example, when $N = 9$ we have $2^N - 2 = 510$. With this many inequality constraints quadprog could suffer numerical difficulties. This raises the following question.

Are all of these $2^N - 2$ inequality constraints needed?

Quadratic Programming

The answer is yes if we insist on setting $\mathbf{x} = \mathbf{f}$. However, the answer is no if we enlarge the dimension of $\mathbf{x}$.

To understand why the answer is yes if we insist on setting $\mathbf{x} = \mathbf{f}$, consider any of these inequality constraints written along with the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ as

$$\pm f_1 \pm f_2 \pm \cdots \pm f_{N-1} \pm f_N \leq 1 + 2\ell \,,$$
$$f_1 + f_2 + \cdots + f_{N-1} + f_N = 1 \,.$$

By adding these and dividing by 2 we obtain

$$\sum_{i \in S} f_i \leq 1 + \ell \,,$$

where S is the subset of indices i with a plus in the inequality constraint.

Quadratic Programming

For every $S \subset \{1, 2, \cdots, N\}$ define the i^{th} entry of $\mathbf{1}_S \in \mathbb{R}^N$ by

$$\mathrm{ent}_i(\mathbf{1}_S) = \begin{cases} 1 & \text{if } i \in S\,, \\ 0 & \text{if } i \notin S\,. \end{cases}$$

Then the $2^N - 2$ inequality conatraints can be expressed as

$$\mathbf{1}_S^{\mathrm{T}}\mathbf{f} \leq 1 + \ell \quad \text{for every nonempty, proper } S \subset \{1, 2, \cdots, N\}\,.$$

The equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ can be used to show that these $2^N - 2$ inequality conatraints can also be expressed as

$$-\ell \leq \mathbf{1}_S^{\mathrm{T}}\mathbf{f} \quad \text{for every nonempty, proper } S \subset \{1, 2, \cdots, N\}\,.$$

Quadratic Programming

To understand why the answer is no if we enlarge the dimension of $\mathbf{x}$, consider the following equivalences.

$$\Pi_\ell = \left\{ \mathbf{f} \in \mathbb{R}^N \ : \ \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \mathbf{s} \in \mathbb{R}^N,\ \mathbf{s} \geq \mathbf{0}\,,\ (\mathbf{f} + \mathbf{s}) \geq \mathbf{0}\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{s} \leq \ell \right\}$$
$$= \left\{ \mathbf{f} \in \mathbb{R}^N \ : \ \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \mathbf{g} \in \mathbb{R}^N,\ (\mathbf{g} \pm \mathbf{f}) \geq \mathbf{0}\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{g} \leq 1 + 2\ell \right\}.$$

The two sets on the right-hand side above are equal by the relations

$$\mathbf{s} = \tfrac{1}{2}(\mathbf{g} - \mathbf{f})\,, \qquad \mathbf{g} = \mathbf{f} + 2\mathbf{s}\,.$$

We must show that they are also equal to Π_ℓ. This is left as an exercise.

Quadratic Programming

If we use the first equivalence then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{\ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\ :\ \mathbf{s} \geq \mathbf{0}\,,\ (\mathbf{f}+\mathbf{s}) \geq \mathbf{0}\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{s} \leq \ell\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\ \right\}.$$

By comparing this with the standard quadratic programming problem we see that if we set $\mathbf{x} = (\mathbf{f}\ \ \mathbf{s})^{\mathrm{T}}$ then $M = 2N$, $K = 2N+1$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -\mathbf{I} & -\mathbf{I} \\ \mathbf{O} & -\mathbf{I} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \ell \end{pmatrix}, \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where $\mathbf{O}$ and $\mathbf{I}$ are the $N\times N$ zero and identity matrices.

Quadratic Programming

If we use the second equivalence then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{g}\pm\mathbf{f}) \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{g} \leq 1+2\ell\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{f}=1\,,\; \mathbf{m}^{\mathrm{T}}\mathbf{f}=\mu \right\}.$$

By comparing this with the standard quadratic programming problem we see that if we set $\mathbf{x} = (\mathbf{f}\;\mathbf{g})^{\mathrm{T}}$ then $M = 2N$, $K = 2N+1$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \begin{pmatrix} \mathbf{V} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix},$$

$$\mathbf{C} = \begin{pmatrix} -\mathbf{I} & -\mathbf{I} \\ \mathbf{I} & -\mathbf{I} \\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ 1+2\ell \end{pmatrix}, \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}} \end{pmatrix}, \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where $\mathbf{O}$ and $\mathbf{I}$ are the $N \times N$ zero and identity matrices.

Quadratic Programming

In either case $\mathbf{f}^{\ell}_{\mathrm{f}}(\mu)$ can be obtained as the first N entries of the output x of a quadprog command that is formated as

$$\mathrm{x} = \mathsf{quadprog}(\mathrm{A}, \mathrm{b}, \mathrm{C}, \mathrm{d}, \mathrm{Ceq}, \mathrm{deq})\,,$$

where the matrices A, C, and Ceq, and the vectors b, d, and deq are given on the previous slides.

Remark. By doubling the dimension of the vector $\mathbf{x}$ from N to $2N$ we have reduced the number of inequality constraints from $2^N - 2$ to $2N + 1$. When $N = 9$ this is a reduction from 510 to 19!

Remark. There are other ways to use quadprog to obtain $\mathbf{f}^{\ell}_{\mathrm{f}}(\mu)$. Documentation for this command is easy to find on the web.

Computing Limited-Leverage Frontiers

When computing an ℓ-limited frontier, it helps to know some general properties of the function $\sigma_{\mathrm{f}}^{\ell}(\mu)$. These include:

- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is *continuous* over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is *strictly convex* over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is *piecewise hyperbolic* over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$.

This means that $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell})$ where $\sigma_{\mathrm{f}}^{\ell}(\mu)$ has either *a jump discontinuity in its first derivative* or *a jump discontinuity in its second derivative.*

Guided by these facts we now show how *an ℓ-limited frontier can be approximated numerically with the Matlab command quadprog.*

Computing Limited-Leverage Frontiers

First, partition the interval $[\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ as

$$\mu^\ell_{\mathrm{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu^\ell_{\mathrm{mx}}\,.$$

For example, set $\mu_k = \mu^\ell_{\mathrm{mn}} + k(\mu^\ell_{\mathrm{mx}} - \mu^\ell_{\mathrm{mn}})/n$ for a uniform partition. Pick n large enough to resolve all the features of the ℓ-limited frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

Second, for every $k = 0, \cdots, n$ use quadprog to compute $\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)\}_{k=0}^n$ should be saved.

Third, for every $k = 0, \cdots, n$ compute σ_k by

$$\sigma_k = \sigma^\ell_{\mathrm{f}}(\mu_k) = \sqrt{\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)^{\mathrm{T}}\mathbf{V}\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)}\,.$$

Computing Limited-Leverage Frontiers

Remark. There is typically a unique m_i such that $\mu^{\ell}_{\mathrm{mn}} = m_i$, in which case we have

$$\mathbf{f}^{\ell}_{\mathrm{f}}(\mu_0) = \mathbf{e}_i \,, \qquad \sigma_0 = \sqrt{v_{ii}} \,.$$

Similarly, there is typically a unique m_j such that $\mu^{\ell}_{\mathrm{mx}} = m_j$, in which case we have

$$\mathbf{f}^{\ell}_{\mathrm{f}}(\mu_n) = \mathbf{e}_j \,, \qquad \sigma_n = \sqrt{v_{jj}} \,.$$

Finally, we "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the ℓ-limited frontier in the $\sigma\mu$-plane. This can be done by linear interpolation. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\sigma}^{\ell}_{\mathrm{f}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \sigma_k \,.$$

Computing Limited-Leverage Frontiers

A better way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}} \, \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}} \, \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k) \,,$$

and then set

$$\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu)^{\mathrm{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu)} \,.$$

Remark. This will be a very good approximation if n is large enough. Over each interval (μ_{k-1}, μ_k) it approximates $\sigma_{\mathrm{f}}^{\ell}(\mu)$ with a hyperbola rather than with a line.

Computing Limited-Leverage Frontiers

Remark. Because $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k) \in \Pi_\ell(\mu_k)$ and $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_{k-1}) \in \Pi_\ell(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu) \in \Pi_\ell(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k)\,.$$

Therefore $\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu)$ gives an approximation to the ℓ-limited frontier that lies on or to the right of the ℓ-limited frontier in the $\sigma\mu$-plane.

Remark. When there are no nodes in the interval (μ_{k-1}, μ_k) then we can use the two-fund property to show that $\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \sigma_{\mathrm{f}}^{\ell}(\mu)$.

General Portfolio with Two Risky Assets

Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\text{mn}} = m_1$, $\mu_{\text{mx}} = m_2$ and

$$\mu^\ell_{\text{mn}} = m_1 - \ell(m_2 - m_1), \qquad \mu^\ell_{\text{mn}} = m_2 + \ell(m_2 - m_1).$$

Recall that for every $\mu \in \mathbb{R}$ the unique portfolio allocation that satisfies the constraints $\mathbf{1}^{\text{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\text{T}}\mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu) \in \Pi_\ell$ if and only if $\mu \in [\mu^\ell_{\text{mn}}, \mu^\ell_{\text{mx}}]$.

General Portfolio with Two Risky Assets

Therefore the set $\Pi_\ell(\mu)$ is given by

$$\Pi_\ell = \left\{ \mathbf{f}(\mu) \,:\, \mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}] \right\}.$$

In other words, the set Π_ℓ is the line segment in $\mathbb{R}^2$ that is the image of the interval $[\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$.

Because for every $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ the set $\Pi_\ell(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ over $\Pi_\ell(\mu)$ is $\mathbf{f}(\mu)$. Therefore the ℓ-limited frontier portfolios are

$$\mathbf{f}^\ell_{\mathrm{f}}(\mu) = \mathbf{f}(\mu) \qquad \text{for } \mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}],$$

and the ℓ-limited frontier is given by

$$\sigma = \sigma^\ell_{\mathrm{f}}(\mu) = \sqrt{\mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\,\mathbf{f}(\mu)} \qquad \text{for } \mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}].$$

Hence, the ℓ-limited frontier is simply a segment of the frontier hyperbola. It has no nodes.

General Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \le m_2 \le m_3 , \qquad m_1 < m_3 .$$

Then $\mu_{\text{mn}} = m_1$, $\mu_{\text{mx}} = m_3$ and

$$\mu^{\ell}_{\text{mn}} = m_1 - \ell(m_3 - m_1), \qquad \mu^{\ell}_{\text{mn}} = m_3 + \ell(m_3 - m_1).$$

General Portfolio with Three Risky Assets

Recall that for every $\mu \in \mathbb{R}$ the portfolios that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\, \mathbf{n}\,, \qquad \text{for some } \phi \in \mathbb{R}\,,$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

It can be shown that $\mathbf{f}(\mu, \phi) \in \Pi_\ell$ if and only if $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$, $\phi \in [-\ell, 1 + \ell]$, and

$$-\ell \le \frac{m_3 - \mu}{m_3 - m_1} - \phi\, \frac{m_3 - m_2}{m_3 - m_1} \le 1 + \ell\,,$$

$$-\ell \le \frac{\mu - m_1}{m_3 - m_1} - \phi\, \frac{m_2 - m_1}{m_3 - m_1} \le 1 + \ell\,.$$

General Portfolio with Three Risky Assets

This region can be expressed as

$$\phi^{\ell}_{\mathrm{mn}}(\mu) \leq \phi \leq \phi^{\ell}_{\mathrm{mx}}(\mu)\,,$$

where

$$\phi^{\ell}_{\mathrm{mn}}(\mu) = -\min\left\{ \frac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_3 - m_2}\,,\ \ell\,,\ \frac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_2 - m_1} \right\}\,,$$

$$\phi^{\ell}_{\mathrm{mx}}(\mu) = \min\left\{ \frac{\mu - \mu^{\ell}_{\mathrm{mn}}}{m_2 - m_1}\,,\ 1+\ell\,,\ \frac{\mu^{\ell}_{\mathrm{mx}} - \mu}{m_3 - m_2} \right\}\,.$$

When $\ell > 0$ it is the hexagon $\mathcal{H}_\ell$ in the $\mu\phi$-plane whose vertices are the six distinct points

$$\begin{aligned} &(\mu^{\ell}_{\mathrm{mn}}, 0)\,, \quad (m_1 - \ell(m_2 - m_1), -\ell)\,, \quad (m_2 - \ell(m_3 - m_2), 1+\ell)\,, \\ &(\mu^{\ell}_{\mathrm{mx}}, 0)\,, \quad (m_3 + \ell(m_3 - m_2), -\ell)\,, \quad (m_2 + \ell(m_2 - m_1), 1+\ell)\,. \end{aligned}$$

General Portfolio with Three Risky Assets

Therefore the set Π_ℓ is given by

$$\Pi_\ell = \{\mathbf{f}(\mu,\phi) \,:\, (\mu,\phi) \in \mathcal{H}_\ell\}\,.$$

In other words, the set Π_ℓ is the hexagon in $\mathbb{R}^3$ that is the image of the hexagon $\mathcal{H}_\ell$ under the affine mapping $(\mu,\phi) \mapsto \mathbf{f}(\mu,\phi)$.

Because for every $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ the set $\Pi_\ell(\mu)$ is the intersection of the hexagon Π_ℓ with the plane $\{\mathbf{f} \in \mathbb{R}^3 \,:\, \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. This is a line segment that might be a single point. It is given by

$$\Pi_\ell(\mu) = \{\mathbf{f}(\mu,\phi) \,:\, \phi^\ell_{\mathrm{mn}}(\mu) \le \phi \le \phi^\ell_{\mathrm{mx}}(\mu)\}\,.$$

In other words, the line segment $\Pi_\ell(\mu)$ in $\mathbb{R}^3$ is the image of the interval $[\phi^\ell_{\mathrm{mn}}(\mu), \phi^\ell_{\mathrm{mx}}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu,\phi)$.

General Portfolio with Three Risky Assets

Hence, the point on the ℓ-limited frontier associated with $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ is $(\sigma^\ell_{\mathrm{f}}(\mu), \mu)$ where $\sigma^\ell_{\mathrm{f}}(\mu)$ solves the constrained minimization problem

$$\begin{aligned}\sigma^\ell_{\mathrm{f}}(\mu)^2 &= \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Pi_\ell(\mu) \right\} \\ &= \min\left\{ \mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) \;:\; \phi^\ell_{\mathrm{mn}}(\mu) \leq \phi \leq \phi^\ell_{\mathrm{mx}}(\mu) \right\}.\end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\,\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}$$

is a quadratic in ϕ, we see that it has a unique global minimizer at

$$\phi = \phi_{\mathrm{f}}(\mu) = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}.$$

This global minimizer corresponds to the frontier. It will be the minimizer of our constrained minimization problem for the ℓ-limited frontier if and only if $\phi^\ell_{\mathrm{mn}} \leq \phi_{\mathrm{f}}(\mu) \leq \phi^\ell_{\mathrm{mx}}(\mu)$.

General Portfolio with Three Risky Assets

If $\phi_{\rm f}(\mu) < \phi^{\ell}_{\rm mn}(\mu)$ then the objective function is increasing over $[\phi^{\ell}_{\rm mn}(\mu), \phi^{\ell}_{\rm mx}(\mu)]$, whereby its minimizer is $\phi = \phi^{\ell}_{\rm mn}(\mu)$.

If $\phi^{\ell}_{\rm mx}(\mu) < \phi_{\rm f}(\mu)$ then the objective function is decreasing over $[\phi^{\ell}_{\rm mn}(\mu), \phi^{\ell}_{\rm mx}(\mu)]$, whereby its minimizer is $\phi = \phi^{\ell}_{\rm mx}(\mu)$.

Hence, the minimizer $\phi^{\ell}_{\rm f}(\mu)$ of our constrained minimization problem is

$$
\begin{aligned}
\phi^{\ell}_{\rm f}(\mu) &= \begin{cases} \phi^{\ell}_{\rm mn}(\mu) & \text{if } \phi_{\rm f}(\mu) < \phi^{\ell}_{\rm mn}(\mu) \\ \phi_{\rm f}(\mu) & \text{if } \phi^{\ell}_{\rm mn}(\mu) \le \phi_{\rm f}(\mu) \le \phi^{\ell}_{\rm mx}(\mu) \\ \phi^{\ell}_{\rm mx}(\mu) & \text{if } \phi^{\ell}_{\rm mx}(\mu) < \phi_{\rm f}(\mu) \end{cases} \\
&= \max\left\{\phi^{\ell}_{\rm mn}(\mu)\,,\, \min\left\{\phi_{\rm f}(\mu)\,,\, \phi^{\ell}_{\rm mx}(\mu)\right\}\right\} \\
&= \min\left\{\max\left\{\phi^{\ell}_{\rm mn}(\mu)\,,\, \phi_{\rm f}(\mu)\right\}\,,\, \phi^{\ell}_{\rm mx}(\mu)\right\}\,.
\end{aligned}
$$

Therefore $\sigma^{\ell}_{\rm f}(\mu)^2 = \mathbf{f}(\mu, \phi^{\ell}_{\rm f}(\mu))^{\rm T}\mathbf{V}\mathbf{f}(\mu, \phi^{\ell}_{\rm f}(\mu))$.