

Portfolios that Contain Risky Assets 6: Long Portfolios and Their Frontiers

C. David Levermore

University of Maryland, College Park, MD

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Portfolios that Contain Risky Assets

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Long Portfolios and Their Frontiers

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Long Portfolios

Because the value of any portfolio with short positions can become negative, many investors will not hold a short position in any risky asset.

Portfolios that hold no short positions are called *long portfolios*.

A Markowitz portfolio with allocation \mathbf{f} is long if and only if $f_i \geq 0$ for every i . This can be expressed compactly as

$$\mathbf{f} \geq \mathbf{0}, \quad (1.1)$$

where $\mathbf{0}$ denotes the N -vector with each entry equal to 0 and the inequality is understood entrywise. Therefore the set of all long Markowitz portfolio allocations Λ is given by

$$\Lambda = \{\mathbf{f} \in \mathbb{R}^N : \mathbf{1}^T \mathbf{f} = 1, \mathbf{f} \geq \mathbf{0}\}. \quad (1.2)$$

Long Portfolios

Let \mathbf{e}_i denote the vector whose i^{th} entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^N f_i \mathbf{e}_i,$$

where $f_i \geq 0$ for every $i = 1, \dots, N$ and

$$\sum_{i=1}^N f_i = \mathbf{1}^T \mathbf{f} = 1.$$

This shows that Λ is simply all convex combinations of the vectors $\{\mathbf{e}_i\}_{i=1}^N$. We can visualize Λ when N is small.

When $N = 2$ it is the line segment that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Long Portfolios

When $N = 3$ it is the triangle that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general N it is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$.

Long Portfolios

Remark. When $N = 4$ it is easy to check that the tetrahedron $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ given by

$$\mathcal{T} = \left\{ \mathbf{z} \in \mathbb{R}^3 : \mathbf{w}_k \cdot \mathbf{z} \leq 1 \text{ for } k = 1, 2, 3, 4 \right\},$$

where

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

under the one-to-one affine mapping $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ given by

$$\Phi(\mathbf{z}) = \frac{1}{4} \begin{pmatrix} 1 - \mathbf{w}_1 \cdot \mathbf{z} \\ 1 - \mathbf{w}_2 \cdot \mathbf{z} \\ 1 - \mathbf{w}_3 \cdot \mathbf{z} \\ 1 - \mathbf{w}_4 \cdot \mathbf{z} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -z_1 \\ -z_2 \\ -z_3 \end{pmatrix}.$$

Long Portfolios

Because Λ is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$, it is a nonempty, convex, and bounded set. In addition, Λ is a closed set.

Proof. For any \mathbf{f} in the closure of Λ there exists a sequence $\{\mathbf{f}_n\}_{n \in \mathbb{N}} \subset \Lambda$ such that

$$\mathbf{f} = \lim_{n \rightarrow \infty} \mathbf{f}_n.$$

Because $\mathbf{f}_n \geq \mathbf{0}$ and $\mathbf{1}^T \mathbf{f}_n = 1$ for every $n \in \mathbb{N}$, we see that

$$\mathbf{f} = \lim_{n \rightarrow \infty} \mathbf{f}_n \geq \mathbf{0}, \quad \mathbf{1}^T \mathbf{f} = \lim_{n \rightarrow \infty} \mathbf{1}^T \mathbf{f}_n = 1.$$

Hence, $\mathbf{f} \in \Lambda$. Therefore Λ is a closed set. □

Therefore Λ is a nonempty, closed, bounded, convex set.

Long Portfolios

Because Λ is a bounded set, its return means are bounded. Let

$$\begin{aligned}\mu_{\min} &= \min\{m_1, m_2, \dots, m_N\}, \\ \mu_{\max} &= \max\{m_1, m_2, \dots, m_N\}.\end{aligned}$$

Then because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^T \mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda$ we have

$$\begin{aligned}\mu &= \mathbf{m}^T \mathbf{f} \geq \mu_{\min} \mathbf{1}^T \mathbf{f} = \mu_{\min}, \\ \mu &= \mathbf{m}^T \mathbf{f} \leq \mu_{\max} \mathbf{1}^T \mathbf{f} = \mu_{\max}.\end{aligned}$$

Therefore the return mean μ of any long portfolio satisfies the bounds

$$\mu_{\min} \leq \mu \leq \mu_{\max}.$$

We will show that these bounds are sharp.

Long Portfolios

Because Λ is a bounded set, its return variances are bounded. Let
The return variance of a long portfolio is bounded. Let

$$v_{\max} = \max\{v_{11}, v_{22}, \dots, v_{NN}\}.$$

Because $v_{ij} = c_{ij}\sqrt{v_{ii}v_{jj}}$ and $|c_{ij}| \leq 1$ we see that

$$|v_{ij}| = |c_{ij}|\sqrt{v_{ii}v_{jj}} \leq \sqrt{v_{ii}v_{jj}} \leq v_{\max}.$$

Then because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^T \mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda$ we have

$$v = \mathbf{f}^T \mathbf{V} \mathbf{f} \leq v_{\max} \mathbf{f}^T \mathbf{1} \mathbf{1}^T \mathbf{f} = v_{\max}.$$

Therefore the return variance v of any long portfolio satisfies the bounds

$$v_{\min} < v \leq v_{\max}.$$

We will show that this upper bound is sharp. The lower bound is not. It will be improved soon.

Long Constraints

Let $\Lambda(\mu)$ be the set of all long portfolio allocations with return mean μ . This set is given by

$$\Lambda(\mu) = \left\{ \mathbf{f} \in \Lambda : \mathbf{m}^T \mathbf{f} = \mu \right\}.$$

It is the intersection of the simplex Λ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{m}^T \mathbf{f} = \mu\}$. Clearly $\Lambda(\mu) \subset \Lambda$ for every $\mu \in \mathbb{R}$. We now characterize those μ for which $\Lambda(\mu)$ is nonempty.

Fact. *The set $\Lambda(\mu)$ is nonempty if and only if $\mu \in [\mu_{\min}, \mu_{\max}]$.*

Remark. Because we have assumed that \mathbf{m} is not proportional to $\mathbf{1}$, the return means $\{m_i\}_{i=1}^N$ are not identical. This implies that $\mu_{\min} < \mu_{\max}$, which implies that the interval $[\mu_{\min}, \mu_{\max}]$ does not reduce to a point.

Long Constraints

Proof. Because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^T \mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda(\mu)$ we have the inequalities

$$\mu_{\min} = \mu_{\min} \mathbf{1}^T \mathbf{f} = \mu_{\min} \sum_{i=1}^N f_i \leq \sum_{i=1}^N m_i f_i = \mathbf{m}^T \mathbf{f} = \mu,$$

$$\mu = \mathbf{m}^T \mathbf{f} = \sum_{i=1}^N m_i f_i \leq \mu_{\max} \sum_{i=1}^N f_i = \mu_{\max} \mathbf{1}^T \mathbf{f} = \mu_{\max}.$$

Therefore if $\Lambda(\mu)$ is nonempty then $\mu \in [\mu_{\min}, \mu_{\max}]$.

Conversely, first choose \mathbf{e}_{\min} and \mathbf{e}_{\max} so that

$$\mathbf{e}_{\min} = \mathbf{e}_i \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\min},$$

$$\mathbf{e}_{\max} = \mathbf{e}_j \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\max}.$$

Long Constraints

Now let $\mu \in [\mu_{mn}, \mu_{mx}]$ and set

$$\mathbf{f} = \frac{\mu_{mx} - \mu}{\mu_{mx} - \mu_{mn}} \mathbf{e}_{mn} + \frac{\mu - \mu_{mn}}{\mu_{mx} - \mu_{mn}} \mathbf{e}_{mx}.$$

Clearly $\mathbf{f} \geq \mathbf{0}$. Because $\mathbf{1}^T \mathbf{e}_{mn} = \mathbf{1}^T \mathbf{e}_{mx} = 1$, $\mathbf{m}^T \mathbf{e}_{mn} = \mu_{mn}$, and $\mathbf{m}^T \mathbf{e}_{mx} = \mu_{mx}$, we see that

$$\begin{aligned} \mathbf{1}^T \mathbf{f} &= \frac{\mu_{mx} - \mu}{\mu_{mx} - \mu_{mn}} \mathbf{1}^T \mathbf{e}_{mn} + \frac{\mu - \mu_{mn}}{\mu_{mx} - \mu_{mn}} \mathbf{1}^T \mathbf{e}_{mx} \\ &= \frac{\mu_{mx} - \mu}{\mu_{mx} - \mu_{mn}} + \frac{\mu - \mu_{mn}}{\mu_{mx} - \mu_{mn}} = 1, \\ \mathbf{m}^T \mathbf{f} &= \frac{\mu_{mx} - \mu}{\mu_{mx} - \mu_{mn}} \mathbf{m}^T \mathbf{e}_{mn} + \frac{\mu - \mu_{mn}}{\mu_{mx} - \mu_{mn}} \mathbf{m}^T \mathbf{e}_{mx} \\ &= \frac{\mu_{mx} - \mu}{\mu_{mx} - \mu_{mn}} \mu_{mn} + \frac{\mu - \mu_{mn}}{\mu_{mx} - \mu_{mn}} \mu_{mx} = \mu. \end{aligned}$$

Hence, $\mathbf{f} \in \Lambda(\mu)$. *Therefore if $\mu \in [\mu_{mn}, \mu_{mx}]$ then $\Lambda(\mu)$ is nonempty.* □

Long Constraints

For every $\mu \in [\mu_{\min}, \mu_{\max}]$ the set $\Lambda(\mu)$ is the nonempty intersection in \mathbb{R}^N of the $N - 1$ dimensional simplex Λ with the $N - 1$ dimensional hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{m}^T \mathbf{f} = \mu\}$. *Therefore $\Lambda(\mu)$ will be a nonempty, closed, bounded, convex polytope of dimension at most $N - 2$.*

Remark. When there are n assets with $m_i > \mu$ and $N - n$ assets with $m_i < \mu$ then $\Lambda(\mu)$ will have $n(N - n)$ vertices. This means that $\Lambda(\mu)$ can have at most $\frac{1}{4}N^2$ vertices when N is even and can have at most $\frac{1}{4}(N^2 - 1)$ vertices when N is odd.

Long Constraints

We can visualize the polytope $\Lambda(\mu)$ when N is small.

- When $N = 2$ it is a point because it is the intersection of the line segment Λ with a transverse line.
- When $N = 3$ it is either a point or line segment because it is the intersection of the triangle Λ with a transverse plane.
- When $N = 4$ it is either a point, line segment, triangle, or convex quadrilateral because it is the intersection of the tetrahedron Λ with a transverse hyperplane.

Long Constraints

Remark. Recall from our last remark that when $N = 4$ the set $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ under the one-to-one affine mapping $\Phi : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ given there. The set $\Lambda(\mu) \subset \mathbb{R}^4$ is thereby the image under Φ of the intersection of \mathcal{T} with the hyperplane H_μ given by

$$H_\mu = \left\{ \mathbf{z} \in \mathbb{R}^3 ; \mathbf{m}^T \Phi(\mathbf{z}) = \mu \right\} .$$

Hence, the set $\Lambda(\mu)$ in \mathbb{R}^4 can be visualized in \mathbb{R}^3 as the set $\mathcal{T}_\mu = \mathcal{T} \cap H_\mu$. Because Φ is one-to-one and \mathbf{m} is arbitrary, H_μ can be any hyperplane in \mathbb{R}^3 . Therefore \mathcal{T}_μ can be the intersection of the tetrahedron \mathcal{T} with any hyperplane in \mathbb{R}^3 .

Long Constraints

When such an intersection is nonempty it can be either

1. a *point* that is a vertex of \mathcal{T} ,
2. a *line segment* that is an edge of \mathcal{T} ,
3. a *triangle* with vertices on edges of \mathcal{T} ,
4. a *convex quadrilateral* with vertices on edges of \mathcal{T} .

These are each convex polytopes of dimension at most 2.

Long Frontiers

The set Λ in \mathbb{R}^N of all long portfolios is associated with the set $\Sigma(\Lambda)$ in the $\sigma\mu$ -plane of volatilities and return means given by

$$\Sigma(\Lambda) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 : \sigma = \sqrt{\mathbf{f}^T \mathbf{V} \mathbf{f}}, \mu = \mathbf{m}^T \mathbf{f}, \mathbf{f} \in \Lambda \right\}.$$

The set $\Sigma(\Lambda)$ is the image in \mathbb{R}^2 of the simplex Λ in \mathbb{R}^N under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set Λ is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set $\Sigma(\Lambda)$ is compact.

We have seen that the set $\Lambda(\mu)$ of all long portfolios with return mean μ is nonempty if and only if $\mu \in [\mu_{\min}, \mu_{\max}]$. Hence, $\Sigma(\Lambda)$ can be expressed as

$$\Sigma(\Lambda) = \left\{ \left(\sqrt{\mathbf{f}^T \mathbf{V} \mathbf{f}}, \mu \right) : \mu \in [\mu_{\min}, \mu_{\max}], \mathbf{f} \in \Lambda(\mu) \right\}.$$

The points on the boundary of $\Sigma(\Lambda)$ that correspond to those long portfolios that have less volatility than every other long portfolio with the same return mean is called the *long frontier*.

Long Frontiers

The long frontier is the curve in the $\sigma\mu$ -plane given by the equation

$$\sigma = \sigma_{\text{lf}}(\mu) \quad \text{over} \quad \mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}],$$

where the value of $\sigma_{\text{lf}}(\mu)$ is obtained for each $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ by solving the constrained minimization problem

$$\sigma_{\text{lf}}(\mu)^2 = \min \left\{ \sigma^2 : (\sigma, \mu) \in \Sigma \right\} = \min \left\{ \mathbf{f}^T \mathbf{V} \mathbf{f} : \mathbf{f} \in \Lambda(\mu) \right\}.$$

Because the function $\mathbf{f} \mapsto \mathbf{f}^T \mathbf{V} \mathbf{f}$ is continuous over the compact set $\Lambda(\mu)$, a *minimizer exists*.

Because \mathbf{V} is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^T \mathbf{V} \mathbf{f}$ is strictly convex over the convex set $\Lambda(\mu)$, whereby *the minimizer is unique*.

Long Frontiers

If we denote this unique minimizer by $\mathbf{f}_{\text{lf}}(\mu)$ then for every $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ the function $\sigma_{\text{lf}}(\mu)$ is given by

$$\sigma_{\text{lf}}(\mu) = \sqrt{\mathbf{f}_{\text{lf}}(\mu)^{\text{T}} \mathbf{V} \mathbf{f}_{\text{lf}}(\mu)},$$

where $\mathbf{f}_{\text{lf}}(\mu)$ can be expressed as

$$\mathbf{f}_{\text{lf}}(\mu) = \arg \min \left\{ \frac{1}{2} \mathbf{f}^{\text{T}} \mathbf{V} \mathbf{f} : \mathbf{f} \in \mathbb{R}^N, \mathbf{f} \geq \mathbf{0}, \mathbf{1}^{\text{T}} \mathbf{f} = 1, \mathbf{m}^{\text{T}} \mathbf{f} = \mu \right\}.$$

Here $\arg \min$ is read *“the argument that minimizes”*. It means that $\mathbf{f}_{\text{lf}}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2} \mathbf{f}^{\text{T}} \mathbf{V} \mathbf{f}$ subject to the given constraints.

Remark. This problem can not be solved by Lagrange multipliers because of the inequality constraints $\mathbf{f} \geq \mathbf{0}$ associated with the set $\Lambda(\mu)$. It is harder to solve analytically than the analogous minimization problem for portfolios with unlimited leverage. Therefore we will first present a numerical approach that can generally be applied.

Long Frontiers

Because the function being minimized is quadratic in \mathbf{f} while the constraints are linear in \mathbf{f} , this is called a *quadratic programming problem*. It can be solved for a particular \mathbf{V} , \mathbf{m} , and μ by using either the Matlab command “**quadprog**” or an equivalent command in some other language.

The Matlab command `quadprog(A, b, C, d, Ceq, deq)` returns the solution of a quadratic programming problem in the standard form

$$\arg \min \left\{ \frac{1}{2} \mathbf{x}^T \mathbf{A} \mathbf{x} + \mathbf{b}^T \mathbf{x} : \mathbf{x} \in \mathbb{R}^M, \mathbf{C} \mathbf{x} \leq \mathbf{d}, \mathbf{C}_{\text{eq}} \mathbf{x} = \mathbf{d}_{\text{eq}} \right\},$$

where $\mathbf{A} \in \mathbb{R}^{M \times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^M$, $\mathbf{C} \in \mathbb{R}^{K \times M}$, $\mathbf{d} \in \mathbb{R}^K$, $\mathbf{C}_{\text{eq}} \in \mathbb{R}^{K_{\text{eq}} \times M}$, and $\mathbf{d}_{\text{eq}} \in \mathbb{R}^{K_{\text{eq}}}$. Here K and K_{eq} are the number of inequality and equality constraints respectively.

Long Frontiers

Given \mathbf{V} , \mathbf{m} , and $\mu \in [\mu_{\min}, \mu_{\max}]$, the problem that we want to solve to obtain $\mathbf{f}_{\text{If}}(\mu)$ is

$$\arg \min \left\{ \frac{1}{2} \mathbf{f}^T \mathbf{V} \mathbf{f} : \mathbf{f} \in \mathbb{R}^N, \mathbf{f} \geq \mathbf{0}, \mathbf{1}^T \mathbf{f} = 1, \mathbf{m}^T \mathbf{f} = \mu \right\}.$$

By comparing the standard quadratic programming problem given on the previous slide we see that we can set $\mathbf{x} = \mathbf{f}$ then $M = N$, $K = N$, $K_{\text{eq}} = 2$, and

$$\mathbf{A} = \mathbf{V}, \quad \mathbf{b} = \mathbf{0}, \quad \mathbf{C} = -\mathbf{I}, \quad \mathbf{d} = \mathbf{0}, \quad \mathbf{C}_{\text{eq}} = \begin{pmatrix} \mathbf{1}^T \\ \mathbf{m}^T \end{pmatrix}, \quad \mathbf{d}_{\text{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where \mathbf{I} is the $N \times N$ identity. Notice that

- $M = N$ because $\mathbf{x} = \mathbf{f} \in \mathbb{R}^N$,
- $K = N$ because $\mathbf{f} \geq \mathbf{0}$ gives N inequality constraints,
- $K_{\text{eq}} = 2$ because $\mathbf{1}^T \mathbf{f} = 1$ and $\mathbf{m}^T \mathbf{f} = \mu$ are two equality constraints.

Long Frontiers

Therefore $\mathbf{f}_{\text{lf}}(\mu)$ can be obtained as the output \mathbf{f} of a quadprog command that is formatted as

$$\mathbf{f} = \text{quadprog}(\mathbf{V}, \mathbf{z}, -\mathbf{I}, \mathbf{z}, \text{Ceq}, \text{deq}),$$

where the matrices \mathbf{V} , \mathbf{I} , and Ceq , and vectors \mathbf{z} and deq are given by

$$\mathbf{V} = \mathbf{V}, \quad \mathbf{z} = \mathbf{0}, \quad \mathbf{I} = \mathbf{I}, \quad \text{Ceq} = \begin{pmatrix} \mathbf{1}^T \\ \mathbf{m}^T \end{pmatrix}, \quad \text{deq} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

Remark. There are other ways to use quadprog to obtain $\mathbf{f}_{\text{lf}}(\mu)$. Documentation for this command is easy to find on the web.

Long Frontiers

When computing a long frontier, it helps to know some general properties of the function $\sigma_{\text{lf}}(\mu)$. These include:

- $\sigma_{\text{lf}}(\mu)$ is *continuous* over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$;
- $\sigma_{\text{lf}}(\mu)$ is *strictly convex* over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$;
- $\sigma_{\text{lf}}(\mu)$ is *piecewise hyperbolic* over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$.

This means that $\sigma_{\text{lf}}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\text{mn}}, \mu_{\text{mx}})$ where $\sigma_{\text{lf}}(\mu)$ has either *a jump discontinuity in its first derivative* or *a jump discontinuity in its second derivative*.

Guided by these facts we now show how *a long frontier can be approximated numerically with the Matlab command quadprog*.

Long Frontiers

First, partition the interval $[\mu_{\min}, \mu_{\max}]$ as

$$\mu_{\min} = \mu_0 < \mu_1 < \dots < \mu_{n-1} < \mu_n = \mu_{\max}.$$

For example, set $\mu_k = \mu_{\min} + k(\mu_{\max} - \mu_{\min})/n$ for a uniform partition. Pick n large enough to resolve all the features of the long frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

Second, for every $k = 0, \dots, n$ use quadprog to compute $\mathbf{f}_{\text{lf}}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}_{\text{lf}}(\mu_k)\}_{k=0}^n$ should be saved.

Third, for every $k = 0, \dots, n$ compute σ_k by

$$\sigma_k = \sigma_{\text{lf}}(\mu_k) = \sqrt{\mathbf{f}_{\text{lf}}(\mu_k)^T \mathbf{V} \mathbf{f}_{\text{lf}}(\mu_k)}.$$

Long Frontiers

Remark. There is typically a unique m_i such that $\mu_{\min} = m_i$, in which case we have

$$\mathbf{f}_{\text{lf}}(\mu_0) = \mathbf{e}_i, \quad \sigma_0 = \sqrt{v_{ii}}.$$

Similarly, there is typically a unique m_j such that $\mu_{\max} = m_j$, in which case we have

$$\mathbf{f}_{\text{lf}}(\mu_n) = \mathbf{e}_j, \quad \sigma_n = \sqrt{v_{jj}}.$$

Finally, we “connect the dots” between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ to build an approximation to the long frontier in the $\sigma\mu$ -plane. This can be done by linear interpolation. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\sigma}_{\text{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}} \sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}} \sigma_k.$$

Long Frontiers

A better way to “connect the dots” between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\text{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}} \mathbf{f}_{\text{lf}}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}} \mathbf{f}_{\text{lf}}(\mu_k),$$

and then set

$$\tilde{\sigma}_{\text{lf}}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\text{lf}}(\mu)^T \mathbf{V} \tilde{\mathbf{f}}_{\text{lf}}(\mu)}.$$

Remark. This will be a very good approximation if n is large enough. Over each interval (μ_{k-1}, μ_k) it approximates $\sigma_{\text{f}}^{\ell}(\mu)$ with a hyperbola rather than with a line.

Long Frontiers

Remark. Because $\mathbf{f}_{\text{lf}}(\mu_k) \in \Lambda(\mu_k)$ and $\mathbf{f}_{\text{lf}}(\mu_{k-1}) \in \Lambda(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_{\text{lf}}(\mu) \in \Lambda(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k).$$

Therefore $\tilde{\sigma}_{\text{lf}}(\mu)$ gives an approximation to the long frontier that lies on or to the right of the long frontier in the $\sigma\mu$ -plane.

Remark. When there are no nodes in the interval (μ_{k-1}, μ_k) then we can use the two-fund property to show that $\tilde{\sigma}_{\text{lf}}(\mu) = \sigma_{\text{lf}}(\mu)$.

General Portfolio with Two Risky Assets

Recall the portfolio of two risky assets with mean vector \mathbf{m} and covariance matrix \mathbf{V} given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\min} = m_1$ and $\mu_{\max} = m_2$. Recall that for every $\mu \in \mathbb{R}$ the unique portfolio that satisfies the constraints $\mathbf{1}^T \mathbf{f} = 1$ and $\mathbf{m}^T \mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_2] = [\mu_{\min}, \mu_{\max}]$. Therefore the set Λ of long portfolios is given by

$$\Lambda = \{ \mathbf{f}(\mu) : \mu \in [m_1, m_2] \}.$$

General Portfolio with Two Risky Assets

In other words, the line segment Λ in \mathbb{R}^2 is the image of the interval $[m_1, m_2]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$.

Because for every $\mu \in [m_1, m_2]$ the set $\Lambda(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^T \mathbf{V} \mathbf{f}$ over $\Lambda(\mu)$ is $\mathbf{f}(\mu)$. Therefore the long frontier portfolios are

$$\mathbf{f}_{\text{lf}}(\mu) = \mathbf{f}(\mu) \quad \text{for } \mu \in [m_1, m_2],$$

and the long frontier is given by

$$\sigma = \sigma_{\text{lf}}(\mu) = \sqrt{\mathbf{f}(\mu)^T \mathbf{V} \mathbf{f}(\mu)} \quad \text{for } \mu \in [m_1, m_2].$$

Hence, the long frontier is simply a segment of the frontier hyperbola. It has no nodes.

General Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector \mathbf{m} and covariance matrix \mathbf{V} given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \leq m_2 \leq m_3, \quad m_1 < m_3.$$

Then $\mu_{\min} = m_1$ and $\mu_{\max} = m_3$.

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Recall that for every $\mu \in \mathbb{R}$ the portfolios that satisfies the constraints $\mathbf{1}^T \mathbf{f} = 1$ and $\mathbf{m}^T \mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi \mathbf{n}, \quad \text{for some } \phi \in \mathbb{R},$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \quad \mathbf{n} = \frac{1}{m_3 - m_1} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu, \phi) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_3] = [\mu_{\min}, \mu_{\max}]$ and

$$0 \leq \phi \leq \min \left\{ \frac{m_3 - \mu}{m_3 - m_2}, \frac{\mu - m_1}{m_2 - m_1} \right\}.$$

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For every $\mu \in [m_1, m_3]$ we define

$$\phi_{\max}(\mu) = \min \left\{ \frac{m_3 - \mu}{m_3 - m_2}, \frac{\mu - m_1}{m_2 - m_1} \right\}.$$

Then the set Λ of long portfolios is given by

$$\Lambda = \left\{ \mathbf{f}(\mu, \phi) : (\mu, \phi) \in \mathcal{T}_\Lambda \right\},$$

where \mathcal{T}_Λ is the triangle in the $\mu\phi$ -plane given by

$$\mathcal{T}_\Lambda = \left\{ (\mu, \phi) \in \mathbb{R}^2 : \mu \in [m_1, m_3], 0 \leq \phi \leq \phi_{\max}(\mu) \right\}.$$

The base of this triangle is the interval $[m_1, m_3]$ on the μ -axis. Its peak is the point $(m_2, 1)$, so its height is 1.

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Therefore the sets Λ and $\Lambda(\mu)$ in \mathbb{R}^3 can be visualized as follows.

The set Λ is the triangle in \mathbb{R}^3 that is the image of the triangle \mathcal{T}_Λ under the affine mapping $(\mu, \phi) \mapsto \mathbf{f}(\mu, \phi)$.

For every $\mu \in [m_1, m_3]$ the set $\Lambda(\mu)$ is given by

$$\Lambda(\mu) = \{\mathbf{f}(\mu, \phi) : 0 \leq \phi \leq \phi_{\max}(\mu)\}.$$

Therefore the set $\Lambda(\mu)$ is the line segment in \mathbb{R}^3 that is the image of the interval $[0, \phi_{\max}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu, \phi)$.

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Hence, the point on the long frontier associated with $\mu \in [\mu_{\min}, \mu_{\max}]$ is $(\sigma_{lf}(\mu), \mu)$ where $\sigma_{lf}(\mu)$ solves the constrained minimization problem

$$\begin{aligned}\sigma_{lf}(\mu)^2 &= \min \left\{ \mathbf{f}^T \mathbf{V} \mathbf{f} : \mathbf{f} \in \Lambda(\mu) \right\} \\ &= \min \left\{ \mathbf{f}(\mu, \phi)^T \mathbf{V} \mathbf{f}(\mu, \phi) : 0 \leq \phi \leq \phi_{\max}(\mu) \right\}.\end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu, \phi)^T \mathbf{V} \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu)^T \mathbf{V} \mathbf{f}_{13}(\mu) + 2\phi \mathbf{n}^T \mathbf{V} \mathbf{f}_{13}(\mu) + \phi^2 \mathbf{n}^T \mathbf{V} \mathbf{n}$$

is a quadratic in ϕ , we see that it has a unique global minimizer at

$$\phi = \phi_f(\mu) = -\frac{\mathbf{n}^T \mathbf{V} \mathbf{f}_{13}(\mu)}{\mathbf{n}^T \mathbf{V} \mathbf{n}}.$$

This global minimizer corresponds to the frontier. It will be the minimizer of our constrained minimization problem for the long frontier if and only if $0 \leq \phi_f(\mu) \leq \phi_{\max}(\mu)$.

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If $\phi_f(\mu) < 0$ then the objective function is increasing over $[0, \phi_{\text{mx}}(\mu)]$, whereby its minimizer is $\phi = 0$.

If $\phi_{\text{mx}}(\mu) < \phi_f(\mu)$ then the objective function is decreasing over $[0, \phi_{\text{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi_{\text{mx}}(\mu)$.

Hence, the minimizer $\phi_{\text{lf}}(\mu)$ of our constrained minimization problem is

$$\begin{aligned} \phi_{\text{lf}}(\mu) &= \begin{cases} 0 & \text{if } \phi_f(\mu) < 0 \\ \phi_f(\mu) & \text{if } 0 \leq \phi_f(\mu) \leq \phi_{\text{mx}}(\mu) \\ \phi_{\text{mx}}(\mu) & \text{if } \phi_{\text{mx}}(\mu) < \phi_f(\mu) \end{cases} \\ &= \max\{0, \min\{\phi_f(\mu), \phi_{\text{mx}}(\mu)\}\} \\ &= \min\{\max\{0, \phi_f(\mu)\}, \phi_{\text{mx}}(\mu)\}. \end{aligned}$$

Therefore $\sigma_{\text{lf}}(\mu)^2 = \mathbf{f}(\mu, \phi_{\text{lf}}(\mu))^T \mathbf{V} \mathbf{f}(\mu, \phi_{\text{lf}}(\mu))$.

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Understanding the long frontier thereby reduces to understanding $\phi_f(\mu)$. This can be done graphically in the $\mu\phi$ -plane by considering the triangle \mathcal{T}_Λ and the line \mathcal{L}_f given by

$$\phi = \phi_f(\mu).$$

Because

$$\mathbf{f}_{13}(m_1) = \mathbf{e}_1, \quad \mathbf{f}_{13}(m_2) = -\mathbf{n} + \mathbf{e}_2, \quad \text{and} \quad \mathbf{f}_{13}(m_3) = \mathbf{e}_3,$$

we see that

$$\begin{aligned} \phi_f(m_1) &= -\frac{\mathbf{n}^T \mathbf{V} \mathbf{f}_{13}(m_1)}{\mathbf{n}^T \mathbf{V} \mathbf{n}} = -\frac{\mathbf{n}^T \mathbf{V} \mathbf{e}_1}{\mathbf{n}^T \mathbf{V} \mathbf{n}}, \\ \phi_f(m_2) &= -\frac{\mathbf{n}^T \mathbf{V} \mathbf{f}_{13}(m_2)}{\mathbf{n}^T \mathbf{V} \mathbf{n}} = 1 - \frac{\mathbf{n}^T \mathbf{V} \mathbf{e}_2}{\mathbf{n}^T \mathbf{V} \mathbf{n}}, \\ \phi_f(m_3) &= -\frac{\mathbf{n}^T \mathbf{V} \mathbf{f}_{13}(m_3)}{\mathbf{n}^T \mathbf{V} \mathbf{n}} = -\frac{\mathbf{n}^T \mathbf{V} \mathbf{e}_3}{\mathbf{n}^T \mathbf{V} \mathbf{n}}. \end{aligned}$$

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This shows we can read off from the entries of \mathbf{Vn} that:

\mathcal{L}_f lies below the vertex $(m_1, 0)$ of \mathcal{T}_Λ iff $\mathbf{e}_1^T \mathbf{Vn} > 0$;

\mathcal{L}_f lies above the vertex $(m_1, 0)$ of \mathcal{T}_Λ iff $\mathbf{e}_1^T \mathbf{Vn} < 0$;

\mathcal{L}_f lies below the vertex $(m_2, 1)$ of \mathcal{T}_Λ iff $\mathbf{e}_2^T \mathbf{Vn} > 0$;

\mathcal{L}_f lies above the vertex $(m_2, 1)$ of \mathcal{T}_Λ iff $\mathbf{e}_2^T \mathbf{Vn} < 0$;

\mathcal{L}_f lies below the vertex $(m_3, 0)$ of \mathcal{T}_Λ iff $\mathbf{e}_3^T \mathbf{Vn} > 0$;

\mathcal{L}_f lies above the vertex $(m_3, 0)$ of \mathcal{T}_Λ iff $\mathbf{e}_3^T \mathbf{Vn} < 0$.

Below we consider three of the many different cases that can arise. For simplicity we will assume that $m_1 < m_2 < m_3$.

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Case 1. The line \mathcal{L}_f lies below the interior of \mathcal{T}_Λ if and only if

$$\mathbf{e}_1^T \mathbf{V} \mathbf{n} \geq 0, \quad \text{and} \quad \mathbf{e}_3^T \mathbf{V} \mathbf{n} \geq 0.$$

Then $\phi_{\text{lf}}(\mu) = 0$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\text{lf}}(\mu) = \sqrt{\mathbf{f}_{13}(\mu)^T \mathbf{V} \mathbf{f}_{13}(\mu)}.$$

This is the long frontier built from assets 1 and 3.

General Portfolio with Three Risky Assets

Case 2. The line \mathcal{L}_f lies above the interior of \mathcal{T}_Λ if and only if

$$\mathbf{e}_1^T \mathbf{V} \mathbf{n} \leq 0, \quad \mathbf{e}_2^T \mathbf{V} \mathbf{n} \leq 0, \quad \text{and} \quad \mathbf{e}_3^T \mathbf{V} \mathbf{n} \leq 0.$$

Then $\phi_{\text{lf}}(\mu) = \phi_{\text{mx}}(\mu)$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\text{lf}}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^T \mathbf{V} \mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, m_2], \\ \sqrt{\mathbf{f}_{23}(\mu)^T \mathbf{V} \mathbf{f}_{23}(\mu)} & \text{for } \mu \in [m_2, m_3]. \end{cases}$$

This patches the long frontier built from assets 1 and 2 with the long frontier built from assets 2 and 3. It generally has a jump discontinuity in its first derivative at the node $\mu = m_2$.

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Case 3. The line \mathcal{L}_f lies above the base of \mathcal{T}_Λ but intersects the interior of \mathcal{T}_Λ if and only if

$$\mathbf{e}_1^T \mathbf{Vn} < 0, \quad \mathbf{e}_2^T \mathbf{Vn} > 0, \quad \text{and} \quad \mathbf{e}_3^T \mathbf{Vn} < 0.$$

Then there exists $\mu_1 \in [m_1, m_2]$ and $\mu_2 \in [m_2, m_3]$ such that

$$\phi_{\text{lf}}(\mu) = \begin{cases} \frac{\mu - m_1}{m_2 - m_1} & \text{for } \mu \in [m_1, \mu_1], \\ \phi_f(\mu) & \text{for } \mu \in (\mu_1, \mu_2), \\ \frac{m_3 - \mu}{m_3 - m_2} & \text{for } \mu \in [\mu_2, m_3]. \end{cases}$$

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The long frontier is

$$\sigma = \sigma_{lf}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^T \mathbf{V} \mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, \mu_1], \\ \sigma_f(\mu) & \text{for } \mu \in (\mu_1, \mu_2), \\ \sqrt{\mathbf{f}_{23}(\mu)^T \mathbf{V} \mathbf{f}_{23}(\mu)} & \text{for } \mu \in [\mu_2, m_3]. \end{cases}$$

It generally has jump discontinuities in its second derivative at the nodes $\mu = \mu_1$ and $\mu = \mu_2$.

Simple Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector \mathbf{m} and covariance matrix \mathbf{V} given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \quad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$, where the last condition is equivalent to the condition that \mathbf{V} is positive definite given $s > 0$.

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Its frontier parameters are

$$\sigma_{\text{mv}} = \sqrt{\frac{1}{a}} = s \sqrt{\frac{1+2r}{3}}, \quad \mu_{\text{mv}} = \frac{b}{a} = m,$$

$$\nu_{\text{as}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s} \sqrt{\frac{2}{1-r}}.$$

Its minimum volatility portfolio is $\mathbf{f}_{\text{mv}} = \frac{1}{3}\mathbf{1}$, whereby we can take $\mu_0 = m$. Clearly $[\mu_{\text{mn}}, \mu_{\text{mx}}] = [m - d, m + d]$. Its frontier is determined by

$$\sigma_f(\mu) = s \sqrt{\frac{1+2r}{3} + \frac{1-r}{2} \left(\frac{\mu - m}{d}\right)^2} \quad \text{for } \mu \in (-\infty, \infty).$$

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The allocation of the frontier portfolio with return mean μ is

$$\mathbf{f}_f(\mu) = \begin{pmatrix} \frac{1}{3} - \frac{\mu - m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu - m}{2d} \end{pmatrix} = \begin{pmatrix} \frac{m + \frac{2}{3}d - \mu}{2d} \\ \frac{1}{3} \\ \frac{\mu - m + \frac{2}{3}d}{2d} \end{pmatrix}.$$

The frontier portfolio holds long positions when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. Therefore $[\underline{\mu}_1, \bar{\mu}_1] = [m - \frac{2}{3}d, m + \frac{2}{3}d]$ and the long frontier satisfies

$$\sigma_{1f}(\mu) = \sigma_f(\mu) \quad \text{for } \mu \in [m - \frac{2}{3}d, m + \frac{2}{3}d].$$

The allocation of first asset vanishes at the right endpoint while that of the third vanishes at the left endpoint.

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In order to extend the long frontier beyond the right endpoint $\bar{\mu}_1 = m + \frac{2}{3}d$ to $\mu_{\max} = m + d$ we reduce the portfolio by removing the first asset and set

$$\bar{\mathbf{m}}_1 = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m \\ m + d \end{pmatrix}, \quad \bar{\mathbf{V}}_1 = s^2 \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

Then

$$\bar{\mathbf{V}}_1^{-1} = \frac{1}{s^2(1-r^2)} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}, \quad \bar{\mathbf{V}}_1^{-1} \mathbf{1} = \frac{1}{s^2(1+r)} \mathbf{1},$$

whereby

$$\bar{a}_1 = \mathbf{1}^T \bar{\mathbf{V}}_1^{-1} \mathbf{1} = \frac{2}{s^2(1+r)}, \quad \bar{b}_1 = \mathbf{1}^T \bar{\mathbf{V}}_1^{-1} \bar{\mathbf{m}}_1 = \frac{2m+d}{s^2(1+r)},$$

$$\bar{c}_1 = \bar{\mathbf{m}}_1^T \bar{\mathbf{V}}_1^{-1} \bar{\mathbf{m}}_1 = \frac{2m(m+d)}{s^2(1+r)} + \frac{d^2}{s^2(1-r^2)}.$$

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The associated frontier parameters are

$$\sigma_{mv_1} = \sqrt{\frac{1}{\bar{a}_1}} = s \sqrt{\frac{1+r}{2}}, \quad \mu_{mv_1} = \frac{\bar{b}_1}{\bar{a}_1} = m + \frac{1}{2}d,$$
$$\nu_{as_1} = \sqrt{\bar{c}_1 - \frac{\bar{b}_1^2}{\bar{a}_1}} = \frac{d}{2s} \sqrt{\frac{2}{1-r}},$$

whereby the frontier of the reduced portfolio is given by

$$\sigma_{\bar{f}_1}(\mu) = s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d} \right)^2}.$$

Simple Portfolio with Three Risky Assets

Similarly, in order to extend the long frontier beyond the left endpoint $\underline{\mu}_1 = m - \frac{2}{3}d$ to $\mu_{mn} = m - d$ we reduce the portfolio by removing the third asset. We find that the frontier of the reduced portfolio is given by

$$\sigma_{f_1}(\mu) = s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d} \right)^2}.$$

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By putting these pieces together we see that the long frontier is given by

$$\sigma_{\text{lf}}(\mu) = \begin{cases} s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d} \right)^2} & \text{for } \mu \in [m - d, m - \frac{2}{3}d], \\ s \sqrt{\frac{1+2r}{3} + \frac{1-r}{2} \left(\frac{\mu - m}{d} \right)^2} & \text{for } \mu \in [m - \frac{2}{3}d, m + \frac{2}{3}d], \\ s \sqrt{\frac{1+r}{2} + \frac{1-r}{2} \left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d} \right)^2} & \text{for } \mu \in [m + \frac{2}{3}d, m + d]. \end{cases}$$

This is strictly convex and continuously differentiable over $[m - d, m + d]$.

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Its second derivative is defined and positive everywhere in $[m - d, m + d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where it has jump discontinuities. Thus,

$$\sigma_{\text{lf}}(m \pm \frac{2}{3}d) = s \sqrt{\frac{5 + 4r}{9}}, \quad \sigma_{\text{lf}}(m \pm d) = s.$$

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Finally, the long frontier allocations are given by

$$\mathbf{f}_{\text{lf}}(\mu) = \begin{cases} \begin{pmatrix} \frac{m-\mu}{d} \\ \frac{\mu-m+d}{d} \\ 0 \end{pmatrix} & \text{for } \mu \in [m-d, m - \frac{2}{3}d], \\ \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} & \text{for } \mu \in [m - \frac{2}{3}d, m + \frac{2}{3}d], \\ \begin{pmatrix} 0 \\ \frac{m+d-\mu}{d} \\ \frac{\mu-m}{d} \end{pmatrix} & \text{for } \mu \in [m + \frac{2}{3}d, m+d]. \end{cases}$$

Notice that these allocations do not depend on either s or r .

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Remark. These long frontier allocations are continuous and piecewise linear over $[m - d, m + d]$. Their first derivatives are defined everywhere in $[m - d, m + d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where they have jump discontinuities. The allocations at these nodes are

$$\mathbf{f}_{\text{lf}}\left(m - \frac{2}{3}d\right) = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}, \quad \mathbf{f}_{\text{lf}}\left(m + \frac{2}{3}d\right) = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$