Portfolios that Contain Risky Assets 5: Portfolios with Risk-Free Assets

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Portfolios that Contain Risky Assets
Part I: Portfolio Models

Portfolios with Risk-Free Assets

Risk-Free Assets

Until now we have considered portfolios that contain only risky assets. We now consider two kinds of *risk-free* assets (assets that have no volatility associated with them) that can play a major role in portfolio management.

The first is a *safe investment* that pays dividends at a prescribed annual return $\mu_{\text{si}}^{\text{an}}$. This can be an FDIC insured bank account, or safe securities such as US Treasury Bills, Notes, or Bonds. (U.S. Treasury Bills are used most commonly.) *We can only hold a long position in such an asset.*

The second is a *credit line* from which you can borrow at a prescribed annual return $\mu_{\text{cl}}^{\text{an}}$ up to your credit limit. Such a credit line should require you to put up assets like real estate or part of your portfolio (a *margin*) as collateral from which the borrowed money can be recovered if need be. *We can only hold a short position in such an asset.*

Risk-Free Assets

We will assume that $\mu_{\rm cl}^{\rm an} \geq \mu_{\rm si}^{\rm an}$, because otherwise investors would make money by borrowing at return $\mu_{\rm cl}^{\rm an}$ in order to invest at the greater return $\mu_{\rm si}^{\rm an}$. (Here we are again neglecting transaction costs.) Because free money does not sit around for long, market forces would quickly adjust the returns so that $\mu_{\rm cl}^{\rm an} \geq \mu_{\rm si}^{\rm an}$. In practice, $\mu_{\rm cl}^{\rm an}$ is about three points higher than $\mu_{\rm si}^{\rm an}$.

We will also assume that a portfolio will not hold a position in both the safe investment and the credit line when $\mu_{\rm cl}^{\rm an} > \mu_{\rm si}^{\rm an}$. To do so would effectively be borrowing at return $\mu_{\rm cl}^{\rm an}$ in order to invest at the lesser return $\mu_{\rm si}^{\rm an}$. While there can be cash-flow management reasons for holding such a position for a short time, it is not a smart long term position.

These assumptions imply that every portfolio can be viewed as holding a position in at most one risk-free asset: it can hold either a long position at return $\mu_{\rm si}^{\rm an}$, a short position at return $\mu_{\rm cl}^{\rm an}$, or a neutral risk-free position.

Risk-Free Assets

The first step is to find the annual returns $\mu_{\mathrm{si}}^{\mathrm{an}}$ and $\mu_{\mathrm{cl}}^{\mathrm{an}}$ that are available to us at the start of the period over which we are investing. If we are using a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ to guide us and are planning to invest over the year following this history (say, the year that starts on day $D+1$) then we can select $\mu_{\mathrm{si}}^{\mathrm{an}}$ and $\mu_{\mathrm{cl}}^{\mathrm{an}}$ as follows.

- Let $\mu_{\mathrm{si}}^{\mathrm{an}}$ be the return for the 52 week U.S. Treasury Bill on day D. (Return histories for U.S. Treasury Bills, Notes, and Bonds can be found on the U.S. Treasury website. Use the "coupon" rate.)
- Let $\mu_{\mathrm{cl}}^{\mathrm{an}} = \mu_{\mathrm{si}}^{\mathrm{an}} + .03$. (Recall that .03 is 3%. Many sources of credit are available. The best run about 3% above the Treasury Bill rate.)

Risk-Free Assets

The next step is to convert the annual return $\mu_{\mathrm{rf}}^{\mathrm{an}}$ of each risk-free asset to a return μ_{rf} per trading day. If there are D_{y} trading days in a year then we can approximate μ_{rf} by the relation

$$\left(1+\mu_{\mathrm{rf}}^{\mathrm{an}}\right)^{\frac{1}{D_{\mathrm{y}}}} = 1+\mu_{\mathrm{rf}} \,.$$

Here we take the growth of the risk-free asset to be the same between any two successive trading days — namely, the factor $(1+\mu_{\mathrm{rf}})$. Whether this is true or not depends upon the asset, but for most risk-free assets it depends upon the number of calendar days between successive trading days.

Because $\frac{1}{D_{\mathrm{y}}}$ and $\mu_{\mathrm{rf}}^{\mathrm{an}}$ are small, we can make the further approximation

$$\left(1+\mu_{\mathrm{rf}}^{\mathrm{an}}\right)^{\frac{1}{D_{\mathrm{y}}}} \approx 1+\frac{1}{D_{\mathrm{y}}}\mu_{\mathrm{rf}}^{\mathrm{an}} \,,$$

whereby $\mu_{\mathrm{rf}} \approx \frac{1}{D_{\mathrm{y}}}\mu_{\mathrm{rf}}^{\mathrm{an}}$.

Risk-Free Assets

There are several ways to choose D_{y}. Because there are typically 252 trading days in a year, we can simply set $D_{\mathrm{y}} = 252$. Alternatively, if our history is over a multiple h of a full year then we can set $D_{\mathrm{y}} = D/h$. For example, if our history is over two full years and $D = 503$ then we can set $D_{\mathrm{y}} = 251.5$. Typical values of h are $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 5, and 10.

Remark. The value $D_{\mathrm{y}} = 252$ is correct when there are 365 calendar days, 104 weekend days, and 9 holidays, none of which fall on a weekend. For such a year we have $252 = 365 - 104 - 9$. Of course, leap years have 366 days, some years have 105 or 106 weekend days, and holidays fall on weekends in some years. Some years will have unscheduled closures due to bad weather or some other emergency. The day after Thanksgiving (always a Friday) plus July 3 and December 24 when they do not fall on a weekend will be half-days. We will treat half-days the same other trading days.

Markowitz Portfolios

We now extend the notion of Markowitz portfolios to portfolios that might include a single risk-free asset with return μ_{rf}.

Let $b_{\mathrm{rf}}(d)$ denote the balance in the risk-free asset at the start of day d. For a long position $\mu_{\mathrm{rf}} = \mu_{\mathrm{si}}$ and $b_{\mathrm{rf}}(d) > 0$, while for a short position $\mu_{\mathrm{rf}} = \mu_{\mathrm{cl}}$ and $b_{\mathrm{rf}}(d) < 0$.

A *Markowitz portfolio* containing one risk-free asset and N risky assets is uniquely determined by real numbers f_{rf} and $\{f_i\}_{i=1}^N$ that satisfy

$$f_{\mathrm{rf}} + \sum_{i=1}^{N} f_i = 1\,.$$

Here f_{rf} is the allocation of the portfolio in the risk-free asset while f_i is the allocation of the portfolio in the i^{th} risky asset.

Markowitz Portfolios

The portfolio is rebalanced at the start of each day so that

$$\frac{b_{\mathrm{rf}}(d)}{\pi(d-1)} = f_{\mathrm{rf}}\,, \qquad \frac{n_i(d)\,s_i(d-1)}{\pi(d-1)} = f_i \qquad \text{for } i = 1, \cdots, N\,.$$

Its value at the start of day d is

$$\pi(d-1) = b_{\mathrm{rf}}(d) + \sum_{i=1}^{N} n_i(d)\,s_i(d-1)\,,$$

while its value at the end of day d is approximately

$$\pi(d) = b_{\mathrm{rf}}(d)\,(1+\mu_{\mathrm{rf}}) + \sum_{i=1}^{N} n_i(d)\,s_i(d)\,.$$

Markowitz Portfolios

We can thereby approximate the return for day d as

$$
\begin{aligned}
r(d) &= \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} \\
&= \frac{b_{\mathrm{rf}}(d)\,\mu_{\mathrm{rf}}}{\pi(d-1)} + \sum_{i=1}^{N} \frac{n_i(d)\big(s_i(d) - s_i(d-1)\big)}{\pi(d-1)} \\
&= \frac{b_{\mathrm{rf}}(d)\,\mu_{\mathrm{rf}}}{\pi(d-1)} + \sum_{i=1}^{N} \frac{n_i(d)s_i(d-1)}{\pi(d-1)}\,\frac{s_i(d) - s_i(d-1)}{s_i(d-1)} \\
&= f_{\mathrm{rf}}\,\mu_{\mathrm{rf}} + \sum_{i=1}^{N} f_i\, r_i(d) = f_{\mathrm{rf}}\,\mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\,.
\end{aligned}
$$

Markowitz Portfolios

The portfolio return mean μ and variance v are then given by

$$\begin{aligned}\mu &= \sum_{d=1}^{D} w(d)\, r(d) = \sum_{d=1}^{D} w(d) \left(f_{\mathrm{rf}}\, \mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}} \mathbf{r}(d) \right) \\ &= f_{\mathrm{rf}}\, \mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}} \left(\sum_{d=1}^{D} w(d)\, \mathbf{r}(d) \right) = f_{\mathrm{rf}}\, \mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}} \mathbf{m}\,, \\ v &= \sum_{d=1}^{D} w(d) \left(r(d) - \mu \right)^2 = \sum_{d=1}^{D} w(d) \left(\mathbf{f}^{\mathrm{T}} \mathbf{r}(d) - \mathbf{f}^{\mathrm{T}} \mathbf{m} \right)^2 \\ &= \mathbf{f}^{\mathrm{T}} \left(\sum_{d=1}^{D} w(d) \left(\mathbf{r}(d) - \mathbf{m} \right) \left(\mathbf{r}(d) - \mathbf{m} \right)^{\mathrm{T}} \right) \mathbf{f} = \mathbf{f}^{\mathrm{T}} \mathbf{V} \mathbf{f}\,. \end{aligned}$$

We thereby obtain the formulas

$$\mu = \mu_{\mathrm{rf}} \left(1 - \mathbf{1}^{\mathrm{T}} \mathbf{f} \right) + \mathbf{m}^{\mathrm{T}} \mathbf{f}\,, \qquad v = \mathbf{f}^{\mathrm{T}} \mathbf{V} \mathbf{f}\,.$$

Capital Allocation Lines

These formulas can be viewed as describing a point that lies on a certain half-line in the $\sigma\mu$-plane. Let (σ, μ) be the point in the $\sigma\mu$-plane associated with the Markowitz portfolio characterized by the allocation $\mathbf{f} \neq \mathbf{0}$. For now we assume that $\mathbf{1}^{\mathrm{T}}\mathbf{f} > 0$. Define

$$\tilde{\mathbf{f}} = \frac{\mathbf{f}}{\mathbf{1}^{\mathrm{T}}\mathbf{f}} \,.$$

Notice that $\mathbf{1}^{\mathrm{T}}\tilde{\mathbf{f}} = 1$. Let $\tilde{\mu} = \mathbf{m}^{\mathrm{T}}\tilde{\mathbf{f}}$ and $\tilde{\sigma} = \sqrt{\tilde{\mathbf{f}}^{\mathrm{T}}\mathbf{V}\tilde{\mathbf{f}}}$. Then $(\tilde{\sigma}, \tilde{\mu})$ is the point in the $\sigma\mu$-plane associated with the Markowitz portfolio without risk-free assets that is characterized by the allocation $\tilde{\mathbf{f}}$. Because

$$\mu = \left(1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}\right)\mu_{\mathrm{rf}} + \mathbf{1}^{\mathrm{T}}\mathbf{f}\,\tilde{\mu}\,, \qquad \sigma = \mathbf{1}^{\mathrm{T}}\mathbf{f}\,\tilde{\sigma}\,,$$

we see that the point (σ, μ) in the $\sigma\mu$-plane lies on the half-line that starts at the point $(0, \mu_{\mathrm{rf}})$ and passes through the point $(\tilde{\sigma}, \tilde{\mu})$ that corresponds to a portfolio that does not contain the risk-free asset.

Capital Allocation Lines

Conversely, given any point $(\tilde{\sigma}, \tilde{\mu})$ corresponding to a Markowitz portfolio that contains no risk-free assets, consider the half-line

$$(\sigma, \mu) = (\phi\,\tilde{\sigma}\,,\,(1-\phi)\mu_{\mathrm{rf}} + \phi\,\tilde{\mu}) \qquad \text{where } \phi > 0\,.$$

If a portfolio corresponding to $(\tilde{\sigma}, \tilde{\mu})$ has allocation $\tilde{\mathbf{f}}$ then the point on the half-line given by ϕ corresponds to the portfolio with allocation $\mathbf{f} = \phi\tilde{\mathbf{f}}$. This portfolio allocates $1 - \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 - \phi$ of its value to the risk-free asset. The risk-free asset is held long if $\phi \in (0,1)$ and held short if $\phi > 1$ while $\phi = 1$ corresponds to a neutral position. We must restrict ϕ to either $(0,1]$ or $[1,\infty)$ depending on whether the risk-free asset is the safe investment or the credit line. This segment of the half-line is called the *capital allocation line* through $(\tilde{\sigma}, \tilde{\mu})$ associated with the risk-free asset.

Capital Allocation Lines

Therefore we can use the appropriate capital allocation lines to construct the set of all points in the $\sigma\mu$-plane associated with Markowitz portfolios that contain a risk-free asset from the set of all points in the $\sigma\mu$-plane associated with Markowitz portfolios that contain no risk-free assets.

In the next section we will use this capital allocation line construction to see how the efficient frontier for Markowitz portfolios is modified by the inclusion of risk-free assets.

Efficient Frontier

Recall that the efficient frontier associated with $\mathcal{M}$ is

$$\mu = \mu_{\mathrm{mv}} + \nu_{\mathrm{as}} \sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2} \qquad \text{for } \sigma \geq \sigma_{\mathrm{mv}} \,. \tag{4.1}$$

Every point on this curve has a unique frontier portfolio associated with it.

When $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ there is a unique half-line that starts at the point $(0, \mu_{\mathrm{rf}})$ and is tangent to this curve. Denote this half-line by

$$\mu = \mu_{\mathrm{rf}} + \nu_{\mathrm{tg}}\, \sigma \qquad \text{for } \sigma \geq 0 \,. \tag{4.2}$$

Let $(\sigma_{\mathrm{tg}}, \mu_{\mathrm{tg}})$ be the point at which this tangency occurs. The unique frontier portfolio associated with this point is called the *tangency portfolio* associated with the risk-free asset; its allocation is $\mathbf{f}_{\mathrm{tg}} = \mathbf{f}_{\mathrm{f}}(\mu_{\mathrm{tg}})$. Then the appropriate capital allocation line will be part of the efficient frontier.

Efficient Frontier

The so-called *tangency parameters*, $\sigma_{\rm tg}$, $\mu_{\rm tg}$, and $\nu_{\rm tg}$, are determined by solving the system of three equations

$$\mu_{\rm tg} = \mu_{\rm rf} + \nu_{\rm tg}\,\sigma_{\rm tg}\,, \qquad \mu_{\rm tg} = \mu_{\rm mv} + \nu_{\rm as}\sqrt{\sigma_{\rm tg}^2 - \sigma_{\rm mv}^2}\,,$$

$$\nu_{\rm tg} = \frac{\nu_{\rm as}\,\sigma_{\rm tg}}{\sqrt{\sigma_{\rm tg}^2 - \sigma_{\rm mv}^2}}\,.$$

The first equation says that $(\sigma_{\rm tg}, \mu_{\rm tg})$ lies on the capital allocation line (4.2). The second says that $(\sigma_{\rm tg}, \mu_{\rm tg})$ also lies on the efficient frontier (4.1) associated with $\mathcal{M}$. The third equates the slope of the capital allocation line to that of the effiecient frontier at the point $(\sigma_{\rm tg}, \mu_{\rm tg})$.

Efficient Frontier

By using the last equation to eliminate ν_{tg} from the first, and then using the resulting equation to eliminate μ_{tg} from the second, we find that

$$\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\nu_{\mathrm{as}}} = \frac{\sigma_{\mathrm{tg}}^2}{\sqrt{\sigma_{\mathrm{tg}}^2 - \sigma_{\mathrm{mv}}^2}} - \sqrt{\sigma_{\mathrm{tg}}^2 - \sigma_{\mathrm{mv}}^2} = \frac{\sigma_{\mathrm{mv}}^2}{\sqrt{\sigma_{\mathrm{tg}}^2 - \sigma_{\mathrm{mv}}^2}} .$$

We thereby obtain

$$\sigma_{\mathrm{tg}} = \sigma_{\mathrm{mv}} \sqrt{1 + \left(\frac{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}\right)^2}, \qquad \mu_{\mathrm{tg}} = \mu_{\mathrm{mv}} + \frac{\nu_{\mathrm{as}}^2\,\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}},$$
$$\nu_{\mathrm{tg}} = \nu_{\mathrm{as}} \sqrt{1 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}}\right)^2}. \tag{4.3}$$

Efficient Frontier

The allocation $\mathbf{f}_{\rm tg}$ of the tangency portfolio is then given by

$$\begin{aligned} \mathbf{f}_{\rm tg} = \mathbf{f}_{\rm f}(\mu_{\rm tg}) &= \mathbf{f}_{\rm mv} + \frac{\mu_{\rm tg} - \mu_{\rm mv}}{\nu_{\rm as}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm mv}\mathbf{1}) \\ &= \sigma_{\rm mv}^2 \mathbf{V}^{-1}\mathbf{1} + \frac{\sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm mv}\mathbf{1}) \\ &= \frac{\sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\rm rf}\mathbf{1}) \,. \end{aligned} \tag{4.4}$$

Remark. When $\mu_{\rm rf} \geq \mu_{\rm mv}$ the capital allocation line construction fails. *Moreover, the model becomes unrealistic as $\mu_{\rm rf}$ approaches $\mu_{\rm mv}$ because the resulting tangent portfolios take on short positions without restriction.* In reality short positions are restricted by *credit limits*.

Efficient Frontier

Formulas (4.3) and (4.4) can be applied to either the safe investment or the credit line by simply choosing μ_{rf} to be either μ_{si} or μ_{cl}. If $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}}$ then the tangency parameters associated with the safe investment are

$$\sigma_{\mathrm{st}} = \sigma_{\mathrm{mv}} \sqrt{1 + \left(\frac{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}} \right)^2} \,, \qquad \mu_{\mathrm{st}} = \mu_{\mathrm{mv}} + \frac{\nu_{\mathrm{as}}^2\,\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}} \,,$$
$$\nu_{\mathrm{st}} = \nu_{\mathrm{as}} \sqrt{1 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}} \right)^2} \,. \tag{4.5}$$

The tangency portfolio allocation associated with the safe investment is

$$\mathbf{f}_{\mathrm{st}} = \frac{\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1}) \,. \tag{4.6}$$

Efficient Frontier

If $\mu_{\mathrm{cl}} < \mu_{\mathrm{mv}}$ then the tangency parameters associated with the credit line are

$$\sigma_{\mathrm{ct}} = \sigma_{\mathrm{mv}} \sqrt{1 + \left(\frac{\nu_{\mathrm{as}}\, \sigma_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{cl}}} \right)^2} \,, \qquad \mu_{\mathrm{ct}} = \mu_{\mathrm{mv}} + \frac{\nu_{\mathrm{as}}^2\, \sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{cl}}} \,,$$
$$\nu_{\mathrm{ct}} = \nu_{\mathrm{as}} \sqrt{1 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{cl}}}{\nu_{\mathrm{as}}\, \sigma_{\mathrm{mv}}} \right)^2} \,. \tag{4.7}$$

The tangency portfolio allocation associated with the credit line is

$$\mathbf{f}_{\mathrm{ct}} = \frac{\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{cl}}} \mathbf{V}^{-1} \left(\mathbf{m} - \mu_{\mathrm{cl}} \mathbf{1} \right) . \tag{4.8}$$

Efficient Frontier

Remark. The safe investment capital allocation line exists if and only if $\mu_{\rm si} < \mu_{\rm mv}$. If it exists the its slope is called the *Sharp ratio*. This slope is

$$\nu_{\rm st} = \frac{\mu_{\rm st} - \mu_{\rm si}}{\sigma_{\rm st}} \,,$$

where $(\sigma_{\rm st}, \mu_{\rm st})$ is the point of tangency of the safe investment capital allocation line. It satisfies

$$\nu_{\rm st} > \nu_{\rm as} \,.$$

In other words, the Sharp ratio is greater than the slope of the asymptote of the efficient frontier.

Efficient Frontier

The addition of the associated capital allocation lines to the efficient frontier associated with $\mathcal{M}$ yields the efficient frontier when the risk-free assets are included.

- If $\mu_{\mathrm{si}} \leq \mu_{\mathrm{cl}} < \mu_{\mathrm{mv}}$ then both capital allocation lines are added.
- If $\mu_{\mathrm{si}} < \mu_{\mathrm{mv}} \leq \mu_{\mathrm{cl}}$ then the only capital allocation line is the one associated with the safe investment.
- If $\mu_{\mathrm{mv}} \leq \mu_{\mathrm{si}} \leq \mu_{\mathrm{cl}}$ then there are no capital allocation lines.

We will do this in the next two sections for two models. The so-called *One Risk-Free Rate Model* sets $\mu_{\mathrm{si}} = \mu_{\mathrm{cl}}$. The so-called *Two Risk-Free Rate Model* sets $\mu_{\mathrm{si}} < \mu_{\mathrm{cl}}$.

One Risk-Free Rate Model

One Risk-Free Rate Model. First consider the case when $\mu_{\rm si} = \mu_{\rm cl} < \mu_{\rm mv}$. Set $\mu_{\rm rf} = \mu_{\rm si} = \mu_{\rm cl}$ and let $\nu_{\rm tg}$ and $\sigma_{\rm tg}$ be the slope and volatility of the tangency portfolio that is common to both the safe investment and the credit line. The efficient frontier is then given by

$$\mu_{\rm ef}(\sigma) = \mu_{\rm rf} + \nu_{\rm tg}\,\sigma \qquad \text{for } \sigma \in [0,\infty)\,.$$

Let $\mathbf{f}_{\rm tg}$ be the allocation of the tangency portfolio that is common to both the safe investment and the credit line. The allocation of the associated portfolio is then given by

$$\mathbf{f}_{\rm ef}(\sigma) = \frac{\sigma}{\sigma_{\rm tg}}\,\mathbf{f}_{\rm tg} \qquad \text{for } \sigma \in [0,\infty)\,,$$

These portfolios are constructed as follows:

One Risk-Free Rate Model

1. if $\sigma = 0$ then the investor holds only the safe investment;
2. if $\sigma \in (0, \sigma_{\text{tg}})$ then the investor places
   - $\dfrac{\sigma_{\text{tg}} - \sigma}{\sigma_{\text{tg}}}$ of the portfolio value in the safe investment,
   - $\dfrac{\sigma}{\sigma_{\text{tg}}}$ of the portfolio value in the tangency portfolio $\mathbf{f}_{\text{tg}}$;
3. if $\sigma = \sigma_{\text{tg}}$ then the investor holds only the tangency portfolio $\mathbf{f}_{\text{tg}}$;
4. if $\sigma \in (\sigma_{\text{tg}}, \infty)$ then the investor places
   - $\dfrac{\sigma}{\sigma_{\text{tg}}}$ of the portfolio value in the tangency portfolio $\mathbf{f}_{\text{tg}}$,
   - by borrowing $\dfrac{\sigma - \sigma_{\text{tg}}}{\sigma_{\text{tg}}}$ of this value from the credit line.

One Risk-Free Rate Model

Remark. Some of those who believe in market efficiency assert that a capitalization weighted market average should fall on the tangent portfolio. Because such an average is dominated by the S&P 500, this amounts to asserting that an S&P 500 index fund should track the tangent portfolio. If this is indeed the case then all that an investor would have to do to acheive optimal performance over the long term is to find the correct balance between the risk-free asset and an S&P 500 index fund. While this is a good investment strategy, it is not optimal. It has three weaknesses.

- The tangent portfolio may not exist or may take short positions.
- The S&P 500 index does not always track the tangent portfolio.
- The one risk-free rate model is not very good.

Two Risk-Free Rate Model

Two Risk-Free Rates Model. Next consider the case when $\mu_{\rm si} < \mu_{\rm cl} < \mu_{\rm mv}$. Let $\nu_{\rm st}$ and $\sigma_{\rm st}$ be the slope and volatility of the so-called *safe tangency portfolio* associated with the safe investment. Let $\nu_{\rm ct}$ and $\sigma_{\rm ct}$ be the slope and volatility of the so-called *credit tangency portfolio* associated with the credit line. These are given by

$$\nu_{\rm st} = \nu_{\rm as}\sqrt{1 + \left(\frac{\mu_{\rm mv} - \mu_{\rm si}}{\nu_{\rm as}\,\sigma_{\rm mv}}\right)^2}\,, \qquad \sigma_{\rm st} = \sigma_{\rm mv}\sqrt{1 + \left(\frac{\nu_{\rm as}\,\sigma_{\rm mv}}{\mu_{\rm mv} - \mu_{\rm si}}\right)^2}\,,$$

$$\nu_{\rm ct} = \nu_{\rm as}\sqrt{1 + \left(\frac{\mu_{\rm mv} - \mu_{\rm cl}}{\nu_{\rm as}\,\sigma_{\rm mv}}\right)^2}\,, \qquad \sigma_{\rm ct} = \sigma_{\rm mv}\sqrt{1 + \left(\frac{\nu_{\rm as}\,\sigma_{\rm mv}}{\mu_{\rm mv} - \mu_{\rm cl}}\right)^2}\,.$$

Two Risk-Free Rate Model

The efficient frontier is then given by

$$\mu_{\mathrm{ef}}(\sigma) = \begin{cases} \mu_{\mathrm{si}} + \nu_{\mathrm{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\mathrm{st}}]\,, \\ \mu_{\mathrm{mv}} + \nu_{\mathrm{as}}\sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2} & \text{for } \sigma \in [\sigma_{\mathrm{st}}, \sigma_{\mathrm{ct}}]\,, \\ \mu_{\mathrm{cl}} + \nu_{\mathrm{ct}}\,\sigma & \text{for } \sigma \in [\sigma_{\mathrm{ct}}, \infty)\,. \end{cases}$$

The return means for the tangency portfolios are

$$\mu_{\mathrm{st}} = \mu_{\mathrm{mv}} + \frac{\nu_{\mathrm{as}}^2\,\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}}\,, \qquad \mu_{\mathrm{ct}} = \mu_{\mathrm{mv}} + \frac{\nu_{\mathrm{as}}^2\,\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{cl}}}\,.$$

Two Risk-Free Rate Model

The allocations of risky assets for the tangency portfolios are

$$\mathbf{f}_{\mathrm{st}} = \frac{\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{si}}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{si}}\mathbf{1}) \,, \quad \mathbf{f}_{\mathrm{ct}} = \frac{\sigma_{\mathrm{mv}}^2}{\mu_{\mathrm{mv}} - \mu_{\mathrm{cl}}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{cl}}\mathbf{1}) \,.$$

By the two fund property, the allocation of risky assets for any efficient frontier portfolio is then given by

$$\mathbf{f}_{\mathrm{ef}}(\sigma) = \begin{cases} \dfrac{\sigma}{\sigma_{\mathrm{st}}} \mathbf{f}_{\mathrm{st}} & \text{for } \sigma \in [0, \sigma_{\mathrm{st}}] \,, \\ \dfrac{\mu_{\mathrm{ct}} - \mu_{\mathrm{ef}}(\sigma)}{\mu_{\mathrm{ct}} - \mu_{\mathrm{st}}} \mathbf{f}_{\mathrm{st}} + \dfrac{\mu_{\mathrm{ef}}(\sigma) - \mu_{\mathrm{st}}}{\mu_{\mathrm{ct}} - \mu_{\mathrm{st}}} \mathbf{f}_{\mathrm{ct}} & \text{for } \sigma \in (\sigma_{\mathrm{st}}, \sigma_{\mathrm{ct}}) \,, \\ \dfrac{\sigma}{\sigma_{\mathrm{ct}}} \mathbf{f}_{\mathrm{ct}} & \text{for } \sigma \in [\sigma_{\mathrm{ct}}, \infty) \,. \end{cases}$$

Two Risk-Free Rate Model

These portfolios are constructed as follows:

1. if $\sigma = 0$ then the investor holds only the safe investment;
2. if $\sigma \in (0, \sigma_{\mathrm{st}})$ then the investor places
   - $\dfrac{\sigma_{\mathrm{st}} - \sigma}{\sigma_{\mathrm{st}}}$ of the portfolio value in the safe investment,
   - $\dfrac{\sigma}{\sigma_{\mathrm{st}}}$ of the portfolio value in the safe tangency portfolio $\mathbf{f}_{\mathrm{st}}$;
3. if $\sigma = \sigma_{\mathrm{st}}$ then the investor holds only the safe tangency portfolio $\mathbf{f}_{\mathrm{st}}$;

Two Risk-Free Rate Model

4. if $\sigma \in (\sigma_{\mathrm{st}}, \sigma_{\mathrm{ct}})$ then the investor holds
   - $\dfrac{\mu_{\mathrm{ct}} - \mu_{\mathrm{ef}}(\sigma)}{\mu_{\mathrm{ct}} - \mu_{\mathrm{st}}}$ of the portfolio value in the safe tangency portfolio $\mathbf{f}_{\mathrm{st}}$,
   - $\dfrac{\mu_{\mathrm{ef}}(\sigma) - \mu_{\mathrm{st}}}{\mu_{\mathrm{ct}} - \mu_{\mathrm{st}}}$ of the portfolio value in the credit tangency portfolio $\mathbf{f}_{\mathrm{ct}}$;
5. if $\sigma = \sigma_{\mathrm{ct}}$ then the investor holds only the credit tangency portfolio $\mathbf{f}_{\mathrm{ct}}$;
6. if $\sigma \in (\sigma_{\mathrm{ct}}, \infty)$ then the investor places
   - $\dfrac{\sigma}{\sigma_{\mathrm{ct}}}$ of the portfolio value in the credit tangency portfolio $\mathbf{f}_{\mathrm{ct}}$,
   - by borrowing $\dfrac{\sigma - \sigma_{\mathrm{ct}}}{\sigma_{\mathrm{ct}}}$ of this value from the credit line.

Two Risk-Free Rate Model

Remark. Some of those who believe in market efficiency assert that a capitalization weighted market average should fall on the efficient frontier. This means is should fall somewhere between the safe tangent portfolio and the credit tangent portfolio. Because such an average is dominated by the S&P 500, this amounts to asserting that an S&P 500 index fund will fall near the efficient frontier between the two tangent portfolios. If this is indeed the case then all that an investor would have to do to acheive optimal performance over the long term is to find the correct balance between the risk-free assets, an S&P 500 index fund, and at least one other index fund that lies near the Markowitz efficient frontier. While this is a better investment strategy, it is not optimal. It has two weaknesses.

- The tangent portfolios may not exist or may take short positions.
- Index funds do not always stay near the Markowitz frontier.

Two Risk-Free Rate Model

Remark. The Two Risk-Free Rate Model is more realistic than the One Risk-Free Rate Model.

Remark. Some brokers will simply ask investors to select a σ that reflects the risk they are willing to take and then will build a portfolio for them that is near the efficient frontier portfolio $\mathbf{f}_{\mathrm{ef}}(\sigma)$. To guide this selection of σ, the broker will describe certain choices of σ as being "very conservative", "conservative", "aggressive", or "very aggressive" without quantifying what these terms mean. The performance of the resulting portfolio will often be disappointing to those who selected a "very conservative" σ and painful to those who selected a "very aggressive" σ. We will see reasons for this when we develop probability-based methods for selecting a σ.