Portfolios that Contain Risky Assets 1: Risk and Reward

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Part I: Portfolio Models

Risk and Reward

Risky Assets

The risk associated with an investment is the uncertainty of its outcome. Every investment has risk associated with it. Hiding your cash under a mattress puts it at greater risk of loss to theft or fire than depositing it in a bank, and is a sure way to not make money.

Depositing your cash into an FDIC insured bank account is the safest investment that you can make — the only risk of loss would be to an extreme national calamity.

However, a bank account generally will yield a lower reward on your investment than any asset that has more risk associated with it. Such assets include stocks (equities), bonds, commodities (gold, oil, corn, etc.), private equity (venture capital), hedge funds, and real estate. With the exception of real estate, the prices of these assets can fluctuate several percent in a single day. Such assets are called **risky assets.**

Risky Assets

Remark. *Market forces generally will insure that assets associated with higher potential reward are also associated with greater risk and vice versa.* Investment offers that seem to violate this principle are always scams.

We will consider three basic claases of risky assets:

- **stocks**,
- **bonds**,
- **commodities**.

We will also consider **funds** that hold many assets drawn from one or more of these basic classes. We start with brief descriptions of these assets.

Risky Assets

Stocks. Stocks (or equities) are part ownership of a company. Their value goes up when the company does well, and goes down when it does poorly. Some stocks pay a periodic (often quarterly) dividend of either cash or more stock. Stocks are traded on exchanges like the NYSE or NASDAQ.

The risk associated with a stock reflects the uncertainty about the future performance of the company. This uncertainty has many facets. For example, there might be questions about the future market share of its products, the availablity of the raw materials needed for its products, or the value of its current assets. Stocks in larger companies are generally less risky than stocks in smaller companies. *Stocks are generally higher reward/higher risk investments compared to bonds.*

Risky Assets

Bonds. Bonds are loans to either a government or a company. The borrower usually makes a periodic (often quarterly) interest payment, and ultimately pays back the principle at a maturity date. Bonds are traded on secondary markets where their value is based upon current interest rates. For example, if interest rates go up then bond values will go down on the secondary market.

The risk associated with a bond reflects the uncertainty about the credit worthiness of the borrower. Short term bonds are generally less risky than long term ones. Bonds from large entities are generally less risky than those from small entities. Bonds from governments are generally less risky than those from companies. (This is even true in some cases where the ratings given by some ratings agencies suggest otherwise.) *Bonds are generally lower reward/lower risk investments compared to stocks.*

Risky Assets

Commodities. Commodities are hard assets such as gold, corn, oil, or real estate. They are bought with the hope that their value will go up. For example, an investor might fear an inflationary period. Some commodities are bought in standard units (troy ounces, bushels, barrels). Others are bought through shares of a partnership. Some (like rental real estate) have a regular income associated with it, but most provide no income.

The risk associated with a commodity is the uncertainty about the future demand for it and supply of it. This uncertainty has many facets because the variety of commodities is huge. For example, farm commodities are perishable, so will become worthless if held too long. The value of oil or gold falls when new supplies are discovered. The demand for oil depends upon the weather. Gold prices often spike during times of uncertainty, but tend to return to inflation adjusted levels. *Because of their variety, commodities can fall anywhere on the reward/risk spectrum.*

Risky Assets

Mutual and Exchange-Traded Funds. Funds can hold a combination of stocks, bonds, and/or commodities. They are set up by investment companies. Shares of mutual funds are bought and sold through the company that set it up. Shares of exchange-traded funds (ETFs) are bought and sold just as you would shares of a stock. *Funds of either type are typically lower reward/lower risk investments compared to the individual assets from which they are composed.*

Funds are managed either *actively* or *passively.* An actively-managed fund has a strategy to outperform some market index like the S&P 500, Russell 1000, or Russell 2000. A passively-managed fund is called an *index fund* because it is designed so that its performance will match some market index. Index funds are often portrayed to be *lower reward/lower risk* investments compared to actively-managed funds. *However, index funds typically will outperform most actively-managed funds.*

Risky Assets

The Problem. Suppose that we are considering how to invest in N risky assets. Let $s_i(d)$ be the share price of the i^{th} asset at the close of the d^{th} trading day of a period that has D trading days. Here we understand that $s_i(0)$ is the share price at the close of the last trading day before the period being considered. We will assume that every $s_i(d)$ is positive. We would like to use the share price history $\{s_i(d)\}_{d=0}^{D}$ to gain insight into how to manage our portfolio over the coming period.

In this course we will examine the following questions.

Can mathematical models be used to help manage investment portfolios?
Can stochastic (random, probabilistic) models be built that quantitatively mimic this price history?

Daily Returns

The first thing to understand that the share price of an asset has very little economic significance. This is because the size of our investment in an asset is the same if we own 100 shares worth 50 dollars each or 25 shares worth 200 dollars each. What is economically significant is how much our investment rises or falls in value. Because our investment in asset i would have changed by the price ratio $s_i(d)/s_i(d-1)$ over the course of day d, this ratio is economically significant. Rather than use the price ratio as the basic variable, it is traditional to use the so-called *daily return*, which is defined by

$$r_i(d) \equiv \frac{s_i(d)}{s_i(d-1)} - 1 = \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} .$$

Therefore the share price of asset i increases on days when $r_i(d) > 0$ and decreases on days when $r_i(d) < 0$.

Daily Returns

One way to understand daily returns is to set $r_i(d)$ equal to a constant μ. Upon solving the resulting relation for $s_i(d)$ we find that

$$s_i(d) = (1+\mu)\, s_i(d-1) \quad \text{for every } d = 1, \cdots, D\,.$$

By induction on d we can then derive the compound interest formula

$$s_i(d) = (1+\mu)^d s_i(0) \quad \text{for every } d = 1, \cdots, D\,.$$

If we assume that $|\mu| << 1$ then we can use the approximation

$$\log(1+\mu) \approx \mu\,,$$

whereby

$$s_i(d) \approx e^{\mu d}\, s_i(0)\,.$$

We thereby see μ is nearly the exponential growth rate in units of "per day" of the share price.

Daily Returns

Remark. The trading day might seem like an arbitrary measure of time. From a theoretical viewpoint we could have used a shorter measure like half-days, hours, quarter hours, or minutes. However, the rules governing trading are different during a trading day than they are at the end of one. So the shorter the measure, the harder it is to extract information from the additional data. This extra work is not worth doing unless you profit sufficiently.

Alternatively, we could have used a longer measure like weeks, months, or quarters. The longer the measure, the less data we use, which means we have less understanding of the market.

Many investors now use daily data. Weekly data was often used when calculations were done by hand.

Daily Returns

Remark. It is not obvious that daily returns are the right quantities upon which to build a theory of markets. For example, another possibility is to use the *daily growth rates* $x_i(d)$ defined by

$$x_i(d) = \log\left(\frac{s_i(d)}{s_i(d-1)}\right) .$$

These are also functions of the price ratio $s_i(d)/s_i(d-1)$. Moreover, they are easier to understand than daily returns. For example, if we set $x_i(d)$ equal to a constant γ then by solving the resulting relation for $s_i(d)$ we find

$$s_i(d) = e^{\gamma}\, s_i(d-1) \quad \text{for every } d = 1, \cdots, D .$$

By induction on d we can show that

$$s_i(d) = e^{\gamma d} s_i(0) .$$

However, daily returns are favored because they have better properties with regard to porfolio statistics.

Historical Data

Share price histories can be gotten from websites like *Yahoo Finance* or *Google Finance*. For example, to compute the daily return history for Apple in 2016, type "Apple" into where is says "get quotes". You will see that Apple has the identifier AAPL and is listed on the NASDAQ. Click on "historical prices" and request share prices between "Dec 31, 2015" and "Dec 31, 2016". You will get a table that can be downloaded as a spreadsheet. Use the *adjusted closing prices* to compute the daily returns.

We will consider N risky assets indexed by i. For each i we obtain the closing share price history $\{s_i(d)\}_{d=0}^{D}$ and compute the daily return history $\{r_i(d)\}_{d=1}^{D}$ by the formula

$$r_i(d) = \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} .$$

Because $r_i(d)$ depends upon $s_i(d-1)$, we will need the adjusted closing share price for day $d = 0$, which is the trading day just before day $d = 1$.

Historical Data

Adjusted closing prices are used to compute daily returns in order to aviod complications that can arise if actual closing prices are used. For example, an asset might undergo a *stock split* or pay a *dividend*. Adjusted closing prices build the effect of such events into the historical price history.

Stock splits can happen after the share price of a stock has risen enough so that a single share might be too expensive to attract small investors. Let $n > m$. An n-to-m stock split will convert m shares of stock into n shares of stock. The share price will be reduced by a factor of $\frac{m}{n}$ so that the total capitalization remains unchanged. For example, if you own 50 shares of stock worth 75$ each then after a 3-to-1 stock split you will own 150 shares of stock worth 25$ each.

Historical Data

Splits of 2-to-1, 3-to-1, and 3-to-2 are the most common, but other ratios are used too. Fractional shares are often converted to cash. For example, if you own 25 shares of stock worth 75$ each then after a 3-to-2 stock split you will own 37 shares of stock worth 50$ each plus get a 25$ cash payment. Reverse stock splits can also happen where $n < m$.

On the day of a n-to-m stock split the daily return should be computed from the actual closing price history as

$$r_i(d) = \frac{ns_i(d) - ms_i(d-1)}{ms_i(d-1)} \,.$$

The adjusted closing price history adjusts the share price history before the split so that we do not need to use this formula.

Historical Data

Dividends are used by companies to distribute profits to shareholders. They can be paid either in cash or in stock. They can be paid either at regular intervals or whenever a company chooses. How best to account for them depends upon how they are being paid. The simplest case is when they are paid in stock at regular intervals. The adjusted closing price history trys to adjust the share price history before the dividend in order to account for it. We will not describe how these adjustments are made.

Remark. Many assets do not split or pay dividends most years.

Historical Data

Remark. There will be 252 trading days scheduled in a year with 365 calendar days, 104 weekend days, and 9 holidays, none of which fall on a weekend. This is because $252 = 365 - 104 - 9$. The number 252 is divisible by 12. We can thereby think of 63 trading days as being a quarter or 21 trading days as being a month.

Of course, the number of trading days varys from year to year for many of reasons. For example, leap years have 366 days, some years have 105 or 106 weekend days, and holidays fall on weekends in some years. In addition, exchanges can close due to extreme events.

The day after Thanksgiving (always a Friday) plus July 3 and December 24 when they do not fall on a weekend will be half-days. We will treat half-days the same as other trading days.

Finally, exchanges in different countries observe different holidays. We will not describe how to handle such cases.

Insights from Simple Models

Consider a market model in which each year every asset either goes up 20% or remains unchanged with equal probability. The return mean for this market is 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.2	1.2	1.44	1.2000
Second	1.2	1.0	1.20	1.0954
Third	1.0	1.2	1.20	1.0954
Fourth	1.0	1.0	1.00	1.0000

Three quarters of the investors see returns below 10%. One quarter of the investors have made no money.

Insights from Simple Models

The previous result did not look to bad for most investors, but perhaps you know that sometimes markets produce negative returns. Now consider a market model in which each year every asset either goes up 30% or goes down 10% with equal probability. The return mean for this market is also 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.3	1.3	1.69	1.3000
Second	1.3	0.9	1.17	1.0817
Third	0.9	1.3	1.17	1.0817
Fourth	0.9	0.9	0.81	0.9000

Three quarters of the investors see returns just above 8%. One quarter of the investors have lost money.

Insights from Simple Models

Next, consider a market model in which each year every asset either goes up 40% or goes down 20% with equal probability. The return mean for this market is also 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.4	1.4	1.96	1.4000
Second	1.4	0.8	1.12	1.0583
Third	0.8	1.3	1.12	1.0583
Fourth	0.8	0.8	0.64	0.8000

Three quarters of the investors see returns below 6%. One quarter of the investors have lost over one third of their investment.

Insights from Simple Models

Consider a market model in which each year every asset either goes up 50% or goes down 30% with equal probability. The return mean for this market is 10%. The table shows the returns seen by individual investors who buy just one asset in this market and hold it for two years.

Quartile	Year 1	Year 2	Total	Per Year
First	1.5	1.5	2.25	1.5000
Second	1.5	0.7	1.05	1.0247
Third	0.7	1.5	1.05	1.0247
Fourth	0.7	0.7	0.49	0.7000

Three quarters of the investors see returns below 2.5%. One quarter of the investors have lost over half of their investment.

We see that fewer investors do well in more volatile markets. Therefore our first insight is that *volatility can be viewed a measure of risk.*

Insights from Simple Models

There is another important insight to be gained from these models. Notice that if an individual investor buys an equal value of each asset in each of these markets then that investor sees a 10% return because

$$\frac{1.44 + 1.20 + 1.20 + 1.00}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,,$$
$$\frac{1.69 + 1.17 + 1.17 + 0.81}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,,$$
$$\frac{1.96 + 1.12 + 1.12 + 0.64}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,,$$
$$\frac{2.25 + 1.05 + 1.05 + 0.49}{4} = \frac{4.84}{4} = 1.21 = (1.1)^2 \,.$$

This gives the insight that *a diverse portfolio reduces risk.* We would like to understand how to apply this insight when designing real portfolios. This will require more realisitic models.

Statistical Approach

Daily returns $r_i(d)$ for asset i can vary wildly!

Sometimes the reasons for such fluctuations are clear because they directly relate to some news about the company, agency, or government that issued the asset. For example, news of the Deepwater Horizon explosion on 20 April 2010 caused the share price of British Petroleum stock to fall.

At other times they relate to news that benefit or hurt entire sectors of assets. For example, a rise in crude oil prices might benefit oil and railroad companies but hurt airline and trucking companies.

And at yet other times they relate to general technological, demographic, or social trends. For example, new internet technology might benefit Google and Amazon (companies that exist because of the internet) but hurt traditional "brick and mortar" retailers.

Statistical Approach

Finally, there is often no evident public reason for a particular stock price to rise or fall. The reason might be a takeover attempt, a rumor, insider information, or the fact a large investor needs cash for reasons unrelated to the asset.

Given the complexity of the dynamics underlying market fluctuations, we adopt a statistical approach to quantifying their trends and correlations. More specifically, we will choose an appropriate set of statistics that will be computed from selected daily return histories of the relevant assets. We will then use these statistics to calibrate a model that will show how a set of ideal portfolios might behave in the future. *The implicit assumption of this approach is that in the future the market will behave statistically as it did in the past.*

Statistical Approach

Taking this approach means that the data should be drawn from a long enough daily return history to sample most of the kinds of market events that we expect to see in the future. However, the history should not be too long because very old data will not be relevant to the current market. *A common choice is to use the daily return history from the most recent twelve month period.* For example, if we are planning our portfolio at the beginning of July 2016 then we might use the return histories for July 2015 through June 2016. Then D would be the number of trading days in this period.

Now suppose that we have computed the daily return history $\{r_i(d)\}_{d=1}^{D}$ for each asset. This data should be ported from the speadsheet into MATLAB, R, or another higher level environment that is well-suited to the task ahead.

Statistical Approach

Mean-Variance Statistics. Next we compute the statistical quantities that we will use in our models: *means*, *variances*, *covariances*, and *correlations*.

The *return mean* for asset i, denoted m_i, is

$$m_i = \frac{1}{D} \sum_{d=1}^{D} r_i(d) \,.$$

This measures the trend of the share price. Unfortunately, it is commonly called the *expected return* for asset i even though it is *higher* than the return that most investors will see, especially in highly volatile markets. We will not use this misleading terminology.

Statistical Approach

The *return variance* for asset i, denoted v_i, is

$$v_i = \frac{1}{D}\sum_{d=1}^{D}\left(r_i(d) - m_i\right)^2 .$$

The *return standard deviation* for asset i, denoted σ_i, is given by $\sigma_i = \sqrt{v_i}$. This is called the *return volatility* of asset i. It measures the uncertainty of the market regarding the share price trend.

The *return covariance* for assets i and j, denoted v_{ij}, is

$$v_{ij} = \frac{1}{D}\sum_{d=1}^{D}\left(r_i(d) - m_i\right)\left(r_j(d) - m_j\right) .$$

Notice that $v_{ii} = v_i$. The $N\times N$ matrix (v_{ij}) is symmetric and nonnegative definite. It will usually be positive definite — so we will assume it to be so.

Statistical Approach

Finally, the *return correlation* for assets i and j, denoted c_{ij}, is

$$c_{ij} = \frac{v_{ij}}{\sigma_i \, \sigma_j} \, .$$

Notice that $c_{ii} = 1$ and $-1 \leq c_{ij} \leq 1$. We say assets i and j are *positively correlated* when $0 < c_{ij} \leq 1$ and *negatively correlated* when $-1 \leq c_{ij} < 0$. Positively correlated assets often move in the same direction, while negatively correlated ones often move in opposite directions.

We will consider the N-vector of means (m_i) and the symmetric $N \times N$ matrix of covariances (v_{ij}) to be our basic statistical quantities. The variances (v_i), volatilities (σ_i), and correlations (c_{ij}) can then be easily obtained from (m_i) and (v_{ij}) by formulas that are given above.

Statistical Approach

We can quantify the risk and reward of each asset as follows.

- We use the volatility σ_i as the measure of risk for asset i.
- We use the mean m_i as the measure of reward for asset i.

By plotting the points (σ_i, m_i) in the $\sigma\mu$-plane, we can visualize where each asset falls on the risk-reward spectrum. By doing this using (σ_i, m_i) computed from return histories over different years we can see the variability of these measures.

Later we will use the statistics (m_i) and (v_{ij}) to quantify the risk and reward for models of portfolios composed of these asset. The computation of the statistics (m_i) and (v_{ij}) from return histories is called the *calibration* of these models.

Statistical Approach

General Calibration. We can consider a daily return history $\{r(d)\}_{d=1}^{D}$ over a period of D trading days and assign day d a weight $w(d) > 0$ such that the weights $\{w(d)\}_{d=1}^{D}$ satisfy

$$\sum_{d=1}^{D} w(d) = 1 .$$

The return means and covariances are then given by

$$m_i = \sum_{d=1}^{D} w(d)\, r_i(d) ,$$

$$v_{ij} = \sum_{d=1}^{D} w(d)\, \big(r_i(d) - m_i\big)\big(r_j(d) - m_j\big) .$$

Statistical Approach

In practice the history can extend over a period of one to five years. There are many ways to choose the weights $\{w(d)\}_{d=1}^{D}$. The most common choice is the so-called *uniform weighting*; this gives each day the same weight by setting $w(d) = 1/D$. On the other hand, we might want to give more weight to more recent data. For example, we can give each trading day a positive weight that depends only on the quarter in which it lies, giving greater weight to more recent quarters. We could also consider giving different weights to different days of the week, but such a complication should be avoided unless it yields a clear benefit.

We will have greater confidence in m_i and v_{ij} when they are relatively insensitive to different choices of D and the weights $w(d)$. We can get an idea of the magnitude of this sensitivity by checking the robustness of m_i and v_{ij} to a range of such choices.