Maximizing Profit with Modeling

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Outline

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1. Introduction: The Problem

The basic problem we will examine is the following.

We run a manufacturing company that is equipped to make n different products. How many of each product should you make over the coming week in order to maximize our profit?

For each $i = 1, 2, \cdots, n$ let q_i be the number (quantity) of the i^{th} product that we plan to make over the coming week. Let $\mathbf{q} = (q_1\ q_2\ \cdots\ q_n)^T$. Roughly speaking, we need to find an expression $P(\mathbf{q})$ for expected profit as a function of $\mathbf{q}$ and then find a point $\mathbf{q}^*$ that maximizes $P(\mathbf{q})$ over all possible $\mathbf{q}$ we might choose. Mathematically, the problem takes the form of solving

$$\mathbf{q}^* = \text{argmax}\Big\{P(\mathbf{q})\ :\ \text{all possible } \mathbf{q}\Big\}\,.$$

Our company will be viable if and only if $P(\mathbf{q}^*) > 0$.

1. Introduction: Profit, Revenue, Cost, and Constraints

If the expected *revenue* generated by selling the products is $R(\mathbf{q})$ and the expected *cost* of making and handling them is $C(\mathbf{q})$ then the expected *profit* is given by

$$P(\mathbf{q}) = R(\mathbf{q}) - C(\mathbf{q}) \,.$$

Therefore we need to find expressions for $R(\mathbf{q})$ and $C(\mathbf{q})$.

We also need to identify the constraints that define the set of all possible $\mathbf{q}$ that we will consider. For example, it is clear that for every $i = 1, 2, \cdots, n$ we must require that $q_i \geq 0$. Maybe each q_i should be a natural number. In addition $\mathbf{q}$ will be constrained by the labor and equipment resources the company employs to do the manufacturing.

2. First Approach: Revenue and Cost

Suppose that we expect to sell the i^{th} product for a price of p_i each. Then the expected revenue will be

$$R(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q}\,, \qquad \text{where } \mathbf{p} = (p_1\ p_2\ \cdots\ p_n)^T\,.$$

Suppose that our fixed costs (salaries, rent, base utilities, etc.) are d while the i^{th} product has an expected marginal cost (materials with breakage factored in, equipment maintainance, additional utilities, etc.) of c_i each. Then the expected cost will be

$$C(\mathbf{q}) = \mathbf{c} \cdot \mathbf{q} + d\,, \qquad \text{where } \mathbf{c} = (c_1\ c_2\ \cdots\ c_n)^T\,.$$

Generally $\mathbf{p} > \mathbf{c}$, where the inequality is understood entrywise. This is because we will only consider making goods from which we can profit.

2. First Approach: Resource Constraints

Labor is a resource. If you have N full-time equivalent (FTE) employees then your workforce can deliver a maximum of $40N$ hours of work per week without incurring overtime. If it takes h_i hours of employee time to produce each of the i^{th} product, then we have the labor constraint

$$\mathbf{h} \cdot \mathbf{q} \leq 40N\,, \qquad \text{where } \mathbf{h} = (h_1\; h_2\; \cdots\; h_n)^T\,.$$

Equipment is another resource. If you have a piece of equipment that can run at most H hours per week and if making each of the i^{th} product requires running that piece of equipment s_i seconds, then we have the constraint

$$3600\mathbf{s} \cdot \mathbf{q} \leq H\,, \qquad \text{where } \mathbf{s} = (s_1\; s_2\; \cdots\; s_n)^T\,.$$

Each such constraint can be brought into the form

$$\mathbf{f} \cdot \mathbf{q} \leq 1\,, \qquad \text{for some } \mathbf{f} = (f_1\; f_2\; \cdots\; f_n)^T\,.$$

2. First Approach: Constained Optimization

Upon collecting the facts from the previous slides, we want to maximize

$$P(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q} - \mathbf{c} \cdot \mathbf{q} - d = (\mathbf{p} - \mathbf{c}) \cdot \mathbf{q} - d\,,$$

over $\mathbf{q} \in \mathbb{N}^n$ subject to m constraints of the form

$$\mathbf{f}^{(k)} \cdot \mathbf{q} \leq 1\,, \qquad \text{where } k = 1, 2, \cdots, m \text{ and } \mathbf{f}^{(k)} \in \mathbb{R}^n\,.$$

Let $\mathbf{F}$ be the $m \times n$ matrix whose k^{th} row is $\mathbf{f}^{(k)}$. Then we can express this constained optimization problem as

$$\mathbf{q}^* = \text{argmax}\Big\{(\mathbf{p} - \mathbf{c}) \cdot \mathbf{q} - d \,:\, \mathbf{q} \in \mathbb{N}^n\,, \mathbf{F}\mathbf{q} \leq \mathbf{1}\Big\}\,,$$

where $\mathbf{1} \in \mathbb{R}^m$ is the column vector with every entry equal to 1 and the inequality $\mathbf{F}\mathbf{q} \leq \mathbf{1}$ is understood entrywise.

2. First Approach: Linear Programming

Rather than solve the previous optimization problem in which we imposed the constraint $\mathbf{q} \in \mathbb{N}^n$, we allow the entries of $\mathbf{q}$ to take on any nonnegative value. This leads to the classical *linear programming* problem

$$\mathbf{q}^* = \text{argmax}\Big\{(\mathbf{p} - \mathbf{c}) \cdot \mathbf{q} - d \,:\, \mathbf{q} \geq \mathbf{0}\,,\, \mathbf{F}\mathbf{q} \leq \mathbf{1}\Big\}\,,$$

where $\mathbf{0} \in \mathbb{R}^m$ is the column vector with every entry equal to 0. *The idea is that by rounding the entries of $\mathbf{q}^*$ to the nearest integer we would have a good approximation to the solution of the original problem.*

The domain $\{\mathbf{q} \in \mathbb{R}^n \,:\, \mathbf{q} \geq \mathbf{0}\,,\, \mathbf{F}\mathbf{q} \leq \mathbf{1}\}$ is closed, bounded, and convex. This insures the existence of a maximizer, which is generically unique. All maximizers lie on the boundary of this domain because $\mathbf{p} - \mathbf{c} > \mathbf{0}$. They can be found either by the classical simplex method or by primal-dual interior point methods. We will not discuss these algorithms here.

3. Supply and Demand: Something Missing in Our Model

The model that we have developed above has many shortcomings. The biggest one might be that it neglects the *law of supply and demand*.

The solution $\mathbf{q}^*$ of our constrained optimization problem is a function of $\mathbf{p}$, $\mathbf{c}$, d, and $\mathbf{F}$ that we denote $\mathbf{q}^* = S(\mathbf{p}, \mathbf{c}, d, \mathbf{F})$. This is a so-called *supply relationship* for our company, because it gives how much product we would like to supply in a market described by $(\mathbf{p}, \mathbf{c}, d, \mathbf{F})$.

Our model assumes that the prices $\mathbf{p}$ that we can get for the goods will not be effected by the amount of goods our company supplies. If our company has a tiny market share then this is not a bad assumption. However, the law of supply and demand says that if we increase the supply of a good then, in order to sell all of them, we will have to drop the price to match the so-called *demand relationship*.

3. Supply and Demand: Demand Relationships

A relationship between the price $\mathbf{p}$ of a good, and the quantity $\mathbf{q}$ that can be sold at that price is called a *demand relationship*. The law of demand states that, all other things being equal, *if the price of a good is raised then generally fewer will be sold.* However, this law is not quantitative, so it does not yield an explicit demand relationship. Rather, we derive demand relationships by fitting data. Suppliers collect such data by occasionally offering discounts and seeing how the demand for their product responds. (Offering discounts looks better to customers than raising the price.) The idea is to find a function D such that $\mathbf{q} = D(\mathbf{p})$ fits the data. This is then a model for the demand relationship.

If all other things were equal, demand relationships would be the same for all suppliers of a good. However, suppliers can increase the demand for their products through advertising or good publicity.

3. Supply and Demand: Two Demand Models

The simplest model for a demand relationship is decoupled and linear. In that case for each $i = 1, 2, \cdots, n$ you seek positive coefficients b_i and a_i such that the data is best fit by the relationship

$$p_i = b_i - a_i q_i \,.$$

In many cases the decoupling assumption is a bad one. For example, if one of your products is a fancier version of another then their demand relationship will couple. This will also happen if one product is an accessory for another. *The simplest model for a coupled demand relationship is linear.* In that case for each $i = 1, 2, \cdots, n$ you seek coefficients b_i and a_{ij} such that the data is best fit by the relationship

$$p_i = b_i - \sum_{j=1}^{n} a_{ij} q_j \,.$$

In each of these models b_i is called the *base price* of the i^{th} product.

3. Supply and Demand: Linear Demand Models

Both of the above demand models can be put into the linear form

$$\mathbf{p} = \mathbf{b} - \mathbf{A}\mathbf{q}\,.$$

The vector $\mathbf{b}$ gives the base prices of each product while the matrix $\mathbf{A}$ gives the linear reponse of their prices to supply. In the first model $\mathbf{A}$ is a *diagonal matrix* with positive diagonal entries a_i. The associated demand model has $2n$ parameters to be fit. In the second model $\mathbf{A}$ is the matrix with entries a_{ij}. The associated demand model has $n(n+1)$ parameters to be fit. Other linear demand models lie in between these. Such models are not valid in regimes where any entry of p becomes negative.

Whenever A is invertible such linear demand models can be expressed as

$$\mathbf{q} = D(\mathbf{p}) = \mathbf{A}^{-1}(\mathbf{b} - \mathbf{p})\,.$$

In practice, $\mathbf{A}$ is usually invertible. When $\mathbf{A}$ is diagonal, it always is.

4. Second Approach: Quadratic Revenue Models

For a linear demand model of the form $\mathbf{p} = \mathbf{b} - \mathbf{A}\mathbf{q}$, the expected revenue will become

$$R(\mathbf{q}) = \mathbf{p} \cdot \mathbf{q} = \mathbf{b} \cdot \mathbf{q} - \mathbf{q} \cdot \mathbf{A}\mathbf{q}\,.$$

This is quadratic in $\mathbf{q}$. Because

$$\mathbf{q} \cdot \mathbf{A}\mathbf{q} = \left(\mathbf{A}^{\mathsf{T}}\mathbf{q}\right) \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{A}^{\mathsf{T}}\mathbf{q}\,,$$

we see that $R(\mathbf{q})$ only depends on the symmetric part of $\mathbf{A}$ — namely, on $\frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)$. Specifically, we see that

$$R(\mathbf{q}) = \mathbf{b} \cdot \mathbf{q} - \tfrac{1}{2}\,\mathbf{q} \cdot \left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)\mathbf{q}\,.$$

4. Second Approach: Constrained Optimization

By combining this quadratic revenue model with our linear cost model

$$C(\mathbf{q}) = \mathbf{c} \cdot \mathbf{q} + d\,,$$

we see that the associated expected profit is modeled by

$$\begin{aligned} P(\mathbf{q}) &= R(\mathbf{q}) - C(\mathbf{q}) \\ &= \mathbf{b} \cdot \mathbf{q} - \tfrac{1}{2}\mathbf{q} \cdot \left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)\mathbf{q} - \mathbf{c} \cdot \mathbf{q} - d \\ &= (\mathbf{b} - \mathbf{c}) \cdot \mathbf{q} - d - \tfrac{1}{2}\mathbf{q} \cdot \left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)\mathbf{q}\,. \end{aligned}$$

If we maximize profit subject to the inequality constraints

$$\mathbf{F}\mathbf{q} \leq \mathbf{1}\,,$$

we are led to the constrained optimization problem

$$\mathbf{q}^* = \mathrm{argmax}\Big\{(\mathbf{b} - \mathbf{c}) \cdot \mathbf{q} - d - \tfrac{1}{2}\mathbf{q} \cdot \left(\mathbf{A} + \mathbf{A}^{\mathsf{T}}\right)\mathbf{q} \,:\, \mathbf{q} \in \mathbb{N}^n\,,\, \mathbf{F}\mathbf{q} \leq \mathbf{1}\Big\}\,.$$

4. Second Approach: Quadratic Programming

We again remove the constraint $\mathbf{q} \in \mathbb{N}^n$ and allow the entries of $\mathbf{q}$ to take on any nonnegative value. This yields the *quadratic programming* problem

$$\mathbf{q}^* = \text{argmax}\Big\{(\mathbf{b} - \mathbf{c}) \cdot \mathbf{q} - d - \tfrac{1}{2}\mathbf{q} \cdot \left(\mathbf{A} + \mathbf{A}^\top\right)\mathbf{q} \,:\, \mathbf{q} \geq \mathbf{0}\,,\, \mathbf{F}\mathbf{q} \leq \mathbf{1}\Big\}\,,$$

Once again the idea is to round the entries of this $\mathbf{q}^*$ to the nearest integer to get a good approximation to the solution of the original problem.

The above quadratic programming problem can be solved with the MatLab command “quadprog” or by primal-dual interior point methods. We will not discuss these algorithms here.

4. Second Approach: Strictly Concave Quadratic Case

If $\mathbf{A}+\mathbf{A}^{\top}$ is positive definite then $P(\mathbf{q})$ is strictly concave and has a unique global maximizer without imposing any constraints. This global maximizer is given by

$$\mathbf{q}^{**} = \left(\mathbf{A} + \mathbf{A}^{\top}\right)^{-1} (\mathbf{b} - \mathbf{c}) \, .$$

Whenever $\mathbf{q}^{**}$ satisfies the constraints it will also be the solution to the constrained optimzation problem — i.e. $\mathbf{q}^{*} = \mathbf{q}^{**}$.

Remark. The condition that $\mathbf{A} + \mathbf{A}^{\top}$ is positive definite implies that $\mathbf{A}$ is invertible. This condition will always be satisfied when $\mathbf{A}$ is diagonal.

4. Second Approach: Strictly Concave Quadratic Case

Remark. Even if the global maximizer $\mathbf{q}^{**}$ does not satisfy the constraints, because $P(\mathbf{q}^{**}) \geq P(\mathbf{q}^{*})$, a necessary condition for our company to be viable is

$$P(\mathbf{q}^{**}) = \frac{1}{2}(\mathbf{b}-\mathbf{c}) \cdot \left(\mathbf{A}+\mathbf{A}^{\top}\right)^{-1}(\mathbf{b}-\mathbf{c}) - d \geq 0\,.$$

5. Further Questions

We have seen that how different models of the demand relationship change the constrained optimization problem associated with maximizing profit. Some natural questions arise.

- How sensitive is $\mathbf{q}^*$ to the choice of a demand model?
- Can more complicated demand models lead to poorer answers?
- Is there some way to find the best demand model?
- What do “poorer” and “best” mean in this context?

5. Further Questions: Uncertainty

There is considerable uncertainty in or demand model. Perhaps it is better to introduce a shochastic demand model in the form

$$\mathbf{p} = \mathbf{b} - \mathbf{A}\mathbf{q} + \mathbf{z}\,,$$

where $\mathbf{z}$ is a random variable drawn from an unknown distribution with mean zero and known variance matrix $\mathbf{V}$. For example, the matrix $\mathbf{V}$ can be computed from the residuals of the least squares fit that was used to determine $\mathbf{A}$ and $\mathbf{b}$.

This modification transforms our maximization problem into a stochastic maximization problem. Building an objective can be done in a way similar to the way in which cautious objectives were built for portfolio management.