Portfolios that Contain Risky Assets 17: Fortune's Formulas

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Portfolios that Contain Risky Assets
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Fortune's Formulas

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Introduction

Given a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$, a choice of positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1, and a risk-free rate $\mu_{\rm rf}$, a cautious investor might select a portfolio allocation $\mathbf{f}$ from a set Π that maximizes a cautious objective

$$\widehat{\Gamma}^{\chi}(\mathbf{f}) = \widehat{\gamma}(\mathbf{f}) - \chi \sqrt{\widehat{\theta}(\mathbf{f})}\,, \qquad (1.1a)$$

where $\chi \geq 0$ is a caution coefficient chosen by the investor,

$$\begin{aligned} \widehat{\gamma}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \log\big(1 + r(d,\mathbf{f})\big)\,, \\ \widehat{\theta}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \Big(\log\big(1 + r(d,\mathbf{f})\big) - \hat{\gamma}(\mathbf{f})\Big)^2\,, \end{aligned} \qquad (1.1b)$$

with $r(d,\mathbf{f})$ given by

$$r(d,\mathbf{f}) = \mu_{\rm rf}\big(1 - \mathbf{1}^{\rm T}\mathbf{f}\big) + \mathbf{r}(d)^{\rm T}\mathbf{f}\,. \qquad (1.1c)$$

Introduction

We now consider some settings in which mean-variance approximations to this optimization problem can be solved analytically. These approximations replace the objective (1.1) with estimators that depend only on:

- the return for the risk-free assets μ_{rf},
- the return sample mean vector $\mathbf{m}$,
- the return sample covariance matrix $\mathbf{V}$,
- the nonnegative caution coefficient χ.

Recall that $\mathbf{m}$ and $\mathbf{V}$ are computed from a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and a choice of positive weights $\{w_d\}_{d=1}^{D}$ that sum to 1 by

$$\mathbf{m} = \sum_{d=1}^{D} w_d\, \mathbf{r}(d)\,, \qquad \mathbf{V} = \sum_{d=1}^{D} w_d \left(\mathbf{r}(d) - \mathbf{m}\right)\left(\mathbf{r}(d) - \mathbf{m}\right)^{\mathrm{T}}. \tag{1.2}$$

Introduction

In the previous lecture we saw that the maximizer $\mathbf{f}_*$ for such a problem corresponds to a point (σ_*, μ_*) on the efficient frontier. Moreover, we saw that (σ_*, μ_*) is the point in the $\sigma\mu$-plane where the level curves of the objective are tangent to the efficient frontier. While this geometric picture gave insight into how optimal portfolio allocations arise, we have not yet computed them.

The explicit formulas derived in this lecture for the maximizer $\mathbf{f}_*$ will confirm the general picture developed in the previous lecture. They will also give insight into the relative merits of the different families of approximate objectives In particular, the maximizers when $\chi = 0$ give different realizations of the Kelly Criterion — so-called *fortune's formulas*. The maximizers when $\chi > 0$ will be associated fractional Kelly strategies. We will derive and analyze these formulas after reviewing the efficient frontier for unlimited leverage portfolios with the one risk-free rate model.

Efficient Frontier

Recall that for unlimited leverage portfolios without risk-free assets the frontier is the hyperbola in the right-half of the $\sigma\mu$-plane given by

$$\sigma = \sqrt{\sigma_{\rm mv}^2 + \left(\frac{\mu - \mu_{\rm mv}}{\nu_{\rm as}}\right)^2}\,, \qquad (2.3a)$$

where the so-called frontier parameters $\sigma_{\rm mv}$, $\mu_{\rm mv}$, and $\nu_{\rm as}$ are given by

$$\frac{1}{\sigma_{\rm mv}^2} = \mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}\,, \qquad \mu_{\rm mv} = \frac{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{m}}{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}}\,,$$
$$\nu_{\rm as}^2 = \mathbf{m}^{\rm T}\mathbf{V}^{-1}\mathbf{m} - \frac{(\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{m})^2}{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}}\,. \qquad (2.3b)$$

The positive definiteness of $\mathbf{V}$ insures that $\sigma_{\rm mv} > 0$ and $\nu_{\rm as} > 0$. The so-called frontier hyperbola given by (2.3a) has vertex $(\sigma_{\rm mv}, \mu_{\rm mv})$ and asymptotes

$$\mu = \mu_{\rm mv} \pm \nu_{\rm as}\,\sigma \qquad \text{for } \sigma \geq 0\,.$$

Efficient Frontier

If risk-free assets are added using the one risk-free rate model with risk-free return μ_{rf} then when $\mu_{\text{rf}} < \mu_{\text{mv}}$ the *efficient frontier* is the tangent half-line given by

$$\mu = \mu_{\text{rf}} + \nu_{\text{tg}}\,\sigma \qquad \text{for } \sigma \geq 0\,, \tag{2.4a}$$

where the slope is

$$\begin{aligned} \nu_{\text{tg}} &= \sqrt{(\mathbf{m} - \mu_{\text{rf}}\mathbf{1})^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\text{rf}}\mathbf{1})} \\ &= \nu_{\text{as}}\sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{rf}}}{\nu_{\text{as}}\,\sigma_{\text{mv}}}\right)^2}\,. \end{aligned} \tag{2.4b}$$

This slope is the so-called *Sharpe ratio* of the efficient frontier. It will be the slope at $\sigma = 0$ of the efficient frontier associated will any set of leveraged portfolios.

Efficient Frontier

The efficient frontier (2.4a) is tangent to the frontier hyperbola (2.3a) at the point $(\sigma_{\mathrm{tg}}, \mu_{\mathrm{tg}})$ where

$$\sigma_{\mathrm{tg}} = \sigma_{\mathrm{mv}} \sqrt{1 + \left(\frac{\nu_{\mathrm{as}}\, \sigma_{\mathrm{mv}}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}} \right)^2}\,, \qquad \mu_{\mathrm{tg}} = \mu_{\mathrm{mv}} + \frac{\nu_{\mathrm{as}}^{\,2}\, \sigma_{\mathrm{mv}}^{\,2}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}\,.$$

The unique *tangency portfolio* associated with this point has allocation

$$\mathbf{f}_{\mathrm{tg}} = \frac{\sigma_{\mathrm{mv}}^{\,2}}{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}\, \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{rf}} \mathbf{1})\,. \tag{2.5}$$

Every portfolio on the efficient frontier (2.4a) can be viewed as holding a position in this tangency portfolio and a position in a risk-free asset.

Mean-Variance Approximations

We can select a portfolio on this efficient frontier by maximizing a mean-variance objective that approximates the cautious objective (1.1). These objectives are contructed by replacing the $\widehat{\gamma}(\mathbf{f})$ and $\widehat{\theta}(\mathbf{f})$ that appear in (1.1a) and that are defined by (1.1b) with mean-variance estimators that depend only on:

- the return for the risk-free assets μ_{rf},
- the return sample mean vector $\mathbf{m}$,
- the return sample covariance matrix $\mathbf{V}$.

Here we study three such approximations. Each of these approximations will respect an important symmetry of the cautious objective. This symmetry becomes evident upon rewriting the cautious objective.

Mean-Variance Approximations

It is easy to check from (1.1c) that

$$\log\left(1 + r(d,\mathbf{f})\right) = \log(1 + \mu_{\mathrm{rf}}) + \log\left(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\right) , \qquad (3.6)$$

where $\tilde{\mathbf{r}}(d)$ is defined by

$$\tilde{\mathbf{r}}(d) = \frac{1}{1 + \mu_{\mathrm{rf}}} \left(\mathbf{r}(d) - \mu_{\mathrm{rf}}\mathbf{1}\right) . \qquad (3.7)$$

The i^{th} entry of $\tilde{\mathbf{r}}(d)$ is called the *relative return* of asset i on day d with respect to the risk-free rate μ_{rf}. The so-called *relative growth rate* of the portfolio with allocation $\mathbf{f}$ is

$$\tilde{x}(d,\mathbf{f}) = \log\left(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\right) . \qquad (3.8)$$

It is the growth rate of the portfolio relative to that of the safe investment.

Mean-Variance Approximations

It then follows from (1.1b) and (3.6) that

$$\widehat{\gamma}(\mathbf{f}) = \log(1 + \mu_{\mathrm{rf}}) + \tilde{\gamma}(\mathbf{f})\,, \qquad \widehat{\theta}(\mathbf{f}) = \tilde{\theta}(\mathbf{f})\,, \tag{3.9a}$$

where

$$\begin{aligned} \tilde{\gamma}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \, \log\Big(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\Big)\,, \\ \tilde{\theta}(\mathbf{f}) &= \sum_{d=1}^{D} w_d \, \Big(\log\Big(1 + \tilde{\mathbf{r}}(d)^{\mathrm{T}}\mathbf{f}\Big) - \tilde{\gamma}(\mathbf{f})\Big)^2 \,. \end{aligned} \tag{3.9b}$$

These are the relative growth rate sample mean and sample variance respectively.

Mean-Variance Approximations

It then follows from (1.1a) that the cautious objective can be rewritten as

$$\widehat{\Gamma}^{\chi}(\mathbf{f}) = \log(1+\mu_{\rm rf}) + \widetilde{\Gamma}^{\chi}(\mathbf{f})\,, \tag{3.10a}$$

where

$$\widetilde{\Gamma}^{\chi}(\mathbf{f}) = \tilde{\gamma}(\mathbf{f}) - \chi\sqrt{\tilde{\theta}(\mathbf{f})}\,. \tag{3.10b}$$

The key observation is that

$$\tilde{\gamma}(\mathbf{f}), \qquad \tilde{\theta}(\mathbf{f}), \qquad \widetilde{\Gamma}^{\chi}(\mathbf{f})\,,$$

are the same as

$$\widehat{\gamma}(\mathbf{f}), \qquad \widehat{\theta}(\mathbf{f}), \qquad \widehat{\Gamma}^{\chi}(\mathbf{f})\,,$$

with $\mu_{\rm rf}$ replaced by 0 and $\mathbf{r}(d)$ replaced by $\tilde{\mathbf{r}}(d)$.

Mean-Variance Approximations

- It is clear from (3.10a) that the maximizer of $\widehat{\Gamma}^{\chi}(\mathbf{f})$ is also the maximizer of $\widetilde{\Gamma}^{\chi}(\mathbf{f})$.
- Our observation then implies that this maximizer can depend only on the relative return history $\{\tilde{\mathbf{r}}(d)\}_{d=1}^{D}$.

Therefore a mean-variance approximation of the cautious objective (1.1) should have a maximizer that depends only on:

- the relative return sample mean vector $\widetilde{\mathbf{m}}$,
- the relative return sample covariance matrix $\widetilde{\mathbf{V}}$,
- the nonnegative caution coefficient χ,

where

$$\widetilde{\mathbf{m}} = \sum_{d=1}^{D} w_d \, \tilde{\mathbf{r}}(d) \,, \qquad \widetilde{\mathbf{V}} = \sum_{d=1}^{D} w_d \left(\tilde{\mathbf{r}}(d) - \widetilde{\mathbf{m}}\right) \left(\tilde{\mathbf{r}}(d) - \widetilde{\mathbf{m}}\right)^{\mathrm{T}} . \tag{3.11}$$

Mean-Variance Approximations

It follows from the definitions (3.11) of $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{V}}$, the relation (3.7) between $\tilde{\mathbf{r}}(d)$ and $\mathbf{r}(d)$, and the definitions (1.2) of $\mathbf{m}$ and $\mathbf{V}$ that

$$\widetilde{\mathbf{m}} = \frac{1}{1+\mu_{\mathrm{rf}}}\left(\mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}\right) , \qquad \widetilde{\mathbf{V}} = \frac{1}{(1+\mu_{\mathrm{rf}})^2}\,\mathbf{V} . \tag{3.12}$$

It is clear from (2.4b) and the above relations that the Sharpe ratio is given by

$$\nu_{\mathrm{tg}} = \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}} . \tag{3.13}$$

Therefore mean-variance approximations of the cautious objective $\widehat{\Gamma}^{\chi}(\mathbf{f})$ that respect this symmetry can be constructed from mean-variance approximations of $\widetilde{\Gamma}^{\chi}(\mathbf{f})$ that depend on the relative return sample mean $\widetilde{\mathbf{m}}$ and sample variance $\widetilde{\mathbf{V}}$ and formula (3.10).

Mean-Variance Approximations

We will derive explicit formulas for the solutions to the maximization problems for the family of parabolic objectives

$$\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,, \tag{3.14a}$$

the family of quadratic objectives

$$\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,, \tag{3.14b}$$

and the family of reasonable objectives

$$\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,, \tag{3.14c}$$

considered over their natural domains of allocations **f** for unlimited leverage portfolios and one risk-free rate.

Parabolic Objectives

First we consider the maximization problem

$$\mathbf{f}_* = \arg\max\left\{\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f}\in\mathbb{R}^N\right\}, \tag{4.15a}$$

where $\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f})$ is the family of parabolic objectives (3.14a) given by

$$\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,. \tag{4.15b}$$

If $\mathbf{f}\neq 0$ then the gradient of $\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}} - \widetilde{\mathbf{V}}\mathbf{f} - \frac{\chi}{\sigma}\,\widetilde{\mathbf{V}}\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} > 0$.

Parabolic Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = \widetilde{\mathbf{m}} - \frac{\sigma_* + \chi}{\sigma_*} \widetilde{\mathbf{V}} \mathbf{f}_* \,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_*} > 0$. Upon solving this equation for $\mathbf{f}_*$ we obtain

$$\mathbf{f}_* = \frac{\sigma_*}{\sigma_* + \chi} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \,. \tag{4.16}$$

All that remains is to determine σ_*.

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_* = \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} = \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \nu_{\mathrm{tg}}^2 \,.$$

Parabolic Objectives

We conclude that σ_* satisfies

$$(\sigma_* + \chi)^2 = \nu_{\mathrm{tg}}^2 .$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that χ must satisfy the bounds

$$0 \leq \chi < \nu_{\mathrm{tg}} , \tag{4.17}$$

and that σ_* is determined by

$$\sigma_* + \chi = \nu_{\mathrm{tg}} .$$

Then the maximizer $\mathbf{f}_*$ given by (4.16) becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} . \tag{4.18}$$

Parabolic Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\text{tg}}$. To treat that case we will use the *Cauchy inequality* in the form

$$\left|\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right| \leq \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,. \tag{4.19}$$

When $\chi \geq \nu_{\text{tg}}$ the positive definiteness of $\widetilde{\mathbf{V}}$, the fact $\chi \geq \nu_{\text{tg}}$, the *Sharpe ratio* formula (3.13), and the above *Cauchy inequality* imply

$$\begin{aligned}
\widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f}) &= \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \nu_{\text{tg}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&= \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \leq 0 = \widetilde{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{0})\,.
\end{aligned}$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\text{tg}}$.

Parabolic Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (4.15) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \frac{\chi}{\nu_{\rm tg}}\right) \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\rm tg}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\rm tg}\,. \end{cases} \tag{4.20}$$

This solution lies on the efficient frontier (2.4a). When $\chi < \nu_{\rm tg}$ it allocates $f_{\rm tg}^{\chi}$ times the portfolio value in the tangent portfolio $\mathbf{f}_{\rm tg}$ given by (2.5) and $(1 - f_{\rm tg}^{\chi})$ times the portfolio value in a risk-free asset, where

$$f_{\rm tg}^{\chi} = \mathbf{1}^{\rm T}\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\rm tg}}\right) (1 + \mu_{\rm rf}) \frac{\mu_{\rm mv}}{\sigma_{\rm mv}^2}\,. \tag{4.21}$$

Parabolic Objectives

Remark. Kelly investors take $\chi = 0$, in which case (4.20) reduces to

$$\mathbf{f}_* = \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{4.22}$$

This is often called *fortune's formula* in the belief that it is a good approximation to the Kelly strategy. In this view formula (4.20) gives a fractional Kelly strategy for every $\chi \in (0, \nu_{\mathrm{tg}})$. However, we will see that formula (4.22) gives an allocation that can be far from the Kelly strategy, and can lead to overbetting.

Quadratic Objectives

Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\left\{ \widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N \right\}, \tag{5.23a}$$

where $\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f})$ is the family of quadratic objectives (3.14b) given by

$$\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,. \tag{5.23b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}} - \widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \widetilde{\mathbf{V}}\mathbf{f} - \frac{\chi}{\sigma}\,\widetilde{\mathbf{V}}\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} > 0$.

Quadratic Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = \widetilde{\mathbf{m}} - \widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* - \frac{\sigma_* + \chi}{\sigma_*}\,\widetilde{\mathbf{V}}\mathbf{f}_*\,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}_*} > 0$.

After multiplying this relation by $\widetilde{\mathbf{V}}^{-1}$ and bringing the terms involving $\mathbf{f}_*$ to the left-hand side, we obtain

$$\frac{\sigma_* + \chi}{\sigma_*}\,\mathbf{f}_* + \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* = \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{5.24}$$

Quadratic Objectives

Now multiply this by $\sigma_* \, \mathbf{m}^{\mathrm{T}}$ and use the *Sharpe ratio* formula (3.13), $\widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} = \nu_{\mathrm{tg}}^2$, to obtain

$$\left(\sigma_* + \chi + \nu_{\mathrm{tg}}^2 \, \sigma_*\right) \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* = \nu_{\mathrm{tg}}^2 \, \sigma_* \,,$$

which implies that

$$\widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_* = \frac{\nu_{\mathrm{tg}}^2 \, \sigma_*}{\sigma_* + \chi + \nu_{\mathrm{tg}}^2 \, \sigma_*} \,.$$

When this expression is placed into (5.24) we can solve for $\mathbf{f}_*$ to find

$$\mathbf{f}_* = \frac{\sigma_*}{\sigma_* + \chi + \nu_{\mathrm{tg}}^2 \, \sigma_*} \, \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \,. \qquad (5.25)$$

All that remains is to determine σ_*.

Quadratic Objectives

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_* = \frac{\sigma_*^2}{\left((1 + \nu_{\mathrm{tg}}^2)\, \sigma_* + \chi \right)^2} \, \widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} = \frac{\sigma_*^2}{\left((1 + \nu_{\mathrm{tg}}^2)\, \sigma_* + \chi \right)^2} \, \nu_{\mathrm{tg}}^2 \,,$$

we conclude that σ_* satisfies

$$\left((1 + \nu_{\mathrm{tg}}^2)\, \sigma_* + \chi \right)^2 = \nu_{\mathrm{tg}}^2 \,.$$

Quadratic Objectives

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that χ must satisfy the bounds

$$0 \leq \chi < \nu_{\mathrm{tg}} \,, \tag{5.26}$$

and that σ_* is determined by

$$(1 + \nu_{\mathrm{tg}}^2)\, \sigma_* + \chi = \nu_{\mathrm{tg}} \,.$$

Therefore the maximizer $\mathbf{f}_*$ given by (5.25) becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{1}{1 + \nu_{\mathrm{tg}}^2}\, \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \,. \tag{5.27}$$

Quadratic Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\text{tg}}$. In that case the positive definiteness of $\widetilde{\mathbf{V}}$, the fact $\chi \geq \nu_{\text{tg}}$, the *Sharpe ratio* formula (3.13), and the *Cauchy inequality* (4.19) imply

$$
\begin{aligned}
\widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f}) &= \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \nu_{\text{tg}}\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \\
&= \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} \leq 0 = \widetilde{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{0})\,.
\end{aligned}
$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\text{tg}}$.

Quadratic Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (5.23) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{1 + \nu_{\mathrm{tg}}^2} & \text{if } \chi < \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}}\,. \end{cases} \tag{5.28}$$

This solution lies on the efficient frontier (2.4a). When $\chi < \nu_{\mathrm{tg}}$ it allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ given by (2.5) and $(1 - f_{\mathrm{tg}}^{\chi})$ times the portfolio value in a risk-free asset, where

$$f_{\mathrm{tg}}^{\chi} = \mathbf{1}^{\mathrm{T}}\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{1 + \mu_{\mathrm{rf}}}{1 + \nu_{\mathrm{tg}}^2} \frac{\mu_{\mathrm{mv}}}{\sigma_{\mathrm{mv}}^2}\,. \tag{5.29}$$

Quadratic Objectives

Remark. If $\chi < \nu_{\rm tg}$ then the maximizer $\mathbf{f}_*$ given by (5.28) satisfies

$$\widetilde{\mathbf{m}}^{\rm T}\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\rm tg}}\right)\frac{\widetilde{\mathbf{m}}^{\rm T}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{1+\nu_{\rm tg}^2} = \left(1 - \frac{\chi}{\nu_{\rm tg}}\right)\frac{\nu_{\rm tg}^2}{1+\nu_{\rm tg}^2}\,,$$

Because $\widetilde{\mathbf{m}}^{\rm T}\mathbf{f}_* < 1$, we see that $\mathbf{f}_*$ lies within the interior of the set

$$\Pi_{\rm q} = \left\{\mathbf{f} \in \mathbb{R}^N \,:\, \widetilde{\mathbf{m}}^{\rm T}\mathbf{f} \leq 1\right\},$$

which lies within the domain from the maximization problem (5.23a). Therefore $\mathbf{f}_*$ is also the maximizer of $\widetilde{\Gamma}_{\rm q}^{\chi}(\mathbf{f})$ over the domain $\Pi_{\rm q}$, which is the maximization problem that we considered for the quadratic objective in the previous lecture.

Quadratic Objectives

Remark. Kelly investors take $\chi = 0$, in which case (5.28) reduces to

$$\mathbf{f}_* = \frac{1}{1+\nu_{\mathrm{tg}}^2}\,\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{5.30}$$

Formula (5.30) differs significantly from formula (4.22) whenever the Sharpe ratio ν_{tg} is not small. Sharpe ratios are often near 1 and sometimes can be as large as 3. So which of these should be called *fortune's formula*? Certainly not formula (4.22)! To see why, set $\mathbf{f} = \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}$ into the quadratic objective (5.23b) with $\chi = 0$ to obtain

$$\widetilde{\Gamma}_{\mathrm{q}}^{0}(\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}) = \tfrac{1}{2}\,\nu_{\mathrm{tg}}^2 - \tfrac{1}{2}\,\nu_{\mathrm{tg}}^4\,,$$

which is negative when $\nu_{\mathrm{tg}} > 1$. So formula (4.22) might overbet!

Reasonable Objectives

Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\left\{ \widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N \,,\, 1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} > 0 \right\}, \tag{6.31a}$$

where $\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f})$ is the family of reasonable objectives (3.14c) given by

$$\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}}\,. \tag{6.31b}$$

Because $\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) \to -\infty$ as $\mathbf{f}$ approaches the boundary of the domain being considered in (6.31a), the maximizer must lie in the interior of the domain. If $\mathbf{f} \neq 0$ then the gradient of $\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\widetilde{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f}) = \frac{1}{1+\mu}\,\widetilde{\mathbf{m}} - \widetilde{\mathbf{V}}\mathbf{f} - \frac{\chi}{\sigma}\,\widetilde{\mathbf{V}}\mathbf{f}\,,$$

where $\mu = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}$ and $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}} > 0$.

Reasonable Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{f}_* = \frac{1}{1+\mu_*} \, \frac{\sigma_*}{\sigma_* + \chi} \, \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \, , \tag{6.32}$$

where $\mu_* = \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{f}_*$ and $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_*} > 0$.

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_*}$ we have

$$\begin{aligned} \sigma_*^2 = \mathbf{f}_*^{\mathrm{T}} \widetilde{\mathbf{V}} \mathbf{f}_* &= \frac{1}{(1+\mu_*)^2} \, \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \, \widetilde{\mathbf{m}}^{\mathrm{T}} \widetilde{\mathbf{V}}^{-1} \widetilde{\mathbf{m}} \\ &= \frac{1}{(1+\mu_*)^2} \, \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \, \nu_{\mathrm{tg}}^2 \, . \end{aligned}$$

Reasonable Objectives

From this we conclude that μ_* and σ_* satisfy

$$(\sigma_* + \chi)^2 = \frac{\nu_{\rm tg}^2}{(1+\mu_*)^2}\,.$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < \frac{\nu_{\rm tg}}{1+\mu_*}\,, \tag{6.33}$$

and that we can determine σ_* in terms of μ_* from

$$\sigma_* + \chi = \frac{\nu_{\rm tg}}{1+\mu_*}\,.$$

Then the maximizer $\mathbf{f}_*$ given by (6.32) becomes

$$\mathbf{f}_* = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\rm tg}}\right)\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,, \tag{6.34}$$

Reasonable Objectives

Because $\mu_* = \mathbf{m}^{\mathrm{T}}\mathbf{f}_*$, by the *Sharpe ratio* formula (3.13) we have

$$\mu_* = \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathrm{tg}}}\right)\widetilde{\mathbf{m}}^{\mathrm{T}}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}$$
$$= \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathrm{tg}}}\right)\nu_{\mathrm{tg}}^2 \,.$$

This can be reduced to the quadratic equation

$$\left(\frac{\nu_{\mathrm{tg}}}{1+\mu_*}\right)^2 + \left(\frac{1}{\nu_{\mathrm{tg}}} - \chi\right)\frac{\nu_{\mathrm{tg}}}{1+\mu_*} = 1\,,$$

which has the unique positive root

$$\frac{\nu_{\mathrm{tg}}}{1+\mu_*} = -\frac{1}{2}\left(\frac{1}{\nu_{\mathrm{tg}}} - \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1}{\nu_{\mathrm{tg}}} - \chi\right)^2}\,. \qquad (6.35)$$

Reasonable Objectives

Then condition (6.33) is satisfied if and only if

$$0 < \frac{\nu_{\mathrm{tg}}}{1+\mu_*} - \chi = -\frac{1}{2}\left(\frac{1}{\nu_{\mathrm{tg}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1}{\nu_{\mathrm{tg}}} - \chi\right)^2}\,.$$

This inequality holds if and only if

$$0 < 1 + \frac{1}{4}\left(\frac{1}{\nu_{\mathrm{tg}}} - \chi\right)^2 - \frac{1}{4}\left(\frac{1}{\nu_{\mathrm{tg}}} + \chi\right)^2 = 1 - \frac{\chi}{\nu_{\mathrm{tg}}}\,.$$

This holds if and only if χ satisfies the bounds

$$0 \le \chi < \nu_{\mathrm{tg}}\,. \tag{6.36}$$

Reasonable Objectives

By using (6.35) to eliminate μ_* from the maximizer $\mathbf{f}_*$ given by (6.34) we find

$$\mathbf{f}_* = \left[-\frac{1}{2}\left(\frac{1}{\nu_{\text{tg}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1}{\nu_{\text{tg}}} - \chi\right)^2}\, \right] \frac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{\nu_{\text{tg}}} \,.$$

This becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\text{tg}}}\right) \frac{1}{D(\chi, \nu_{\text{tg}})} \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} \,, \tag{6.37a}$$

where

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2} \,. \tag{6.37b}$$

Reasonable Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\rm tg}$. In that case the concavity of log(x), the positive definiteness of $\widetilde{\mathbf{V}}$, the fact $\chi \geq \nu_{\rm tg}$, the *Sharpe ratio* formula (3.13), and the *Cauchy inequality* (4.19) imply

$$
\begin{aligned}
\widetilde{\Gamma}_{\rm r}^{\chi}(\mathbf{f}) &= \log\left(1 + \widetilde{\mathbf{m}}^{\rm T}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\rm T}\widetilde{\mathbf{V}}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\rm T}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\rm T}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\rm T}\widetilde{\mathbf{V}}\mathbf{f}} \\
&\leq \widetilde{\mathbf{m}}^{\rm T}\mathbf{f} - \nu_{\rm tg}\,\sqrt{\mathbf{f}^{\rm T}\widetilde{\mathbf{V}}\mathbf{f}} \\
&= \widetilde{\mathbf{m}}^{\rm T}\mathbf{f} - \sqrt{\widetilde{\mathbf{m}}^{\rm T}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\rm T}\widetilde{\mathbf{V}}\mathbf{f}} \leq 0 = \widetilde{\Gamma}_{\rm r}^{\chi}(\mathbf{0})\,.
\end{aligned}
$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\rm tg}$.

Reasonable Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (6.31) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\rm tg}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{D(\chi, \nu_{\rm tg})} & \text{if } \chi < \nu_{\rm tg}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\rm tg}\,, \end{cases} \tag{6.38}$$

where $D(\chi, y)$ was defined by (6.37b).

This solution lies on the efficient frontier (2.4a). When $\chi < \nu_{\rm tg}$ it allocates $f_{\rm tg}^{\chi}$ times the portfolio value in the tangent portfolio $\mathbf{f}_{\rm tg}$ given by (2.5) and $(1 - f_{\rm tg}^{\chi})$ times the portfolio value in a risk-free asset, where

$$f_{\rm tg}^{\chi} = \mathbf{1}^{\rm T}\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\rm tg}}\right) \frac{1 + \mu_{\rm rf}}{D(\chi, \nu_{\rm tg})} \, \frac{\mu_{\rm mv}}{\sigma_{\rm mv}^2}\,. \tag{6.39}$$

Reasonable Objectives

Remark. If $\chi < \nu_{\rm tg}$ then the maximizer $\mathbf{f}_*$ given by (6.38) satisfies

$$\widetilde{\mathbf{m}}^{\rm T}\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\rm tg}}\right)\frac{\widetilde{\mathbf{m}}^{\rm T}\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{D(\chi,\nu_{\rm tg})} = \left(1 - \frac{\chi}{\nu_{\rm tg}}\right)\frac{\nu_{\rm tg}^2}{D(\chi,\nu_{\rm tg})}\,.$$

Because $1 + \widetilde{\mathbf{m}}^{\rm T}\mathbf{f}_* \geq 1 > 0$, we confirm that $\mathbf{f}_*$ lies within the interior of the domain from the maximization problem (6.31a).

Remark. Kelly investors take $\chi = 0$, in which case (6.38) reduces to

$$\mathbf{f}_* = \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\rm tg}^2}}\,\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}\,. \tag{6.40}$$

This candidate for *fortune's formula* will be compared with the others next.

Comparisons

The maximizers for the parabolic, quadratic, and reasonable objectives are given by (4.20), (5.28), and (6.38) respectively. They are

$$\mathbf{f}_*^{\mathrm{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}}\,, \end{cases} \tag{7.41a}$$

$$\mathbf{f}_*^{\mathrm{q}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{1+\nu_{\mathrm{tg}}^2} & \text{if } \chi < \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}}\,, \end{cases} \tag{7.41b}$$

$$\mathbf{f}_*^{\mathrm{r}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\widetilde{\mathbf{V}}^{-1}\widetilde{\mathbf{m}}}{D(\chi,\nu_{\mathrm{tg}})} & \text{if } \chi < \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}}\,. \end{cases} \tag{7.41c}$$

where $D(\chi, y)$ was defined by (6.37b).

Comparisons

Fact 1. Of these, $\mathbf{f}_*^{\mathrm{q}}$ is the most conservative and $\mathbf{f}_*^{\mathrm{p}}$ is the most agressive.

Proof. Recall from (6.37b) that

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2}\,. \tag{7.42}$$

This is a strictly increasing function of χ because for every $y > 0$ we have

$$\partial_\chi D(\chi, y) = \tfrac{1}{2} y \left(1 - \frac{1 - \chi\, y}{\sqrt{(1 - \chi\, y)^2 + 4y^2}} \right) > 0\,.$$

Hence, for every $\chi \in [0, y)$ we have

$$1 < D(0, y) \le D(\chi, y) < D(y, y) = 1 + y^2\,. \tag{7.43}$$

Therefore $1 < D(\chi, \nu_{\mathrm{tg}}) < 1 + \nu_{\mathrm{tg}}^2$ when $\chi < \nu_{\mathrm{tg}}$. □

Comparisons

We now compare the dependence of $\mathbf{f}_*^{\mathrm{q}}$ and $\mathbf{f}_*^{\mathrm{r}}$ upon χ and ν_{tg}.

Fact 2. For every $\chi \in [0, \nu_{\mathrm{tg}})$ we have

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}}{1 + \nu_{\mathrm{tg}}^2} \leq \frac{D(\chi, \nu_{\mathrm{tg}})}{1 + \nu_{\mathrm{tg}}^2} < 1\,, \tag{7.44}$$

where the left-hand side is a strictly decreasing function of ν_{tg}.

Proof. By setting $y = \nu_{\mathrm{tg}}$ in (7.43) we obtain

$$1 + \nu_{\mathrm{tg}}^2 > D(\chi, \nu_{\mathrm{tg}}) \geq D(0, \nu_{\mathrm{tg}}) = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}\,.$$

The inequalities (7.44) follow. The task of proving the left-hand side of (7.44) is a strictly decreasing function of ν_{tg} is left as an exercise. □

Comparisons

We now use **Fact 2** to show that $\mathbf{f}_*^{\mathrm{q}}$ and $\mathbf{f}_*^{\mathrm{r}}$ are close when $\nu_{\mathrm{tg}} \leq \frac{2}{3}$.

Fact 3. If $\nu_{\mathrm{tg}} \leq \frac{2}{3}$ then for every $\chi \in [0, \nu_{\mathrm{tg}})$ we have

$$\tfrac{12}{13} \leq \frac{D(\chi, \nu_{\mathrm{tg}})}{1 + \nu_{\mathrm{tg}}^2} < 1\,. \tag{7.45}$$

Proof. By the monotonicity asserted in **Fact 2** if $\nu_{\mathrm{tg}} \leq \frac{2}{3}$ then

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}}{1 + \nu_{\mathrm{tg}}^2} \geq \frac{\frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3}}{1 + \frac{4}{9}} = \frac{\frac{4}{3}}{\frac{13}{9}} = \frac{12}{13}\,.$$

Then (7.45) follows from inequality (7.44) of **Fact 2**. □

Comparisons

Remark. We see from (7.41) that when $\chi = 0$

$$\mathbf{f}_*^{\mathrm{q}} = \frac{1}{1+\nu_{\mathrm{tg}}^2}\,\mathbf{f}_*^{\mathrm{p}}\,, \qquad \mathbf{f}_*^{\mathrm{r}} = \frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1+4\nu_{\mathrm{tg}}^2}}\,\mathbf{f}_*^{\mathrm{p}}\,.$$

This is the case when the difference between $\mathbf{f}_*^{\mathrm{q}}$ and $\mathbf{f}_*^{\mathrm{r}}$ is at its greatest. To get a feel for this difference, when $\nu_{\mathrm{tg}} = \sqrt{2}$ these are

$$\mathbf{f}_*^{\mathrm{q}} = \tfrac{1}{3}\,\mathbf{f}_*^{\mathrm{p}}\,, \qquad \mathbf{f}_*^{\mathrm{r}} = \tfrac{1}{2}\,\mathbf{f}_*^{\mathrm{p}}\,,$$

while when $\nu_{\mathrm{tg}} = \sqrt{6}$ these are

$$\mathbf{f}_*^{\mathrm{q}} = \tfrac{1}{7}\,\mathbf{f}_*^{\mathrm{p}}\,, \qquad \mathbf{f}_*^{\mathrm{r}} = \tfrac{1}{3}\,\mathbf{f}_*^{\mathrm{p}}\,.$$

We see that this difference becomes quite large for Sharpe ratios $\nu_{\mathrm{tg}} > 2$.

Seven Lessons Learned

Here are seven lessons learned about these mean-variance objectives.

1. The return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and risk-free return $\mu_{\rm rf}$ play roles in determining the optimal allocation entirely through $\widetilde{\mathbf{m}}$ and $\widetilde{\mathbf{V}}$.

2. The Sharpe ratio $\nu_{\rm tg}$ and the caution coefficient χ play a huge role in determining the optimal allocation. In particular, when $\chi \geq \nu_{\rm tg}$ the optimal allocation is entirely in the safe investment.

3. For any choice of χ the maximizer for the quadratic objective is more conservative than the maximizer for the reasonable objective, which is more conservative than the maximizer for the parabolic objective.

4. The maximizer for a parabolic objective is agressive and will likely overbet when the Sharpe ratio $\nu_{\rm tg}$ is not small.

Seven Lessons Learned

5. The maximizers for quadratic and reasonable objectives are close when the Sharpe ratio ν_{tg} is not large. As χ approaches ν_{tg}, the maximizers for the quadratic and reasonable objectives get closer.

6. We will have greater confidence in the computed Sharpe ratio ν_{tg} when the tangency portfolio lies towards the "nose" of the efficient frontier. This translates into having greater confidence in the maximizers for the quadratic and reasonable objectives.

7. Analyzing the maximizers for both the quadratic and reasonable objectives gave greater insights than analyzing them separately. Together they are fortune's formulas!