Portfolios that Contain Risky Assets 16: Optimization of Mean-Variance Objectives

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Portfolios that Contain Risky Assets
Part II: Stochastic Models

Optimization of Mean-Variance Objectives

Mean-Variance Objectives

We consider portfolios that contains N risky assets along with a risk-free safe investment and possibly a risk-free credit line. Given the return mean vector $\mathbf{m}$, the return covariance matrix $\mathbf{V}$, and the risk-free returns μ_{si} and μ_{cl}, a mean-variance objective for a portfolio allocation $\mathbf{f}$ has the form

$$\hat{\Gamma}(\mathbf{f}) = G(\hat{\sigma}(\mathbf{f}), \hat{\mu}(\mathbf{f})) , \tag{1.1a}$$

where the return mean and variance estimators are given by

$$\begin{aligned} \hat{\sigma}(\mathbf{f}) &= \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} , \\ \hat{\mu}(\mathbf{f}) &= \mu_{\mathrm{rf}}(\mathbf{f}) \left(1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}\right) + \mathbf{m}^{\mathrm{T}}\mathbf{f} , \end{aligned} \qquad \mu_{\mathrm{rf}}(\mathbf{f}) = \begin{cases} \mu_{\mathrm{si}} & \text{if } \mathbf{1}^{\mathrm{T}}\mathbf{f} \leq 1 , \\ \mu_{\mathrm{cl}} & \text{if } \mathbf{1}^{\mathrm{T}}\mathbf{f} > 1 . \end{cases} \tag{1.1b}$$

Here we show how to optimize such objectives over a class Π of Markowitz portfolio allocations.

Mean-Variance Objectives

If $\hat{\Gamma}(\mathbf{f})$ is $\hat{\Gamma}_{\mathrm{p}}^{\chi}(\mathbf{f})$, $\hat{\Gamma}_{\mathrm{q}}^{\chi}(\mathbf{f})$, $\hat{\Gamma}_{\mathrm{r}}^{\chi}(\mathbf{f})$, $\hat{\Gamma}_{\mathrm{s}}^{\chi}(\mathbf{f})$, or $\hat{\Gamma}_{\mathrm{t}}^{\chi}(\mathbf{f})$, for some $\chi \geq 0$ then $G(\sigma, \mu)$ is respectively

$$G_{\mathrm{p}}^{\chi}(\sigma, \mu) = \mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,, \tag{1.2a}$$

$$G_{\mathrm{q}}^{\chi}(\sigma, \mu) = \mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,, \tag{1.2b}$$

$$G_{\mathrm{r}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\sigma^2 - \chi\,\sigma\,, \tag{1.2c}$$

$$G_{\mathrm{s}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{1+2\mu} - \chi\,\sigma\,. \tag{1.2d}$$

$$G_{\mathrm{t}}^{\chi}(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\,\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,. \tag{1.2e}$$

Mean-Variance Objectives

These are the parabolic, quadratic, reasonable, sensible, and Taylor estimators respectively. They are respectively defined over the convex sets

$$\Sigma_{\mathrm{p}} = \{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\}\,, \tag{1.3a}$$

$$\Sigma_{\mathrm{q}} = \{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\,,\ \mu < 1\}\,, \tag{1.3b}$$

$$\Sigma_{\mathrm{r}} = \{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\,,\ \mu > -1\}\,, \tag{1.3c}$$

$$\Sigma_{\mathrm{s}} = \{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\,,\ \mu > -\tfrac{1}{2}\}\,. \tag{1.3d}$$

$$\Sigma_{\mathrm{t}} = \{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\,,\ 1 + \mu > \sigma\}\,. \tag{1.3e}$$

It is evident that each $G(\sigma, \mu)$ given in (1.2) is infinitely differentiable over the convex set Σ that is respectively given in (1.3).

Mean-Variance Objectives

We will consider mean-variance objectives (1.1) given by a function $G(\sigma, \mu)$ that is defined over a convex set $\Sigma \subset \mathbb{R}^2$ which is consistent with the class of Markowitz portfolio allocations Π in the sense that

$$\Sigma(\Pi) = \left\{ \left(\hat{\sigma}(\mathbf{f}) \, , \, \hat{\mu}(\mathbf{f}) \right) \, : \, \mathbf{f} \in \Pi \right\} \subset \Sigma \, . \tag{1.4}$$

For example, it can be shown that

$$\Sigma(\Omega_{(0,2)}) \subset \Sigma^{\mathrm{q}} \, ,$$

where $\Omega_{(0,2)}$ is the set of all portfolio allocations with value-ratios in $(0, 2)$;

$$\Sigma(\Omega) \subset \Sigma^{\mathrm{r}} \, ,$$

where Ω is the set of all solvent portfolio allocations; and

$$\Sigma(\Omega_{\frac{1}{2}}) \subset \Sigma^{\mathrm{s}} \, ,$$

where $\Omega_{\frac{1}{2}}$ is the set of all portfolio allocations with value-ratios in $(\frac{1}{2}, \infty)$.

Explicit Level Sets of some Objectives

For every such mean-variance objective (1.1) given by a function $G(\sigma, \mu)$ that is defined over a convex set $\Sigma \subset \mathbb{R}^2$ we define its level set associated with a possible value $\Gamma \in \mathbb{R}$ by

$$\Sigma(\Gamma) = \{(\sigma, \mu) \in \Sigma \, : \, G(\sigma, \mu) = \Gamma\} \, . \tag{2.5}$$

This set will be empty when there are no points $(\sigma, \mu) \in \Sigma$ that satisfy $G(\sigma, \mu) = \Gamma$. The consistency condition (1.4) implies that

$$\left\{ (\hat{\sigma}(\mathbf{f}) \, , \, \hat{\mu}(\mathbf{f})) \, : \, \mathbf{f} \in \Pi(\Gamma) \right\} \subset \Sigma(\Gamma) \, , \tag{2.6a}$$

where $\Pi(\Gamma)$ is defined by

$$\Pi(\Gamma) = \{\mathbf{f} \in \Pi \, : \, \hat{\Gamma}(\mathbf{f}) = \Gamma\} \, . \tag{2.6b}$$

Explicit Level Sets of some Objectives

For the parabolic estimator the points (σ, μ) in the level set $\Sigma_{\mathrm{p}}(\Gamma)$ satisfy

$$\mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma = \Gamma\,.$$

Upon solving this for μ we obtain

$$\begin{aligned}\mu &= \tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma \\ &= \tfrac{1}{2}(\sigma + \chi)^2 + \Gamma - \tfrac{1}{2}\chi^2\,.\end{aligned}$$

This equation yields a parabola with vertex

$$\big(-\chi\,,\,\Gamma - \tfrac{1}{2}\chi^2\big)\,,$$

focal length is $\frac{1}{2}$, and focus

$$\big(-\chi\,,\,\Gamma - \tfrac{1}{2}\chi^2 + \tfrac{1}{2}\big)\,.$$

Explicit Level Sets of some Objectives

The level set $\Sigma_\mathrm{p}(\Gamma)$ is the restriction of this parabola to Σ_p. Because

$$\Sigma_\mathrm{p} = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \, : \, \sigma \geq 0 \right\},$$

we have

$$\Sigma_\mathrm{p}(\Gamma) = \left\{ \left(\sigma \, , \, \mu_\mathrm{p}^\chi(\sigma, \Gamma)\right) \, : \, \sigma \geq 0 \right\}. \tag{2.7}$$

where $\mu = \mu_\mathrm{p}^\chi(\sigma, \Gamma)$ is given by

$$\mu_\mathrm{p}^\chi(\sigma, \Gamma) = \tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma \, .$$

We thereby see that Σ_p is foilated by segments of the family of parabolas given by $\mu = \mu_\mathrm{p}^\chi(\sigma, \Gamma)$. These parabolas shift upward with increasing Γ.

Explicit Level Sets of some Objectives

For the quadratic estimator the points (σ, μ) in the level set $\Sigma_{\mathrm{q}}(\Gamma)$ satisfy

$$\mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma = \Gamma\,.$$

By completing squares we see that this equation has the form

$$\tfrac{1}{2}(\sigma + \chi)^2 + \tfrac{1}{2}(\mu - 1)^2 = \tfrac{1}{2}\chi^2 + \tfrac{1}{2} - \Gamma\,.$$

This equation clearly has no solution unless $\chi^2 + 1 \geq 2\Gamma$. When $\chi^2 + 1 \geq 2\Gamma$ it yields a circle in the $\sigma\mu$-plane with center

$$(\,-\chi\,,\,1)\,,$$

and radius

$$\sqrt{\chi^2 + 1 - 2\Gamma}\,.$$

Explicit Level Sets of some Objectives

The level set $\Sigma_{\mathrm{q}}(\Gamma)$ is the restriction of this circle to Σ_{q}. Because

$$\Sigma_{\mathrm{q}} = \left\{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0\,,\; \mu < 1\right\},$$

we can show that $\Sigma_{\mathrm{q}}(\Gamma)$ is empty when $\Gamma \geq \frac{1}{2}$, and that when $\Gamma < \frac{1}{2}$ we have

$$\Sigma_{\mathrm{q}}(\Gamma) = \left\{(\sigma\,,\,\mu_{\mathrm{q}}^{\chi}(\sigma,\Gamma)) \,:\, 0 \leq \sigma \leq \sqrt{\chi^2 + 1 - 2\Gamma} - \chi\right\}, \tag{2.8}$$

where $\mu = \mu_{\mathrm{q}}^{\chi}(\sigma,\Gamma)$ is given by

$$\mu_{\mathrm{q}}^{\chi}(\sigma,\Gamma) = 1 - \sqrt{\chi^2 + 1 - 2\Gamma - (\sigma+\chi)^2}\,.$$

We thereby see that Σ_{q} is foilated by arcs of the family of semicircles centered at $(-\chi, 1)$ given by $\mu = \mu_{\mathrm{q}}^{\chi}(\sigma,\Gamma)$ for every $\Gamma < \frac{1}{2}$. The radius of these circles decreases with increasing Γ.

Explicit Level Sets of some Objectives

For the reasonable estimator the points (σ, μ) in the level set $\Sigma_{\mathrm{r}}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\sigma^2 - \chi\,\sigma = \Gamma\,.$$

Upon solving this for μ we obtain

$$\begin{aligned}\mu &= \exp\left(\tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma\right) - 1 \\ &= \exp\left(\tfrac{1}{2}(\sigma+\chi)^2 + \Gamma - \tfrac{1}{2}\chi^2\right) - 1\,.\end{aligned}$$

The graph of this function is strictly convex with a minimum at

$$\Big(-\chi\,,\ \exp\left(\Gamma - \tfrac{1}{2}\chi^2\right) - 1\Big)\,.$$

Because $e^x - 1 > x$ for every $x \neq 0$, we see that this curve lies above the corresponding parabola associated with the parabolic estimator.

Explicit Level Sets of some Objectives

The level set $\Sigma_{\rm r}(\Gamma)$ is the restriction of this curve to $\Sigma_{\rm r}$. Because

$$\Sigma_{\rm r} = \{(\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma \geq 0 \,,\, 1 + \mu > 0\} \,,$$

we have

$$\Sigma_{\rm r}(\Gamma) = \left\{ \left(\sigma \,,\, \mu_{\rm r}^{\chi}(\sigma, \Gamma)\right) \,:\, \sigma \geq 0 \right\} , \tag{2.9}$$

where $\mu = \mu_{\rm r}^{\chi}(\sigma, \Gamma)$ is given by

$$\mu_{\rm r}^{\chi}(\sigma, \Gamma) = \exp\left(\tfrac{1}{2}\sigma^2 + \chi\,\sigma + \Gamma\right) - 1 \,.$$

We thereby see that $\Sigma_{\rm r}$ is foilated by segments of the family of curves given by $\mu = \mu_{\rm r}^{\chi}(\sigma, \Gamma)$. These curves shift upward with increasing Γ.

Implicit Level Sets of the Objectives

At this point the explicit approach that we have been taking breaks down. For the sensible estimator the points (σ, μ) in the level set $\Sigma_{\rm s}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{1+2\mu} - \chi\,\sigma = \Gamma\,.$$

For the Taylor estimator the points (σ, μ) in the level set $\Sigma_{\rm t}(\Gamma)$ satisfy

$$\log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma = \Gamma\,.$$

These equations cannot be solved for μ explicitly. Of course, they can be solved for σ explicitly. However, it is easier to analyze the level sets that they define implicitly because is avoids messy formulas.

Implicit Level Sets of the Objectives

We will carry out this implicit analysis in the general setting of an equation in the form

$$G(\sigma, \mu) = \Gamma \,,$$

where we assume that $G_\mu(\sigma, \mu) > 0$ over the interior of the convex set $\Sigma \subset \mathbb{R}^2$. (Here G_μ denotes the partial derivative of G with respect to μ.)

By the Implicit Function Theorem this assumption implies that there exists a unique function $\mu(\sigma, \Gamma)$ such that

$$G(\sigma, \mu(\sigma, \Gamma)) = \Gamma \,. \tag{3.10}$$

Moreover, the function $\mu(\sigma, \Gamma)$ is infinitely differentiable.

Implicit Level Sets of the Objectives

By taking the partial derivative of (3.10) with respect to Γ we find that

$$G_\mu(\sigma, \mu) \, \frac{\partial \mu}{\partial \Gamma} = 1 \, .$$

Because $G_\mu(\sigma, \mu) > 0$, this can be solved to obtain

$$\frac{\partial \mu}{\partial \Gamma} = \frac{1}{G_\mu(\sigma, \mu)} > 0 \, .$$

Therefore $\mu(\sigma, \Gamma)$ is a strictly increasing function of Γ.

Implicit Level Sets of the Objectives

By taking the partial derivative of (3.10) with respect to σ we find that

$$G_\sigma(\sigma,\mu) + G_\mu(\sigma,\mu)\,\frac{\partial\mu}{\partial\sigma} = 0\,,$$

Because $G_\mu(\sigma,\mu) > 0$, this can be solved to obtain

$$\frac{\partial\mu}{\partial\sigma} = -\frac{G_\sigma(\sigma,\mu)}{G_\mu(\sigma,\mu)}\,.$$

(Here G_σ denotes the partial derivative of G with respect to σ.)

Therefore, if we assume that $G_\sigma(\sigma,\mu) < 0$ over the interior of the convex set $\Sigma \subset \mathbb{R}^2$ then $\mu(\sigma,\Gamma)$ is a strictly increasing function of σ.

Implicit Level Sets of the Objectives

Finally, by taking the second partial derivative of (3.10) with respect to σ, using the foregoing result, and again using the fact that $G_\mu(\sigma,\mu) > 0$, we find after some calculation that

$$\frac{\partial^2 \mu}{\partial \sigma^2} = -\frac{1}{G_\mu^3} \begin{pmatrix} G_\mu & -G_\sigma \end{pmatrix} \begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\sigma\mu} & G_{\mu\mu} \end{pmatrix} \begin{pmatrix} G_\mu \\ -G_\sigma \end{pmatrix} ,$$

where the (σ,μ) arguments of all the functions have been suppressed. (Here $G_{\sigma\sigma}$, $G_{\sigma\mu}$, $G_{\mu\mu}$, denote the various second-order partial derivatives of G with respect to σ and μ.)

Therefore if we assume that the right-hand side is positive over the interior of the convex set $\Sigma \subset \mathbb{R}^2$ then $\mu(\sigma,\Gamma)$ is a strictly convex function of σ.

Implicit Level Sets of the Objectives

In summary, if $G(\sigma,\mu)$ considered over the interior of the convex set Σ has the properties

$$G_\sigma < 0\,, \qquad G_\mu > 0\,, \tag{3.11a}$$

$$\begin{pmatrix} G_\mu & -G_\sigma \end{pmatrix} \begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} \begin{pmatrix} G_\mu \\ -G_\sigma \end{pmatrix} < 0\,, \tag{3.11b}$$

then the level sets of $G(\sigma,\mu)$ that lie within the convex set Σ are curves given by $\mu = \mu(\sigma,\Gamma)$ where $\mu(\sigma,\Gamma)$ is:

- a strictly increasing, strictly convex function of σ,
- a strictly increasing function of Γ.

Indeed, the functions $\mu_{\mathrm{p}}^{\chi}(\sigma,\Gamma)$, $\mu_{\mathrm{q}}^{\chi}(\sigma,\Gamma)$, and $\mu_{\mathrm{r}}^{\chi}(\sigma,\Gamma)$ that are given explicitly by (2.7), (2.8), and (2.9), have these properties.

Implicit Level Sets of the Objectives

For completeness, we now verify properties (3.11) for the parabolic, quadratic, reasonable, sensible, and Taylor estimators.

For the parabolic estimator we see from (1.2a) that

$$G(\sigma, \mu) = \mu - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,,$$

whereby

$$\begin{gathered} G_\sigma = -\sigma - \chi\,, \qquad G_\mu = 1\,, \\ G_{\sigma\sigma} = -1\,, \qquad G_{\sigma\mu} = 0\,, \qquad G_{\mu\mu} = 0\,. \end{gathered} \tag{3.12}$$

Hence, because $\chi \geq 0$, properties (3.11) hold for every (σ, μ) in the interior of Σ_{p} given by (1.3a).

Implicit Level Sets of the Objectives

For the quadratic estimator we see from (1.2b) that

$$G(\sigma, \mu) = \mu - \tfrac{1}{2}\mu^2 - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,,$$

whereby

$$\begin{gathered} G_\sigma = -\sigma - \chi\,, \qquad G_\mu = 1 - \mu\,, \\ G_{\sigma\sigma} = -1\,, \qquad G_{\sigma\mu} = 0\,, \qquad G_{\mu\mu} = -1\,. \end{gathered} \tag{3.13}$$

Hence, because $\chi \geq 0$, properties (3.11) hold for every (σ, μ) in the interior of Σ_{q} given by (1.3b).

Implicit Level Sets of the Objectives

For the reasonable estimator we see from (1.2c) that

$$G(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\sigma^2 - \chi\,\sigma\,,$$

whereby

$$\begin{gathered} G_\sigma = -\sigma - \chi\,, \qquad G_\mu = \frac{1}{1+\mu}\,, \\ G_{\sigma\sigma} = -1\,, \qquad G_{\sigma\mu} = 0\,, \qquad G_{\mu\mu} = -\frac{1}{(1+\mu)^2}\,. \end{gathered} \tag{3.14}$$

Hence, because $\chi \geq 0$, properties (3.14) hold for every (σ,μ) in the interior of Σ_{r} given by (1.3c).

Implicit Level Sets of the Objectives

For the sensible estimator we see from (1.2d) that

$$G(\sigma, \mu) = \log(1 + \mu) - \tfrac{1}{2}\frac{\sigma^2}{1 + 2\mu} - \chi\,\sigma\,,$$

whereby

$$\begin{aligned} G_\sigma &= -\frac{\sigma}{1+2\mu} - \chi\,, \qquad G_\mu = \frac{1}{1+\mu} + \frac{\sigma^2}{(1+2\mu)^2}\,, \\ G_{\sigma\sigma} &= -\frac{1}{1+2\mu}\,, \qquad G_{\sigma\mu} = \frac{2\sigma}{(1+2\mu)^2}\,, \\ G_{\mu\mu} &= -\frac{1}{(1+\mu)^2} - \frac{4\sigma^2}{(1+2\mu)^3}\,. \end{aligned} \tag{3.15}$$

Hence, because $\chi \geq 0$, properties (3.11) hold for every (σ, μ) in the interior of Σ_{s} given by (1.3d).

Implicit Level Sets of the Objectives

Because

$$G_{\mathrm{s}}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{1+2\mu} - \chi\,\sigma\,,$$

for every $\sigma > 0$ the mapping $\mu \mapsto G_{\mathrm{s}}^{\chi}(\sigma,\mu)$ is strictly increasing and maps the interval $(\frac{1}{2},\infty)$ onto $\mathbb{R}$. Therefore, for every $\sigma > 0$ and every $\Gamma \in \mathbb{R}$ there exists a unique $\mu_{\mathrm{s}}^{\chi}(\sigma,\Gamma) > -\frac{1}{2}$ such that

$$G_{\mathrm{s}}^{\chi}(\sigma,\mu_{\mathrm{s}}^{\chi}(\sigma,\Gamma)) = \Gamma\,.$$

Moreover, for $\sigma = 0$ and every $\Gamma > \log(\frac{1}{2})$ we have

$$\mu_{\mathrm{s}}^{\chi}(0,\Gamma) = \exp(\Gamma) - 1\,.$$

We thereby see that Σ_{s} is foilated by segments of the family of strictly increasing, strictly convex curves given by $\mu = \mu_{\mathrm{s}}^{\chi}(\sigma,\Gamma)$. These curves shift upward with increasing Γ.

Implicit Level Sets of the Objectives

For the Taylor estimator we see from (1.2e) that

$$G(\sigma, \mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,,$$

whereby

$$\begin{aligned} G_\sigma &= -\frac{\sigma}{(1+\mu)^2} - \chi\,, \qquad & G_\mu &= \frac{1}{1+\mu} + \frac{\sigma^2}{(1+\mu)^3}\,, \\ G_{\sigma\sigma} &= -\frac{1}{(1+\mu)^2}\,, \qquad & G_{\sigma\mu} &= \frac{2\sigma}{(1+\mu)^3}\,, \\ G_{\mu\mu} &= -\frac{1}{(1+\mu)^2} - \frac{3\sigma^2}{(1+\mu)^4}\,. \end{aligned} \tag{3.16}$$

Hence, because $\chi \geq 0$, properties (3.11) hold for every (σ, μ) in the interior of Σ_{t} given by (1.3e).

Implicit Level Sets of the Objectives

Because

$$G_{\mathrm{t}}^{\chi}(\sigma,\mu) = \log(1+\mu) - \tfrac{1}{2}\frac{\sigma^2}{(1+\mu)^2} - \chi\,\sigma\,,$$

for every $\sigma > 0$ the mapping $\mu \mapsto G_{\mathrm{t}}^{\chi}(\sigma,\mu)$ is strictly increasing and maps the interval $(\sigma - 1, \infty)$ onto $\mathbb{R}$. Therefore, for every $\sigma > 0$ and every $\Gamma \in \mathbb{R}$ there exists a unique $\mu_{\mathrm{t}}^{\chi}(\sigma,\Gamma) > \sigma - 1$ such that

$$G_{\mathrm{t}}^{\chi}(\sigma, \mu_{\mathrm{t}}^{\chi}(\sigma,\Gamma)) = \Gamma\,.$$

Moreover, for $\sigma = 0$ and every $\Gamma \in \mathbb{R}$ we have

$$\mu_{\mathrm{t}}^{\chi}(0,\Gamma) = \exp(\Gamma) - 1\,.$$

We thereby see that Σ_{t} is foilated by segments of the family of strictly increasing, strictly convex curves given by $\mu = \mu_{\mathrm{t}}^{\chi}(\sigma,\Gamma)$. These curves shift upward with increasing Γ.

Implicit Level Sets of the Objectives

Remark. Properties (3.11) are implied when $G(\sigma, \mu)$ considered over the interior of the convex set Σ has the properties

$$G_\sigma < 0\,, \qquad G_\mu > 0\,, \tag{3.17a}$$

$$\begin{pmatrix} G_{\sigma\sigma} & G_{\sigma\mu} \\ G_{\mu\sigma} & G_{\mu\mu} \end{pmatrix} \quad \text{is negative definite}\,. \tag{3.17b}$$

It is clear from (3.12) that the parabolic estimator does not have property (3.17b). However, it can be checked from (3.13), (3.14), (3.15), and (3.16) that the quadratic, reasonable, sensible, and Taylor estimators all have properties (3.17). Property (3.17b) implies that $G(\sigma, \mu)$ is a strictly concave function over the convex set Σ, which in turn implies that the mean-variance objective $\hat{\Gamma}(\mathbf{f})$ given by (1.1) is strictly concave over any class Π of portfolio allocations that is consistent with Σ in the sense (1.4).

Reduced Maximization Problem

Mean-variance objectives have the feature that they can be optimized by simply maximizing $G(\sigma, \mu)$ over the efficient frontier of Π in the $\sigma\mu$-plane. Recall that given any choice of Markowitz portfolio allocations Π its efficient frontier is a curve $\mu = \mu_{\mathrm{ef}}(\sigma)$ in the $\sigma\mu$-plane given by an increasing, concave, continuous, piecewise differentiable function $\mu_{\mathrm{ef}}(\sigma)$. The consistency condition (1.4) insures that the efficient frontier of Π lies within Σ.

The function $\mu_{\mathrm{ef}}(\sigma)$ is defined over the interval $[0, \infty)$ for the unlimited leverage, One Risk-Free Rate and Two Risk-Free Rates models, and is defined over some bounded interval $[0, \sigma_{\mathrm{mx}}]$ for every portfolio model with limited leverage. We define the function $\Gamma_{\mathrm{ef}}(\sigma)$ over this interval by

$$\Gamma_{\mathrm{ef}}(\sigma) = G(\sigma, \mu_{\mathrm{ef}}(\sigma)) .$$

Reduced Maximization Problem

Fact. If $G(\sigma, \mu)$ is a strictly decreasing function of σ and a strictly increasing function of μ over Σ then we have

$$\max\{ G(\hat{\sigma}(\mathbf{f}), \hat{\mu}(\mathbf{f})) \ : \ \mathbf{f} \in \Pi\} = \max\{\Gamma_{\mathrm{ef}}(\sigma) \ : \ \sigma \in [0, \sigma_{\mathrm{mx}}]\} \ .$$

Reason. Because frontier portfolios minimize $\hat{\sigma}$ for a given value of $\hat{\mu}$, and because $G(\hat{\mu}, \hat{\sigma})$ is a strictly decreasing function of $\hat{\sigma}$, the optimal $\mathbf{f}_*$ clearly must be a frontier portfolio. Because the optimal portfolio must also be more efficient than every other portfolio with the same volatility, because $G(\hat{\mu}, \hat{\sigma})$ is a strictly increasing function of $\hat{\mu}$, the optimal portfolio must lie on the efficient frontier.

Reduced Maximization Problem

This reduced maximization problem can be visualized by considering the family of level set curves in the $\sigma\mu$-plane parameterized by Γ as

$$G(\sigma, \mu) = \Gamma \,.$$

When $G(\sigma, \mu)$ has properties (3.11) then these curves are strictly increasing, strictly convex functions of σ. As Γ increases the curve shifts upward in the $\sigma\mu$-plane.

For some values of Γ the corresponding curve will intersect the efficient frontier, which is given by $\mu = \mu_{\text{ef}}(\sigma)$. There is clearly a maximum such Γ. As the level set curve is strictly convex while the efficient frontier is concave, for this maximum Γ the intersection will consist of a single point $(\sigma_{\text{opt}}, \mu_{\text{opt}})$. Then $\sigma = \sigma_{\text{opt}}$ is the maximizer of $\Gamma_{\text{ef}}(\sigma)$.

Reduced Maximization Problem

Remark. This reduction is appealing because the efficient frontier only depends on general information about an investor, like whether he or she will take short positions. Once it is computed, the problem of maximizing any given $\hat{\Gamma}(\mathbf{f})$ over all allocations $\mathbf{f}$ reduces to the problem of maximizing the associated $\Gamma_{\mathrm{ef}}(\sigma)$ over all admissible σ — a problem over one variable.

Remark. The maximum problem

$$\max\{\Gamma_{\mathrm{ef}}(\sigma) \,:\, \sigma \in [0, \sigma_{\mathrm{mx}}]\} \,.$$

is easy to solve numerically. We simply evaluate $G(\sigma, \mu)$ at the points (σ_k, μ_k) that were computed to find the efficient frontier numerically. The maximizer is the point (σ_k, μ_k) at which $G(\sigma_k, \mu_k)$ is largest.

Reduced Maximization Problem

Let us consider what might happen. Because $\mu_{\text{ef}}(\sigma)$ has a piecewise derivative, the function $\Gamma_{\text{ef}}(\sigma)$ has the piecewise derivative

$$\Gamma_{\text{ef}}'(\sigma) = \partial_\mu G(\sigma, \mu_{\text{ef}}(\sigma))\, \mu_{\text{ef}}'(\sigma) + \partial_\sigma G(\sigma, \mu_{\text{ef}}(\sigma))\,.$$

Because $\mu_{\text{ef}}(\sigma)$ is concave, $\Gamma_{\text{ef}}'(\sigma)$ is strictly decreasing.

Reduced Maximization Problem

Because $\Gamma_{\rm ef}'(\sigma)$ is strictly decreasing, there are three possibilities.

- $\Gamma_{\rm ef}(\sigma)$ takes its maximum at $\sigma = 0$, the left endpoint of its interval of definition. This case arises whenever $\Gamma_{\rm ef}'(0) \leq 0$.
- $\Gamma_{\rm ef}(\sigma)$ takes its maximum in the interior of its interval of definition at the unique point $\sigma = \sigma_{\rm opt}$ where $\Gamma_{\rm ef}'(\sigma)$ changes sign. This case arises for the unlimited leverage models whenever $\Gamma_{\rm ef}'(0) > 0$, and for a limited leverage portfolio model whenever $\Gamma_{\rm ef}'(\sigma_{\rm mx}) < 0 < \Gamma_{\rm ef}'(0)$.
- $\Gamma_{\rm ef}(\sigma)$ takes its maximum at $\sigma = \sigma_{\rm mx}$, the right endpoint of its interval of definition. This case arises only for limited leverage portfolio models whenever $\Gamma_{\rm ef}'(\sigma_{\rm mx}) \geq 0$.

Reduced Maximization Problem

In summary, our approach to portfolio selection has six steps:

1. Choose a return rate history for some set of risky assets.
2. Calibrate its mean vector $\mathbf{m}$ and covariance matrix $\mathbf{V}$.
3. Given $\mathbf{m}$, $\mathbf{V}$, μ_{si}, μ_{cl}, and any portfolio constraints, compute $\mu_{\mathrm{ef}}(\sigma)$.
4. Choose a mean-variance objective specificed by some $G(\sigma, \mu)$.
5. Find the maximizer σ_{opt} of the function $\Gamma_{\mathrm{ef}}(\sigma) = G(\sigma, \mu_{\mathrm{ef}}(\sigma))$.
6. Evaluate the unique efficient frontier portfolio allocation $\mathbf{f}_{\mathrm{ef}}(\sigma_{\mathrm{opt}})$.

The third step is the most computationally intensive for most choices of portfolio constraints. This step is simplest for unlimited leverage portfolios with a single risk-free rate model. In that case $\mu_{\mathrm{ef}}(\sigma) = \mu_{\mathrm{rf}} + \nu_{\mathrm{tg}}\sigma$, where ν_{tg} is the Sharpe ratio.