Portfolios that Contain Risky Assets 12: Growth Rates

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Portfolios that Contain Risky Assets
Part II: Stochastic Models

11. Independent, Identically-Distributed Models for Assets
12. Growth Rates
13. Independent, Identically-Distributed Models for Portfolios
14. Kelly Objectives for Markowitz Portfolios
15. Cautious Objectives for Markowitz Portfolios
16. Optimization of Mean-Variance Objectives
17. Fortune's Formulas
18. Utility Function Objectives

Growth Rates

1. Growth Rate Probability Densities
2. Normal Growth Rate Model
3. Moment and Cumulant Generating Functions
4. Estimators from Moment Generating Functions
5. Estimators from Cumulant Generating Functions
6. Interpolation Errors

Growth Rate Probability Densities

Given D samples $\{R_d\}_{d=1}^{D}$ that are drawn from the return probability density $q(R)$, the associated simulated share prices satisfy

$$S_d = (1 + R_d)\, S_{d-1}\,, \quad \text{for } d = 1, \cdots, D\,. \tag{1.1}$$

If we set $S_0 = s(0)$ then you can easily see that

$$S_d = \prod_{d'=1}^{d} (1 + R_{d'})\, s(0)\,. \tag{1.2}$$

Growth Rate Probability Densities

The *growth rate* X_d is related to the return R_d by

$$e^{X_d} = 1 + R_d\,. \tag{1.3}$$

In other words, X_d is the growth rate that yeilds a return R_d on trading day d. The formula for S_d then takes the form

$$S_d = \exp\left(\sum_{d'=1}^{d} X_{d'}\right) s(0)\,. \tag{1.4}$$

Growth Rate Probability Densities

If the samples $\{R_d\}_{d=1}^{D}$ *are drawn from a density* $q(R)$ *over* $(-1, \infty)$ *then the* $\{X_d\}_{d=1}^{D}$ *are drawn from a density* $p(X)$ *over* $(-\infty, \infty)$ *where*

$$p(X)\,\mathrm{d}X = q(R)\,\mathrm{d}R\,,$$

with X *and* R *related by*

$$X = \log(1+R)\,, \qquad R = e^X - 1\,.$$

More explicitly, the densities $p(X)$ and $q(R)$ are related by

$$p(X) = q\left(e^X - 1\right) e^X\,, \qquad q(R) = \frac{p(\log(1+R))}{1+R}\,.$$

Growth Rate Probability Densities

Because our models will involve means and variances, we will require that

$$\int_{-\infty}^{\infty} X^2 p(X)\,\mathrm{d}X = \int_{-1}^{\infty} \log(1+R)^2\, q(R)\,\mathrm{d}R < \infty\,,$$
$$\int_{-\infty}^{\infty} \left(e^X - 1\right)^2 p(X)\,\mathrm{d}X = \int_{-1}^{\infty} R^2 q(R)\,\mathrm{d}R < \infty\,.$$

Then the mean γ and variance θ of X are

$$\gamma = \mathrm{Ex}(X) = \int_{-\infty}^{\infty} X\, p(X)\,\mathrm{d}X\,,$$
$$\theta = \mathrm{Var}(X) = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \int_{-\infty}^{\infty} (X-\gamma)^2\, p(X)\,\mathrm{d}X\,.$$

Growth Rate Probability Densities

The big advantage of working with $p(X)$ rather than $q(R)$ is the fact that

$$\log\left(\frac{S_d}{s(0)}\right) = \sum_{d'=1}^{d} X_{d'} .$$

In other words, $\log(S_d/s(0))$ *is a sum of an IID process.* It is easy to compute the mean and variance of this quantity in terms of those of X.

For the mean of $\log(S_d/s(0))$ we find that

$$\mathrm{Ex}\left(\log\left(\frac{S_d}{s(0)}\right)\right) = \sum_{d'=1}^{d} \mathrm{Ex}(X_{d'}) = d\,\gamma ,$$

Growth Rate Probability Densities

For the variance of $\log(S_d/s(0))$ we find that

$$
\begin{aligned}
\mathrm{Var}\left(\log\left(\frac{S_d}{s(0)}\right)\right) &= \mathrm{Ex}\left(\left(\sum_{d'=1}^{d} X_{d'} - d\,\gamma\right)^2\right) \\
&= \mathrm{Ex}\left(\left(\sum_{d'=1}^{d} (X_{d'} - \gamma)\right)^2\right) \\
&= \mathrm{Ex}\left(\sum_{d'=1}^{d}\sum_{d''=1}^{d} (X_{d'} - \gamma)\,(X_{d''} - \gamma)\right) \\
&= \sum_{d'=1}^{d} \mathrm{Ex}\Big((X_{d'} - \gamma)^2\Big) = d\,\theta\,.
\end{aligned}
$$

Growth Rate Probability Densities

Remark. The off-diagonal terms in the foregoing double sum vanish because

$$\mathrm{Ex}\Big((X_{d'} - \gamma)\,(X_{d''} - \gamma)\Big) = 0 \qquad \text{when } d'' \neq d' .$$

Hence, the growth mean and variance of the IID model asset at day d is

$$\mathrm{Ex}\left(\log\left(\frac{S_d}{s(0)}\right)\right) = \gamma\, d\,, \qquad \mathrm{Var}\left(\log\left(\frac{S_d}{s(0)}\right)\right) = \theta\, d\,.$$

Growth Rate Probability Densities

Remark. *The IID model suggests that the growth rate mean γ is a good proxy for the reward of an asset and that $\sqrt{\theta}$ is a good proxy for its risk. However, these are not the proxies chosen by MPT when it is applied to a portfolio consisting of one risky asset.*

The proxies γ and $\sqrt{\theta}$ can be approximated by $\hat{\gamma}$ and $\sqrt{\hat{\theta}}$ where $\hat{\gamma}$ and $\hat{\theta}$ are the unbiased estimators of γ and θ given by

$$\hat{\gamma} = \sum_{d=1}^{D} w_d \, X_d \,, \qquad \hat{\theta} = \sum_{d=1}^{D} \frac{w_d}{1 - \bar{w}} \left(X_d - \hat{\gamma} \right)^2 .$$

Normal Growth Rate Model

We can illustrate what is going on with the simple IID model where $p(X)$ is the *normal* or *Gaussian* density with mean γ and variance θ, which is given by

$$p(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

Let $\{X(d)\}_{d=1}^{\infty}$ be a sequence of IID random variables drawn from $p(X)$. Let $\{Y_d\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y_d = \frac{1}{d}\sum_{d'=1}^{d} X_{d'} \qquad \text{for every } d = 1, \cdots, \infty .$$

Normal Growth Rate Model

We can easily check that

$$\mathrm{Ex}(Y_d) = \gamma\,, \qquad \mathrm{Var}(Y_d) = \frac{\theta}{d}\,.$$

We can also check that

$$\mathrm{Ex}(Y_d | Y_{d-1}) = \frac{d-1}{d}\, Y_{d-1} + \frac{1}{d}\,\gamma\,.$$

So the variables Y_d are neither independent nor identically distributed.

It can be shown (the details are not given here) that Y_d is drawn from the normal density with mean γ and variance θ/d, which is given by

$$p_d(Y) = \sqrt{\frac{d}{2\pi\theta}}\,\exp\left(-\frac{(Y-\gamma)^2 d}{2\theta}\right)\,.$$

Normal Growth Rate Model

Because $S_d/s(0) = e^{d\,Y_d}$, the mean return at day d is

$$\mathrm{Ex}\left(e^{d\,Y_d}\right) = \sqrt{\frac{d}{2\pi\theta}} \int \exp\left(-\frac{(Y-\gamma)^2 d}{2\theta} + d\,Y\right) \mathrm{d}Y$$
$$= \sqrt{\frac{d}{2\pi\theta}} \int \exp\left(-\frac{(Y-\gamma-\theta)^2 d}{2\theta} + d(\gamma + \tfrac{1}{2}\theta)\right) \mathrm{d}Y$$
$$= \exp\left(d(\gamma + \tfrac{1}{2}\theta)\right) .$$

Because $p_d(Y)$ becomes sharply peaked around $Y = \gamma$ as d increases, most investors will see the lower growth rate γ rather than $\gamma + \frac{1}{2}\theta$.

By setting $d = 1$ in the above formula, we see that the return mean is

$$\mu = \mathrm{Ex}(R) = \mathrm{Ex}\left(e^X - 1\right) = \exp(\gamma + \tfrac{1}{2}\theta) - 1 .$$

Hence, $\mu > \gamma + \frac{1}{2}\theta$, with $\mu \approx \gamma + \frac{1}{2}\theta$ when $(\gamma + \frac{1}{2}\theta) << 1$.

Normal Growth Rate Model

Therefore most investors will see a return that is below the return mean μ — far below in volatile markets. This is because e^X amplifies the tail of the normal density. For a more realistic IID model with a density $p(X)$ that decays more slowly than a normal density as $X \to \infty$, this difference can be more striking. Said another way, most investors will not see the same return as Warren Buffett, but his return will boost the mean.

The normal growth rate model confirms that γ is a better proxy for how well a risky asset might perform than μ because $p_d(Y)$ becomes more peaked around $Y = \gamma$ as d increases. We will extend this result to a general class of IID models that are more realistic.

Moment and Cumulant Generating Functions

Estimators for γ and θ will be constructed from the positive function

$$M(\tau) = \mathrm{Ex}\Big(e^{\tau X}\Big) = \int e^{\tau X}\, p_{\mathsf{f}}(X)\,\mathrm{d}X\,.$$

We will assume $M(\tau)$ is defined for every τ in an open interval $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ that contains the interval $[0, 2]$. It can then be shown that $M(\tau)$ is infinitely differentiable over $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ with

$$M^{(m)}(\tau) = \mathrm{Ex}\Big(X^m\, e^{\tau X}\Big) = \int X^m\, e^{\tau X}\, p_{\mathsf{f}}(X)\,\mathrm{d}X\,.$$

We call $M(\tau)$ the *moment generating function* for X because, by setting $\tau = 0$ in the above expression, we see that the *moments* $\{\mathrm{Ex}(X^m)\}_{m=1}^{\infty}$ are generated from $M(\tau)$ by the formula

$$\mathrm{Ex}(X^m) = \int X^m p_{\mathsf{f}}(X)\,\mathrm{d}X = M^{(m)}(0)\,.$$

Moment and Cumulant Generating Functions

A related inifinitely differentiable function over $(\tau_{\rm mn}, \tau_{\rm mx})$ is

$$K(\tau) = \log(M(\tau)) = \log\Big(\mathrm{Ex}\Big(e^{\tau X}\Big)\Big) .$$

We call $K(\tau)$ the *cumulant generating function* because the *cumulants* $\{\kappa_m\}_{m=1}^{\infty}$ of X are generated by the formula $\kappa_m = K^{(m)}(0)$. We see that

$$K'(\tau) = \frac{\mathrm{Ex}\big(X\, e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} ,$$

$$K''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^2 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} ,$$

$$K'''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^3 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} ,$$

$$K''''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^4 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} - 3K''(\tau)^2 .$$

Moment and Cumulant Generating Functions

By evaluating these at $\tau = 0$ we see that the first four cumulants of X are

$$
\begin{aligned}
\kappa_1 &= K'(0) = \mathrm{Ex}(X) = \gamma \,, \\
\kappa_2 &= K''(0) = \mathrm{Ex}\big((X-\gamma)^2\big) = \theta \,, \\
\kappa_3 &= K'''(0) = \mathrm{Ex}\big((X-\gamma)^3\big) \,, \\
\kappa_4 &= K''''(0) = \mathrm{Ex}\big((X-\gamma)^4\big) - 3\theta^2 \,.
\end{aligned}
$$

These are respectively the mean, variance, skewness, and kurtosis.

Skewness measures an asymmetry in the tails of the distribution. It is positive or negative depending on whether the fatter tail is to the right or to the left respectively.

Kurtosis measures a balance between the tails and the center of the distribution. It is larger for distributions with greater weight in the tails than in the center.

Moment and Cumulant Generating Functions

Remark. The formulas

$$K'(\tau) = \frac{\mathrm{Ex}\left(X\, e^{\tau X}\right)}{\mathrm{Ex}\left(e^{\tau X}\right)}\,,$$

$$K''(\tau) = \frac{\mathrm{Ex}\left((X - K'(\tau))^2 e^{\tau X}\right)}{\mathrm{Ex}\left(e^{\tau X}\right)}\,,$$

$$K'''(\tau) = \frac{\mathrm{Ex}\left((X - K'(\tau))^3 e^{\tau X}\right)}{\mathrm{Ex}\left(e^{\tau X}\right)}\,,$$

$$K''''(\tau) = \frac{\mathrm{Ex}\left((X - K'(\tau))^4 e^{\tau X}\right)}{\mathrm{Ex}\left(e^{\tau X}\right)} - 3K''(\tau)^2\,,$$

show that $K'(\tau)$, $K''(\tau)$, $K'''(\tau)$, and $K''''(\tau)$ are the mean, variance, skewness, and kurtosis for the probability density $e^{\tau X} p_{\mathbf{f}}(X)/\mathrm{Ex}(e^{\tau X})$.

Moment and Cumulant Generating Functions

Remark. If X is normally distributed with mean γ and variance θ then

$$p_{\mathsf{f}}(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

A direct calculation then shows that

$$\begin{aligned} \mathrm{Ex}\left(e^{\tau X}\right) &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma)^2}{2\theta} + \tau X\right) \mathrm{d}X \\ &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma-\tau\theta)^2}{2\theta} + \tau\gamma + \tfrac{1}{2}\tau^2\theta\right) \mathrm{d}X \\ &= \exp\left(\tau\gamma + \tfrac{1}{2}\tau^2\theta\right) , \end{aligned}$$

whereby $K(\tau) = \log(\mathrm{Ex}(e^{\tau X})) = \tau\gamma + \frac{1}{2}\tau^2\theta$. *Hence, when X is normally distributed the skewness, kurtosis, and all higher-order cumulants vanish. Conversely, if all of these cumulants vanish then X is normally distributed.*

Moment and Cumulant Generating Functions

Remark. The cumulant generating function $K(\tau)$ is *strictly convex* over the interval $(\tau_{\rm mn}, \tau_{\rm mx})$ because $K''(\tau) > 0$.

Remark. We can also see that $K(\tau)$ is convex over $(\tau_{\rm mn}, \tau_{\rm mx})$ as follows. Let τ_0, $\tau_1 \in (\tau_{\rm mn}, \tau_{\rm mx})$. By applying the *Hölder inequality* with $p = \frac{1}{1-s}$ and $p^* = \frac{1}{s}$, we see that for every $s \in (0,1)$ we have

$$
\begin{aligned}
M((1-s)\tau_0 + s\tau_1)) &= \int e^{(1-s)\tau_0 X}\, e^{s\tau_1 X}\, p_{\mathbf{f}}(X)\, \mathrm{d}X \\
&\leq \left(\int e^{\tau_0 X} p_{\mathbf{f}}(X)\, \mathrm{d}X \right)^{1-s} \left(\int e^{\tau_1 X} p_{\mathbf{f}}(X)\, \mathrm{d}X \right)^{s} \\
&= M(\tau_0)^{1-s} M(\tau_1)^s \,.
\end{aligned}
$$

By taking the logarithm of this inequality we obtain

$$
K((1-s)\tau_0 + s\tau_1) \leq (1-s)K(\tau_0) + sK(\tau_1) \quad \text{for every } s \in (0,1)\,.
$$

Therefore $K(\tau)$ is a convex function over $(\tau_{\rm mn}, \tau_{\rm mx})$.

Estimators from Moment Generating Functions

We will now construct estimators for γ and θ by using the moment generating function

$$M(\tau) = \mathrm{Ex}\Big(e^{\tau X}\Big) \,.$$

Because $R = e^X - 1$ and $\mathrm{Ex}(e^X) = M(1)$, we have

$$\mu = \mathrm{Ex}(R) = M(1) - 1 \,.$$

Because $R - \mu = e^X - M(1)$ and $\mathrm{Ex}(e^{2X}) = M(2)$, we have

$$\xi = \mathrm{Ex}\Big((R-\mu)^2\Big) = M(2) - M(1)^2 \,.$$

These equations can be solved for $M(1)$ and $M(2)$ as

$$M(1) = 1 + \mu \,, \qquad M(2) = (1+\mu)^2 + \xi \,.$$

Therefore knowing μ and ξ is equivalent to knowing $M(1)$ and $M(2)$.

Estimators from Moment Generating Functions

Because $\mathrm{Ex}(X) = M'(0)$ and $\mathrm{Ex}(X^2) = M''(0)$, we see that

$$\begin{aligned} \gamma &= \mathrm{Ex}(X) = M'(0)\,, \\ \theta &= \mathrm{Ex}\big((X-\gamma)^2\big) \\ &= \mathrm{Ex}(X^2) - \gamma^2 = M''(0) - M'(0)^2\,. \end{aligned}$$

Because $M(0) = 1$, we construct an estimator of $M(\tau)$ by interpolating the values $M(0)$, $M(1)$, and $M(2)$ with a quadratic polynomial as

$$\begin{aligned} \widehat{M}(\tau) &= 1 + \tau\big(M(1) - 1\big) + \tau(\tau-1)\tfrac{1}{2}\big(M(2) - 2M(1) + 1\big) \\ &= 1 + \tau\mu + \tfrac{1}{2}\tau(\tau-1)\left(\mu^2 + \xi\right)\,. \end{aligned}$$

By direct calculation we see that

$$\widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,, \qquad \widehat{M}''(0) = \mu^2 + \xi\,.$$

Estimators from Moment Generating Functions

The idea is to now construct estimators for γ and θ by using

$$\widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,, \qquad \widehat{M}''(0) = \mu^2 + \xi\,, \tag{4.5}$$

as estimators for $M'(0)$ and $M''(0)$ in the formulas

$$\gamma = M'(0)\,, \qquad \theta = M''(0) - M'(0)^2\,.$$

We thereby construct estimators $\hat{\gamma}$ and $\hat{\theta}$ as functions of μ and ξ by

$$\begin{aligned}\hat{\gamma} &= \widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,,\\ \hat{\theta} &= \widehat{M}''(0) - \widehat{M}'(0)^2 = \mu^2 + \xi - \left(\mu - \tfrac{1}{2}(\mu^2 + \xi)\right)^2\,.\end{aligned}$$

Estimators from Moment Generating Functions

By replacing the μ and ξ that appear in the foregoing estimators with the estimators

$$\hat{\mu} = \mu_{\mathrm{rf}}(1 - \mathbf{1}^{\mathrm{T}}\mathbf{f}) + \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \qquad \hat{\xi} = \frac{1}{1-\bar{w}}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,. \tag{4.6a}$$

we obtain the estimators

$$\begin{aligned} \hat{\gamma} &= \hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right)\,, \\ \hat{\theta} &= \hat{\mu}^2 + \hat{\xi} - \left(\hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right)\right)^2\,, \end{aligned} \tag{4.6b}$$

The variance θ is generally positive, but the estimator $\hat{\theta}$ given above is not intrinsically positive.

Estimators from Moment Generating Functions

Expanding the above expression for $\hat{\theta}$ in powers of $\hat{\mu}$ and $\hat{\xi}$ yields

$$\hat{\theta} = \hat{\xi} + \hat{\mu}\left(\hat{\mu}^2 + \hat{\xi}\right) - \tfrac{1}{4}\left(\hat{\mu}^2 + \hat{\xi}\right)^2 .$$

The only term in this expansion that is intrinsically positive is the first one. *Therefore we make the smallness assumptions*

$$|\hat{\mu}| \ll 1 , \qquad \hat{\xi} \ll 1 , \qquad |\hat{\mu}|^3 \ll \xi ,$$

and keep only through quadratic statistics — i.e. through quadratic in $\hat{\mu}$ and linear in $\hat{\xi}$. We thereby arrive at the *quadratic estimators*

$$\hat{\gamma} = \hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right) , \qquad \hat{\theta} = \hat{\xi} , \tag{4.7}$$

where $\hat{\mu}$ and $\hat{\xi}$ are given by (4.6a).

Remark. These smallness assumptions are very easy to check.

Estimators from Moment Generating Functions

Remark. The quadratic estimators $\hat{\gamma}$ and $\hat{\theta}$ given by (4.7) have at least three potential sources of error:

- the estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$ used in (4.5) to approximate γ and θ as functions of μ and ξ,
- the estimators $\hat{\mu}$ and $\hat{\xi}$ used in (4.6a) to approximate μ and ξ,
- the smallness assumptions that lead to (4.7).

The derivation of the first estimators assumes that the returns for each Markowitz portfolio are described by a density $q_{\mathbf{f}}(\mathbf{R})$ that is narrow enough for some moment beyond the second to exist. All of these approximations should be examined carefully, especially when markets are highly volatile.

Estimators from Cumulant Generating Functions

We will now give an alternative derivation of quadratic estimators (4.7) that uses the cumulent generating function $K(\tau) = \log(M(\tau))$ and is based on the fact that $\gamma = K'(0)$ and $\theta = K''(0)$. It begins by observing that

$$K(1) = \log\big(M(1)\big) = \log(1+\mu)\,,$$
$$K(2) = \log\big(M(2)\big) = \log\Big((1+\mu)^2 + \xi\Big)\,.$$

Therefore knowing μ and ξ is equivalent to knowing $K(1)$ and $K(2)$.

Because $K(0) = 0$, we construct an estimator of $K(\tau)$ by interpolating the values $K(0)$, $K(1)$, and $K(2)$ with a quadratic polynomial as

$$\begin{aligned}\hat{K}(\tau) &= \tau K(1) + \tau(\tau-1)\tfrac{1}{2}\big(K(2) - 2K(1)\big) \\ &= \tau \log(1+\mu) + \tau(\tau-1)\tfrac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right)\,.\end{aligned}$$

Estimators from Cumulant Generating Functions

This yields the estimators

$$\hat{\gamma} = \hat{K}'(0) = \log(1+\mu) - \tfrac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right),$$

$$\hat{\theta} = \hat{K}''(0) = \log\left(1 + \frac{\xi}{(1+\mu)^2}\right).$$

By replacing the μ and ξ that appear above with the estimators $\hat{\mu}$ and $\hat{\xi}$ given by (4.6a), we obtain the new estimators

$$\hat{\gamma} = \log(1+\hat{\mu}) - \tfrac{1}{2}\log\left(1 + \frac{\hat{\xi}}{(1+\hat{\mu})^2}\right),$$

$$\hat{\theta} = \log\left(1 + \frac{\hat{\xi}}{(1+\hat{\mu})^2}\right).$$

So long as $1 + \hat{\mu} > 0$ these estimators are well defined and $\hat{\theta}$ is positive.

Estimators from Cumulant Generating Functions

If $1 + \hat{\mu} > 0$ and we make the smallness assumption

$$\frac{\hat{\xi}}{(1+\hat{\mu})^2} \ll 1\,,$$

then we obtain the estimators

$$\hat{\gamma} = \log(1+\hat{\mu}) - \tfrac{1}{2}\frac{\hat{\xi}}{(1+\hat{\mu})^2}\,, \qquad \hat{\theta} = \frac{\hat{\xi}}{(1+\hat{\mu})^2}\,. \tag{5.8}$$

Finally, if we make the additional smallness assumptions

$$|\hat{\mu}| \ll 1\,, \qquad |\hat{\mu}|^3 \ll \hat{\xi}\,,$$

use the fact

$$\log(1+\hat{\mu}) = \hat{\mu} - \tfrac{1}{2}\hat{\mu}^2 + \tfrac{1}{3}\hat{\mu}^3 + \cdots\,,$$

and keep only through quadratic statistics then we obtain the *quadratic estimators* (4.7) derived earlier.

Estimators from Cumulant Generating Functions

Remark. *The fact that both derivations lead to the same estimators gives us greater confidence in the validity the quadratic estimators.*

Remark. If the Markowitz portfolio specified by **f** has growth rates X that are normally distributed with mean γ and variance θ then we have seen that $K(\tau) = \tau\gamma + \frac{1}{2}\tau^2\theta$. In this case we have $\hat{K}(\tau) = K(\tau)$, so the estimators $\hat{\gamma} = \hat{K}'(0)$ and $\hat{\theta} = \hat{K}''(0)$ are exact.

Remark. The biggest uncertainty associated with these estimators for $\hat{\gamma}$ and $\hat{\theta}$ is usually the uncertainty inherited from the estimators for $\hat{\mu}$ and $\hat{\xi}$.

Estimators from Cumulant Generating Functions

Exercise. When the quadratic estimators $\hat{\gamma}$ and $\hat{\theta}$ are applied to a single risky asset, they reduce to

$$\hat{\gamma} = \hat{\mu} - \tfrac{1}{2}\big(\hat{\mu}^2 + \hat{\xi}\big)\,, \qquad \hat{\theta} = \hat{\xi}\,.$$

Use these to estimate γ and θ for each of the following assets given the share price history $\{s(d)\}_{d=0}^{D}$. How do these $\hat{\gamma}$ and $\hat{\theta}$ compare with the unbiased estimators for γ and θ that you obtained in the previous problem?

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Exercise. Compute $\hat{\gamma}$ and $\hat{\theta}$ based on daily data for the Markowitz portfolio with value equally distributed among the assets in each of the groups given in the previous exercise.

Interpolation Errors

Here we examine the errors of the interpolation-based estimators given by

$$\widehat{M}'(0) = 2\big(M(1) - 1\big) - \tfrac{1}{2}\big(M(2) - 1\big)\,,$$
$$\widehat{M}''(0) = M(2) - 2M(1) + 1\,.$$

Let $M(\tau)$ be any thrice continuously differentiable function over $[0, 2]$ that satisfies $M(0) = 1$. The Cauchy form of the Taylor remainder then yields

$$M(1) = 1 + M'(0) + \tfrac{1}{2}M''(0) + \tfrac{1}{2}\int_0^1 (1-s)^2 M'''(s)\,\mathrm{d}s\,,$$
$$M(2) = 1 + 2M'(0) + 2M''(0) + \tfrac{1}{2}\int_0^2 (2-s)^2 M'''(s)\,\mathrm{d}s\,.$$

By placing these into the above formulas for $\widehat{M}'(0)$ and $\widehat{M}''(0)$ we obtain

$$\widehat{M}'(0) = M'(0) + E_1\,, \qquad \widehat{M}''(0) = M''(0) + E_2\,,$$

Interpolation Errors

where the errors E_1 and E_2 are given by

$$E_1 = \left[\int_0^1 (1-s)^2 M'''(s)\,\mathrm{d}s - \tfrac{1}{4}\int_0^2 (2-s)^2 M'''(s)\,\mathrm{d}s \right]$$
$$= -\left[\int_0^1 \left(s - \tfrac{3}{4}s^2\right) M'''(s)\,\mathrm{d}s + \tfrac{1}{4}\int_1^2 (2-s)^2 M'''(s)\,\mathrm{d}s \right],$$
$$E_2 = \left[\tfrac{1}{2}\int_0^2 (2-s)^2 M'''(s)\,\mathrm{d}s - \int_0^1 (1-s)^2 M'''(s)\,\mathrm{d}s \right]$$
$$= \left[\tfrac{1}{2}\int_1^2 (2-s)^2 M'''(s)\,\mathrm{d}s + \int_0^1 \left(1 - \tfrac{1}{2}s^2\right) M'''(s)\,\mathrm{d}s \right].$$

Here the integrals seen in the second expression for each error are written so that the factor multiplying $M'''(s)$ inside each integral is nonnegative. This shows that if $M'''(s) \geq 0$ over $[0,2]$ then $E_1 < 0$ and $E_2 > 0$, while if $M'''(s) \leq 0$ over $[0,2]$ then $E_1 > 0$ and $E_2 < 0$.

Interpolation Errors

The errors E_1 and E_2 may be bounded in terms of

$$\|M'''\|_\infty = \max\left\{|M'''(\tau)| \,:\, \tau \in [0,2]\right\}.$$

Specifically, because

$$\int_0^1 \left(s - \tfrac{3}{4}s^2\right) \mathrm{d}s = \tfrac{1}{4}\,, \qquad \int_1^2 (2-s)^2 \, \mathrm{d}s = \tfrac{1}{3}\,,$$

$$\int_0^1 \left(1 - \tfrac{1}{2}s^2\right) \mathrm{d}s = \tfrac{5}{6}\,,$$

we obtain the bounds

$$|E_1| \leq \tfrac{1}{3}\|M'''\|_\infty\,, \qquad |E_2| \leq \|M'''\|_\infty\,.$$

Interpolation Errors

If we want to use these error bounds then we must find either a bound of or an approximation to $\|M'''\|_\infty$. From the definition of $M(\tau)$ we see that

$$M'''(\tau) = \mathrm{Ex}\big(X^3 e^{\tau X}\big) = \int X^3 e^{\tau X} p_{\mathbf{f}}(X)\,\mathrm{d}X\,.$$

Because

$$M''''(\tau) = \mathrm{Ex}\big(X^4 e^{\tau X}\big) = \int X^4 e^{\tau X} p_{\mathbf{f}}(X)\,\mathrm{d}X > 0\,,$$

we see that $M'''(\tau)$ is a strictly increasing function of τ.

Interpolation Errors

Because $M'''(\tau)$ is a strictly increasing function of τ we have

$$\|M'''\|_\infty = \max\{\, -M'''(0)\,,\; M'''(2)\}\,,$$

where the quantities $M'''(0)$ and $M'''(2)$ can be expressed in terms of the return density as

$$M'''(0) = \int_{-1}^{\infty} \big(\log(1+R)\big)^3\, q_{\mathrm{f}}(R)\,\mathrm{d}R\,,$$

$$M'''(2) = \int_{-1}^{\infty} \big(\log(1+R)\big)^3 (1+R)^2 q_{\mathrm{f}}(R)\,\mathrm{d}R\,.$$

Interpolation Errors

These quantities can be approximated by the sample means

$$\widetilde{M'''}(0) = \sum_{d=1}^{D} w(d)\,\big(\log(1 + r(d))\big)^3 \,,$$

$$\widetilde{M'''}(2) = \sum_{d=1}^{D} w(d)\,\big(\log(1 + r(d))\big)^3\,\big(1 + r(d)\big)^2 \,,$$

where $\{r(d)\}_{d=1}^{D}$ is the portfolio return history given by

$$r(d) = (1 - \mathbf{1}^{\mathrm{T}}\mathbf{f})\mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\,.$$

By arguing as we did for $M'''(\tau)$, we can show that $\widetilde{M'''}(0) < \widetilde{M'''}(2)$. Therefore we can approximate $\|M'''\|_\infty$ by

$$\|M'''\|_\infty \approx \max\{\, -\widetilde{M'''}(0)\,,\ \widetilde{M'''}(2)\}\,.$$