Portfolios that Contain Risky Assets 10: Bounded Portfolios and Leverage Limits

C. David Levermore

University of Maryland, College Park, MD

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Portfolios that Contain Risky Assets
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Bounded Portfolios and Leverage Limits

Introduction

Recall the return $r(d)$ of a portfolio for day d is

$$r(d) = \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} \,,$$

where $\pi(d)$ is the value of the portfolio at the close of day d. Notice that $r(d) > 0$ when $\pi(d) < \pi(d-1) < 0$, which shows that return is a terrible notion of reward when $\pi(d-1) < 0$! Therefore we require that

$$\pi(d) > 0 \qquad \text{for every } d = 0, \cdots, D\,. \tag{1.1}$$

In other words, we require that portfolios stay *solvent* over a given return history $\{\mathbf{r}(d)\}_{d=1}^{D}$. We will characterize solvent portfolios. We will then characterize the allocations of solvent Markowitz portfolios. Finally, we will introduce subsets of solvent Markowitz portfolios that can play a role in risk management.

Price Ratios of Assets

Given a share price history $\{s_i(d)\}_{d=0}^{D}$ for N risky assets that are indexed by $i = 1, \cdots, N$, we define the *price ratio* history $\{\rho_i(d)\}_{d=0}^{D}$ by

$$\rho_i(d) = \frac{s_i(d)}{s_i(d-1)} \quad \text{for every } i = 1, \cdots, N \text{ and } d = 1, \cdots, D\,.$$

Because each share price is positive, every price ratio is positive. Because returns $r_i(d)$ were defined by

$$r_i(d) = \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \frac{s_i(d)}{s_i(d-1)} - 1\,,$$

we see that price ratios are related to returns by

$$\rho_i(d) = 1 + r_i(d) \quad \text{for every } i = 1, \cdots, N \text{ and } d = 1, \cdots, D\,. \tag{2.2}$$

Because share prices typically do not change much on any trading day, most price ratios will be close to 1.

Characterization of Solvent Portfolios

We can characterize solvent portfolios in terms of their *value ratios*. If a portfolio has value history $\{\pi(d)\}_{d=0}^{D}$ then its value ratio on trading day d is

$$\rho(d) = \frac{\pi(d)}{\pi(d-1)} \qquad \text{for every } d = 1, \cdots, D\,. \tag{3.3}$$

The value ratio is undefined whenever there is a division by zero.

Remark. The value of a portfolio that holds short positions can become negative, and is more likely to get close to zero than a portfolio that holds only long positions. The value of such portfolios can change significantly from day to day. Therefore their value ratios may not be close to 1. This is very different than the price ratio of most individual assets.

Characterization of Solvent Portfolios

Fact 1. A portfolio will be solvent over a given return history if and only if

$$\pi(0) > 0 \qquad \text{and} \qquad \rho(d) > 0 \quad \text{for every } d = 1, \cdots, D\,. \tag{3.4}$$

Proof. If a portfolio is solvent then by definition (1.1) we know that $\pi(0) > 0$ and that $\pi(d) > 0$ and $\pi(d-1) > 0$ for every $d = 1, \cdots, D$. It follows by (3.3) that $\rho(d) > 0$ for every $d = 1, \cdots, D$. Therefore (3.4) holds.

Conversely, suppose that (3.4) holds. It follows from (2.2) by induction that

$$\pi(d) = \pi(0) \prod_{d'=1}^{d} \rho(d') \qquad \text{for every } d = 1, \cdots, D\,.$$

Then (3.4) implies that $\pi(d) > 0$ for every $d = 0, \cdots, D$. Therefore the portfolio is solvent by definition (1.1). □

Characterization of Solvent Portfolios

Remark. Because portfolio returns $r(d)$ were defined by

$$r(d) = \frac{\pi(d) - \pi(d-1)}{\pi(d-1)} = \frac{\pi(d)}{\pi(d-1)} - 1\,,$$

we see that portfolio value ratios are related to portfolio returns by

$$\rho(d) = 1 + r(d) \qquad \text{for every } d = 1, \cdots, D\,.$$

Therefore characterization (3.4) of solvent portfolios can be recast as

$$\pi(0) > 0 \qquad \text{and} \qquad 1 + r(d) > 0 \quad \text{for every } d = 1, \cdots, D\,. \tag{3.5}$$

We will only consider portfolios with $\pi(0) > 0$. Therefore a portfolio will be solvent if and only if the second solvency condition in (3.5) is satisfied.

Solvent Markowitz Portfolios

We now specialize the notion of solvent portfolios to Markowitz portfolios. This set of portfolios will contain most portfolio models that we will develop. We begin by characterizing the set of allocations for solvent Markowitz portfolios.

The returns for the Markowitz portfolio with allocation $\mathbf{f}$ are $r(d) = \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$, where $\{\mathbf{r}(d)\}_{d=1}^{D}$ is the return history for the individual assets. Then the characterization given by the second solvency condition in (3.5) becomes

$$1 + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d) > 0 \qquad \text{for every } d = 1, \cdots, D\,. \tag{4.6}$$

These D inequality constraints along with the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ must be satisfied by the allocation $\mathbf{f}$ of a Markowitz portfolio for that portfolio to be solvent. Therefore the set of such allocations Ω is given by

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ 1 + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d) > 0 \ \ \forall d \right\}. \tag{4.7}$$

Solvent Markowitz Portfolios

The set of allocations for solvent Markowitz portfolios is best expressed in terms of the N-vector of price ratios on day d, which is

$$\boldsymbol{\rho}(d) = \begin{pmatrix} \rho_1(d) \\ \vdots \\ \rho_N(d) \end{pmatrix} .$$

It follows from (2.2) that $\boldsymbol{\rho}(d) = \mathbf{1} + \mathbf{r}(d)$. Because Markowitz portfolios are specified by allocations $\mathbf{f}$ that satisfy $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$, we see that

$$\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} = \left(\mathbf{1} + \mathbf{r}(d)\right)^{\mathrm{T}}\mathbf{f} = \mathbf{1}^{\mathrm{T}}\mathbf{f} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} = 1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} .$$

Therefore set of allocations for solvent Markowitz portfolios Ω that was defined by (4.7) can be expressed as

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \,,\; 0 < \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \quad \forall d \right\} . \tag{4.8}$$

Solvent Markowitz Portfolios

Remark. Expression (4.8) gives a geometric way to think about the set Ω. It shows that Ω is the intersection of the hyperplane $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ with the half spaces

$$\rho(d)^{\mathrm{T}}\mathbf{f} > 0\,, \qquad \text{for } d = 1, \cdots, D\,.$$

Because the entries of $\rho(d)$ are usually close to 1, the collection of vectors $\{\rho(d)\}_{d=1}^{D}$ can be thought of as a cloud of vectors clustered around the vector $\mathbf{1}$. The boundary of the half space $\rho(d)^{\mathrm{T}}\mathbf{f} > 0$ is the hyperplane $\rho(d)^{\mathrm{T}}\mathbf{f} = 0$, which is the hyperplane that is orthogonal to the vector $\rho(d)$.

Solvent Markowitz Portfolios

Remark. The solvent Markowitz portfolio with allocation $\mathbf{f}$ has value ratio on trading day d given by

$$\rho(d) = 1 + r(d) = 1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} = \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f}\,.$$

Its return mean μ satisfies

$$\begin{aligned} 1 + \mu &= 1 + \sum_{d=1}^{D} w(d) r(d) \\ &= \sum_{d=1}^{D} w(d)\big(1 + r(d)\big) \\ &= \sum_{d=1}^{D} w(d)\big(1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\big) = \sum_{d=1}^{D} w(d)\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} > 0\,. \end{aligned}$$

Solvent Markowitz Portfolios

Now we will show that *every long Markowitz portfolio is solvent*. In other words, we will show that $\Lambda \subset \Omega$. This shows that many solvent portfolios exist. The proof uses the fact that $\rho(d) > \mathbf{0}$, which states that every entry of $\rho(d)$ is positive.

Fact 2. We have $\Lambda \subset \Omega$.

Proof. Let $\mathbf{f} \in \Lambda$. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{f} \geq \mathbf{0}$, at least one entry of $\mathbf{f}$ must be positive. Because $\rho(d) > \mathbf{0}$, $\mathbf{f} \geq \mathbf{0}$, and at least one entry of $\mathbf{f}$ is positive, we have

$$\rho(d)^{\mathrm{T}}\mathbf{f} > 0 \qquad \text{for every } d = 1, \cdots, D\,.$$

We conclude from (4.8) that $\mathbf{f} \in \Omega$. Therefore $\Lambda \subset \Omega$. $\square$

Bounded Below Value-Ratio Portfolios

All investors want their portfolios to be profitable, not merely solvent. Most are realistic enongh to know that some days portfolios will decrease in valve. However, they would like to design their portfolios to limit their losses on such days. One way to do this is to consider only those Markowitz portfolios with value-ratios that are bounded below by some $\rho \in (0, \infty)$ over the value-ratio history $\{\boldsymbol{\rho}(d)\}_{d=0}^{D}$ being considered. The set of allocations for such portfolios is

$$\Omega_\rho = \{\mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \rho \le \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \quad \forall d\}\,. \tag{5.9}$$

Bacause $\rho > 0$, it is clear that these portfolios are solvent — i.e. it is clear that we have

$$\Omega_\rho \subset \Omega \qquad \text{for every } \rho \in (0, \infty)\,.$$

Therefore every Ω_ρ is a bounded when Ω is bounded.

Bounded Below Value-Ratio Portfolios

It is also clear that if $\rho, \rho' \in (0, \infty)$ then

$$\rho' \leq \rho \qquad \Longrightarrow \qquad \Omega_\rho \subset \Omega_{\rho'} \,.$$

Moreover, it is easy to show that for every $\rho \in (0, \infty)$ we have

$$\Omega = \bigcup \left\{ \Omega_{\rho'} \;:\; \rho' \in (0, \rho) \right\} .$$

The set Ω and the set Ω_ρ for any $\rho > 0$ can be characterized in terms of the function $\underline{\rho}(\mathbf{f})$ that is defined for every $\mathbf{f} \in \mathbb{R}^N$ by

$$\underline{\rho}(\mathbf{f}) = \min\left\{ \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \;:\; d = 1, \cdots, D \right\} . \tag{5.10}$$

It is evident from (4.8), (5.9), and (5.10) that for every $\mathbf{f} \in \mathbb{R}^N$ we have

$$\mathbf{f} \in \Omega \qquad \Longleftrightarrow \qquad \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \;\;\&\;\; \underline{\rho}(\mathbf{f}) > 0 \,,$$
$$\mathbf{f} \in \Omega_\rho \qquad \Longleftrightarrow \qquad \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \;\;\&\;\; \underline{\rho}(\mathbf{f}) \geq \rho \,.$$

Bounded Below Value-Ratio Portfolios

An investor might want to know if a set Π of portfolio allocations lies within some Ω_ρ. When Π is compact the following characterizations reduce answering such questions to solving certain optimization problems.

Fact 3. If $\Pi \subset \{\mathbf{f} \in \mathbb{R}^N : \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\}$ is compact then for every $\rho > 0$

$$\Pi \subset \Omega_\rho \qquad \Longleftrightarrow \qquad \rho \leq \underline{\rho}_\Pi^{\mathrm{mn}} \,,$$
$$\Pi \cap \Omega_\rho = \emptyset \qquad \Longleftrightarrow \qquad \rho > \underline{\rho}_\Pi^{\mathrm{mx}} \,,$$

where $\underline{\rho}_\Pi^{\mathrm{mn}}$ and $\underline{\rho}_\Pi^{\mathrm{mx}}$ are given by the optimization problems

$$\underline{\rho}_\Pi^{\mathrm{mn}} = \min\left\{\underline{\rho}(\mathbf{f}) \,:\, \mathbf{f} \in \Pi\right\} \,, \qquad \underline{\rho}_\Pi^{\mathrm{mx}} = \max\left\{\underline{\rho}(\mathbf{f}) \,:\, \mathbf{f} \in \Pi\right\} \,. \qquad (5.11)$$

Remark. Because $\underline{\rho}(\mathbf{f})$ is defined by (5.10) as the minimum over a finite family of linear functions, it is both continuous and concave over $\mathbb{R}^N$. Because $\underline{\rho}(\mathbf{f})$ is continuous over the compact set Π, both the minimizer and the maximizer being asserted by (5.11) exist.

Bounded Below Value-Ratio Portfolios

Proof. Suppose that $\Lambda \cap \Omega_\rho$ is not empty. Let $\mathbf{f} \in \Lambda \cap \Omega_\rho$. Then

$$\rho \leq \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \qquad \text{for every } d = 1, \cdots, D\,.$$

Hence,

$$\rho \leq \min_d \left\{ \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \right\}\,.$$

Therefore $\rho \leq \underline{\rho}_{\mathrm{mx}}$, which contradicts $\rho > \underline{\rho}_{\mathrm{mx}}$. □

Remark. This implies that its return mean μ satisfies

$$1 + \mu = \sum_{d=1}^{D} w(d)\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \geq \rho\,.$$

This gives a lower bound on μ.

Bounded Value-Ratio Portfolios

There are good reasons for investors to consider only portfolios with value-ratios that are bounded above as well as below. One such reason is to keep the portfolio within regimes where the data and models being used are better understood. The set of allocation vectors for Markowitz portfolios with value ratios bounded within $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ is

$$\Omega_{[\underline{\rho},\overline{\rho}]} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \underline{\rho} \leq \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \leq \overline{\rho} \;\; \forall d \right\}. \tag{6.12}$$

It is clear that these portfolios are solvent and have value ratios that are bounded below. Specifically, we have

$$\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega_{\underline{\rho}} \subset \Omega \qquad \text{for every } [\underline{\rho}, \overline{\rho}] \subset (0, \infty)\,.$$

It is also clear that if $[\underline{\rho}, \overline{\rho}]$ and $[\underline{\rho}', \overline{\rho}']$ are subsets of $(0, \infty)$ then

$$[\underline{\rho}, \overline{\rho}] \subset [\underline{\rho}', \overline{\rho}'] \qquad \Longrightarrow \qquad \Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega_{[\underline{\rho}',\overline{\rho}']}\,.$$

Finally, the set $\Lambda \cap \Omega_{[\underline{\rho},\overline{\rho}]}$ is empty if $\underline{\rho} > \underline{\rho}_{\mathrm{mx}}$.

Bounded Value-Ratio Portfolios

Remark. If $\mathbf{f} \in \Omega_{[\underline{\rho},\overline{\rho}]}$ then its return mean μ satisfies the bounds

$$\underline{\rho} \leq 1 + \mu \leq \overline{\rho}\,.$$

Moreover, its return variance v satisfies the bounds

$$0 < v \leq \left(\overline{\rho} - 1 - \mu\right)\left(1 + \mu - \underline{\rho}\right).$$

The bounds on μ follow from the facts that

$$\underline{\rho} \leq \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \leq \overline{\rho}\,, \qquad 1 + \mu = \sum_{d=1}^{D} w(d)\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f}\,.$$

The upper bound on v follows from the additional facts that

$$\left(\overline{\rho} - 1 - \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\right)\left(1 + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f} - \underline{\rho}\right) \geq 0\,,$$

$$\sum_{d=1}^{D} w(d)\left(\mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\right)^2 = \mu^2 + v\,.$$

Bounded Value-Ratio Portfolios

We now show that *every long Markowitz portfolio has bounded value ratios*. Specifically, we will show the following.

Fact 4. We have $\Lambda \subset \Omega_{[\rho_{\mathrm{mn}},\rho_{\mathrm{mx}}]}$ where

$$\begin{aligned} \rho_{\mathrm{mn}} &= \min\left\{\rho_i(d) \,:\, i = 1, \cdots, N\,;\ d = 1, \cdots, D\right\}, \\ \rho_{\mathrm{mx}} &= \max\left\{\rho_i(d) \,:\, i = 1, \cdots, N\,;\ d = 1, \cdots, D\right\}. \end{aligned} \tag{6.13}$$

Moreover, $\Omega_{[\rho_{\mathrm{mn}},\rho_{\mathrm{mx}}]}$ is the smallest set of bounded value-ratio portfolios that contains Λ.

Remark. Here ρ_{mn} and ρ_{mx} are the worst and best price ratios over the given history. They satisfy $0 < \rho_{\mathrm{mn}} < \rho_{\mathrm{mx}}$.

Bounded Value-Ratio Portfolios

Proof. It follows from the definitions of ρ_{mn} and ρ_{mx} given in (6.13) that $\boldsymbol{\rho}(d)$ satisfies the entrywise inequalities

$$\rho_{\mathrm{mn}}\,\mathbf{1} \le \boldsymbol{\rho}(d) \le \rho_{\mathrm{mx}}\,\mathbf{1} \qquad \text{for every } d = 1, \cdots, D\,.$$

Let $\mathbf{f} \in \Lambda$. Because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{f} \ge \mathbf{0}$, the above entrywise inequalities yield the bounds

$$\rho_{\mathrm{mn}} \le \rho_{\mathrm{mn}}\,\mathbf{1}^{\mathrm{T}}\mathbf{f} \le \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} \le \rho_{\mathrm{mx}}\,\mathbf{1}^{\mathrm{T}}\mathbf{f} \le \rho_{\mathrm{mx}}\,. \tag{6.14}$$

We see from (6.14) that for every $\mathbf{f} \in \Lambda$ a lower bound for $\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f}$ is ρ_{mn} and an upper bound for $\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f}$ is ρ_{mx}. Therefore we see from definition (6.12) that $\Lambda \subset \Omega_{[\rho_{\mathrm{mn}},\rho_{\mathrm{mx}}]}$.

It remains to show that $\Omega_{[\rho_{\mathrm{mn}},\rho_{\mathrm{mx}}]}$ is the smallest such set containing Λ.

Bounded Value-Ratio Portfolios

Let i and d be an asset and a day for which $\rho_i(d) = \rho_{\mathrm{mn}}$ given by (6.13). Let $\mathbf{f} = \mathbf{e}_i$, so that the only nonzero position held is in asset i. Then

$$\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} = \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{e}_i = \rho_i(d) = \rho_{\mathrm{mn}} .$$

Clearly, $\mathbf{f} \in \Lambda$ and $\mathbf{f} \notin \Omega_{\underline{\rho}}$ for every $\underline{\rho} > \rho_{\mathrm{mn}}$ by definition (5.9).

Let i and d be an asset and a day for which $\rho_i(d) = \rho_{\mathrm{mx}}$ given by (6.13). Let $\mathbf{f} = \mathbf{e}_i$, so that the only nonzero position held is in asset i. Then

$$\boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{f} = \boldsymbol{\rho}(d)^{\mathrm{T}}\mathbf{e}_i = \rho_i(d) = \rho_{\mathrm{mx}} .$$

Clearly, $\mathbf{f} \in \Lambda$ and $\mathbf{f} \notin \Omega_{[\underline{\rho},\overline{\rho}]}$ for every $\overline{\rho} < \rho_{\mathrm{mx}}$ by definition (6.12).

Therefore $\Lambda \not\subset \Omega_{[\underline{\rho},\overline{\rho}]}$ if $[\underline{\rho}, \overline{\rho}]$ is a proper subset of $[\rho_{\mathrm{mn}}, \rho_{\mathrm{mx}}]$. $\square$

Bounded Value-Ratio Portfolios

Remark. The bounds (6.14) and the definition (5.11) of $\underline{\rho}_{\rm mx}$ imply that

$$\rho_{\rm mn} \leq \underline{\rho}_{\rm mx} \leq \rho_{\rm mx} \,.$$

Remark. One way to assess risky assets for inclusion in a profolio is to compute for each asset being considered the quantities

$$\begin{aligned} \rho_i^{\rm mn} &= \min\left\{\rho_i(d) \,:\, d = 1, \cdots, D\right\}, \\ \rho_i^{\rm mx} &= \max\left\{\rho_i(d) \,:\, d = 1, \cdots, D\right\}. \end{aligned}$$

This should be done for a long history of five to ten or more years. Asset i is accepted for inclusion in the portfolio if $\rho_i^{\rm mn}$ and $\rho_i^{\rm mx}$ fall within a target interval $[\underline{\rho}, \overline{\rho}]$, and is rejected otherwise. The target interval should be large enough to include major indicies like the S&P 500 and the Russell 2000. It should be small enough that the risk is acceptable. This method will filter out riskier assets.