Portfolios that Contain Risky Assets 19: What Can Go Wrong?

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Portfolios that Contain Risky Assets
Part II: Stochastic Models

What Can Go Wrong?

1. Introduction
2. Uncertainty in the IID Model
3. Uncertainty in Our Estimators
4. Economic Uncertainty

Introduction

In this lecture we consider three sources of uncertainy.

- **Uncertainty in the IID Model:** There are two basic assumptions behind this model — namely, that the distribution of returns is independent from one day to the next, and that the distribution of returns is identical each day.
- **Uncertainty in Our Estimators:** There are three sources of estimator uncertainty. First, there are the errors in our sample estimators of mean and variance. Second, there are the errors in our use of the quadratic estimator. Third, the error in our Central Limit Theorem approximation.
- **Economic Uncertainty:** Excessive leverage, market inefficiency, and other economic factors can cause investors to be more cautious.

Uncertainty in the IID Model

Independent. The independence assumption of IID models can be checked by seeking correlation between returns on successive days. Given a return history $\{r(d)\}_{d=0}^{D}$ and a choice of positive weights $\{w(d)\}_{d=1}^{D}$ that sum to 1, we define the return mean estimators

$$\hat{\mu}_0 = \sum_{d=1}^{D} w(d) r(d) \,, \qquad \hat{\mu}_1 = \sum_{d=1}^{D} w(d) r(d-1) \,,$$

the variance estimators

$$\hat{v}_{00} = \sum_{d=1}^{D} w(d) \big(r(d) - \hat{\mu}_0 \big)^2 \,, \qquad \hat{v}_{11} = \sum_{d=1}^{D} w(d) \big(r(d-1) - \hat{\mu}_1 \big)^2 \,,$$

and the covariance estimators

$$\hat{v}_{01} = \sum_{d=1}^{D} w(d) \big(r(d) - \hat{\mu}_0 \big) \big(r(d-1) - \hat{\mu}_1 \big) \,.$$

Uncertainty in the IID Model

Consider the symmetric 2×2 matrix

$$\widehat{\mathbf{V}} = \begin{pmatrix} \hat{v}_{00} & \hat{v}_{01} \\ \hat{v}_{01} & \hat{v}_{11} \end{pmatrix} .$$

This matrix is generally positive definite. If the data was drawn from an IID process with mean μ and variance ξ then

$$\mathrm{Ex}\left(\widehat{\mathbf{V}}\right) = (1 - \bar{w}) \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} .$$

Therefore dependence is the deviation of $\widehat{\mathbf{V}}$ from this form.

Uncertainty in the IID Model

One measure of this deviation is captured by the *autocorrelation*

$$\omega_{\mathrm{cor}} = \frac{\hat{v}_{01}}{\sqrt{\hat{v}_{00}}\sqrt{\hat{v}_{11}}} .$$

Notice that by the Cauchy inequality $\omega_{\mathrm{cor}} \in [-1, 1]$, that $\omega_{\mathrm{cor}} = 1$ is perfect correlation, that $\omega_{\mathrm{cor}} = -1$ is perfect anticorrelation, and that $\omega_{\mathrm{cor}} = 0$ is perfect decorrelation. Therefore the closer ω_{cor} is to zero, the more confidence we have in the independence assumption.

Remark. Another measure of this deviation is

$$\omega_{\mathrm{diag}} = \frac{\hat{v}_{00} - \hat{v}_{11}}{\hat{v}_{00} + \hat{v}_{11}} .$$

This is typically small, so is less informative.

Uncertainty in the IID Model

Remark. The matrix $\widehat{\mathbf{V}}$ arises when we use least squares to fit the data to an autoregression model in the form

$$r(d) = a + b\, r(d-1) + n(d)\,,$$

where we pick a and b to minimize

$$\sum_{d=1}^{D} w(d)\, n(d)^2\,.$$

Remark. This metric can be applied to any risky asset. It can show differences between stocks of companies and broad-based index funds. The asset could be a tangent portfolio for a class of leverage-limited portfolios, say long portfolios. Reacall that given a return history $\{\mathbf{r}(d)\}_{d=0}^{D}$ of N risky assests and a risk-free rate μ_{rf} the return of the portfolio with allocation $\mathbf{f}$ on day d is $r(d) = (1 - \mathbf{1}^{\mathrm{T}}\mathbf{f})\mu_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}$.

Uncertainty in the IID Model

Identically Distributed. The question we must address is how to tell when two samples, $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$, might be drawn from the same probability density. We start with a simpler question. How to compare two probability densities over $(-1,\infty)$, say $p_1(R)$ and $p_2(R)$ where $p_1(R) \geq 0$, $p_2(R) \geq 0$, and

$$\int_{-1}^{\infty} p_1(R)\,\mathrm{d}R = \int_{-1}^{\infty} p_2(R)\,\mathrm{d}R = 1\,.$$

One idea is to compare their distributions $P_1(R)$ and $P_2(R)$, which are

$$P_1(R) = \int_{-1}^{R} p_1(R')\,\mathrm{d}R'\,, \qquad P_2(R) = \int_{-1}^{R} p_2(R')\,\mathrm{d}R'\,.$$

These are nondecreasing functions of R over $(-1,\infty)$ such that

$$\lim_{R\to-1} P_1(R) = \lim_{R\to-1} P_1(R) = 0\,, \qquad \lim_{R\to\infty} P_1(R) = \lim_{R\to\infty} P_1(R) = 1\,.$$

Uncertainty in the IID Model

The *Kolmogorov-Smirnov* measure of the closeness of P_1 and P_2 is the sup norm of their difference:

$$\|P_2 - P_1\|_{\mathrm{KS}} = \sup\{|P_2(R) - P_1(R)| \,:\, R \in (-1,\infty)\}\,.$$

The *Kuiper* measure of the closeness of P_1 and P_2 is

$$\begin{aligned}\|P_2 - P_1\|_{\mathrm{Ku}} = \sup\{&P_2(R) - P_1(R) \,:\, R \in (-1,\infty)\}\\ &+ \sup\{P_1(R) - P_2(R) \,:\, R \in (-1,\infty)\}\,.\end{aligned}$$

It can be shown that

$$\|P_2 - P_1\|_{\mathrm{KS}} \leq \|P_2 - P_1\|_{\mathrm{Ku}} \leq 1\,.$$

Uncertainty in the IID Model

The *Cramer-von Mises* measure of the closeness of P_1 and P_2 is the L^2-norm of their difference:

$$\|P_2 - P_1\|_{\mathrm{CvM}} = \left(\int_{-1}^{\infty} \big(P_2(R) - P_1(R)\big)^2 \mathrm{d}R \right)^{\frac{1}{2}} .$$

This can clearly be generalized to any L^p-norm with respect to any positive measure over $(-1, \infty)$.

For simplicity we will stick to the Kolmogorov-Smirnov and Kuiper measures.

Uncertainty in the IID Model

Now we return to our original question. Given two samples, $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$, we construct their so-called *emperical distributions*

$$\widehat{P}_1(R) = \frac{\#\{d \,:\, r_1(d) \leq R\}}{D_1}\,, \qquad \widehat{P}_2(R) = \frac{\#\{d \,:\, r_2(d) \leq R\}}{D_2}\,.$$

Here $\#S$ denotes the number of elements in a set S. These approximate the unknown true distributions P_1 and P_2 because

$$P_1(R) = \mathrm{Pr}\{r_1(d) \leq R\}\,, \qquad P_2(R) = \mathrm{Pr}\{r_2(d) \leq R\}\,.$$

Then the Kolmogorov-Smirnov and Kuiper measures of the difference $\widehat{P}_2 - \widehat{P}_1$ give us a way to quantify the likelihood that samples are drawn from similar distributions.

Uncertainty in the IID Model

Because $\widehat{P}_1$ and $\widehat{P}_2$ are step functions, we see that

$$\begin{aligned}
\|\widehat{P}_2 - \widehat{P}_1\|_{\mathrm{KS}} &= \max\{|\widehat{P}_2(R) - \widehat{P}_1(R)| \,:\, R \in (-1,\infty)\}\,. \\
\|\widehat{P}_2 - \widehat{P}_1\|_{\mathrm{Ku}} &= \max\{\widehat{P}_2(R) - \widehat{P}_1(R) \,:\, R \in (-1,\infty)\} \\
&\qquad + \max\{\widehat{P}_1(R) - \widehat{P}_2(R) \,:\, R \in (-1,\infty)\}\,.
\end{aligned}$$

Fortunately statisticians have provided software that efficiently computes these values given any two samples $\{r_1(d)\}_{d=1}^{D_1}$ and $\{r_2(d)\}_{d=1}^{D_2}$. These are called respectively the *two-sample KS test* and the *two-sample Kuiper test*.

Uncertainty in the IID Model

Finally, given return histories over a year $\{r(d)\}_{d=1}^{D}$, we can split the year into quarters and compare the emperical distribution of each quarter with that of another quarter or with that of the other three quarters combined. The maximum of all such comparisons made is the score for the year. For example, for each year we might define

$$\omega_{\mathrm{ID}} = \max\left\{ \|\widehat{P}_2 - \widehat{P}_1\|_{\mathrm{KS}} \; : \; \text{all comparisons made} \right\}.$$

If we choose to compare quarters with each other then there will be six comparisons made. If we choose to compare each quarter with the other three quarters combined then there will be four comparisons made. Notice that $\omega_{\mathrm{ID}} \in [0,1]$ and that ω_{ID} small when the distributions are close.

Uncertainty in the IID Model

Remark. These metrics can be applied to any risky asset. We might see a difference between the stock of a single company and a broad-based index fund. Similarly, the asset could be a tangent portfolio for a class of portfolios with a given leverage limit, for example, long portfolios. Reacall that given a return history $\{\mathbf{r}(d)\}_{d=0}^{D}$ of N risky assests and a risk-free rate model μ_{rf} the return of the Markowitz portfolio with allocation $\mathbf{f}$ on day d is

$$r(d) = (1 - \mathbf{1}^{\mathrm{T}}\mathbf{f})\mu_{\mathrm{rf}} + \mathbf{r}(d)^{\mathrm{T}}\mathbf{f}\,.$$

Uncertainty in Our Estimators

This material was presented earlier in the course, so we summarize it here. There are three sources of error in our estimators:

- the errors in the Central-Limit Theorem approximation used to construct our objectives Γ_χ in terms of the growth-rate mean and variance, γ and θ;
- the errors in our sample growth-rate mean and variance estimators, $\hat{\gamma}$ and $\hat{\theta}$;
- the errors in our approximations of our sample growth-rate mean and variance estimators $\hat{\gamma}$ and $\hat{\theta}$ in terms of our sample return mean and variance estimators and $\hat{\mu}$ and $\hat{\xi}$.

Economic Uncertainty

For each risky asset with we can measure its *efficiency* by

$$\omega_{\mathrm{ef}} = \frac{\mu_{\mathrm{ef}}(\sigma_i) - \mu_i}{\mu_{\mathrm{ef}}(\sigma_i) - \mu_{\mathrm{if}}(\sigma_i)} \,.$$

Notice that $\omega_{\mathrm{ef}} \in [0,1]$, that $\omega_{\mathrm{ef}} = 0$ is perfect efficiency, and that $\omega_{\mathrm{ef}} = 1$ is extreme inefficiency.

For each risky asset with we can measure its *proximity* by

$$\omega_{\mathrm{f}} = \frac{\sigma_i - \sigma_{\mathrm{f}}(\mu_i)}{\sigma_i} \,.$$

Notice that $\omega_{\mathrm{f}} \in [0,1)$, that $\omega_{\mathrm{f}} = 0$ means on the frontier, either efficient or ineffiecient, and that $\omega_{\mathrm{f}} \to 1$ as it moves further from the frontier.

Economic Uncertainty

Remark. While these metrics can be applied to any risky asset, they are more likely to have economic meaning when the assets are broad-based index funds, for example an S&P 500 fund or a bond fund. Similarly, the asset could be a tangent portfolio for a class of portfolios with a given leverage limit, for example, long portfolios.

We can also devise metrics based upon the *leverage of a tangent portfolio* for unlimited efficient frontiers. For example, if $\mathbf{f}_{\mathrm{tg}}$ is the allocation for some tangent portfolio then

$$\omega_{\mathrm{tg}} = 1 - \frac{1}{\|\mathbf{f}_{\mathrm{tg}}\|_1} \,.$$

Notice that $\omega_{\mathrm{tg}} \in [0, 1)$, that $\omega_{\mathrm{tg}} = 0$ means the tangent portfolio is long, and that $\omega_{\mathrm{f}} \to 1$ as the leverage of the tangent portfolio becomes large.

Economic Uncertainty

Other metrics of economic uncertainity that do not depend on means and variances can be built from *leverage limits*. For example, if ℓ_ρ is the leverage limit such that

$$\ell < \ell_\rho \quad \text{if and only if} \quad \Pi_\ell \subset \Omega_\rho \,,$$

then we can set

$$\omega_\rho = \frac{1}{1 + 2\ell_\rho} \,.$$

Notice that $\omega_\rho \in (0, 1]$, that $\omega_\rho \to 0$ as leverage limits become unbounded, and that $\omega_\rho \to 1$ as leverage limits vanish.