Portfolios that Contain Risky Assets 17: Fortune's Formulas

C. David Levermore

University of Maryland, College Park, MD

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Portfolios that Contain Risky Assets
Part II: Stochastic Models

Fortune's Formulas

Introduction

We now consider some settings in which the optimization problem can be solved analytically. Specifically, we will derive explicit formulas for the solutions to the maximization problems for the family of parabolic objectives

$$\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) = \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \tag{1.1a}$$

the family of quadratic objectives

$$\Gamma_{\mathrm{q}}^{\chi}(\mathbf{f}) = \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \tag{1.1b}$$

and the family of reasonable objectives

$$\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \tag{1.1c}$$

considered over their natural domains of allocations **f** for unlimited leverage portfolios with one risk-free asset.

Introduction

Recall that:

- μ_{rf} is the return on the risk-free asset;
- $\widetilde{\mathbf{m}}$ is the sample excess return mean vector, which is given in terms of the sample return mean vector $\mathbf{m}$ by $\widetilde{\mathbf{m}} = \mathbf{m} - \mu_{\mathrm{rf}}\mathbf{1}$;
- $\mathbf{V}$ is the sample return covariance matrix;
- χ is the nonnegative caution coefficient chosen by the investor.

Recall too that $\mathbf{m}$ and $\mathbf{V}$ are computed from a return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and a choice of positive weights $\{w(d)\}_{d=1}^{D}$ that sum to 1 by

$$\mathbf{m} = \sum_{d=1}^{D} w(d)\,\mathbf{r}(d)\,, \qquad \mathbf{V} = \sum_{d=1}^{D} w(d)\,\big(\mathbf{r}(d) - \mathbf{m}\big)\,\big(\mathbf{r}(d) - \mathbf{m}\big)^{\mathrm{T}}\,.$$

Introduction

In the previous lecture we saw that the maximizer $\mathbf{f}_*$ for such a problem will correspond to a point (σ_*, μ_*) on the efficient frontier. Moreover, we saw that (σ_*, μ_*) is the point in the $\sigma\mu$-plane where the level curves of the objective are tangent to the efficient frontier. While this geometric picture gave insight into how optimal portfolio allocations arise, we have not yet computed them.

The explicit formulas derived in this lecture for the maximizer $\mathbf{f}_*$ will confirm the general picture developed in the previous lecture. They will also give insight into the relative merits of the different families of objectives in (1.1). In particular, the maximizers when $\chi = 0$ give different realizations of the Kelly Criterion — so-called *fortune's formulas*. The maximizers when $\chi > 0$ will be corresponding fractional Kelly strategies. We will derive and analyze these formulas after reviewing the efficient frontier for unlimited leverage portfolios with one risk-free asset.

Efficient Frontier

Recall that for unlimited leverage portfolios without risk-free assets the frontier is the hyperbola in the right-half of the $\sigma\mu$-plane given by

$$\sigma = \sqrt{\sigma_{\rm mv}^2 + \left(\frac{\mu - \mu_{\rm mv}}{\nu_{\rm as}}\right)^2}\,, \tag{2.2a}$$

where the so-called frontier parameters $\sigma_{\rm mv}$, $\mu_{\rm mv}$, and $\nu_{\rm as}$ are given by

$$\frac{1}{\sigma_{\rm mv}^2} = \mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}\,, \qquad \mu_{\rm mv} = \frac{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{m}}{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}}\,,$$
$$\nu_{\rm as}^2 = \mathbf{m}^{\rm T}\mathbf{V}^{-1}\mathbf{m} - \frac{(\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{m})^2}{\mathbf{1}^{\rm T}\mathbf{V}^{-1}\mathbf{1}}\,. \tag{2.2b}$$

This so-called frontier hyperbola has vertex $(\sigma_{\rm mv}, \mu_{\rm mv})$ and asymptotes

$$\mu = \mu_{\rm mv} \pm \nu_{\rm as}\,\sigma \qquad \text{for } \sigma \geq 0\,.$$

The positive definiteness of $\mathbf{V}$ insures that $\sigma_{\rm mv} > 0$ and $\nu_{\rm as} > 0$.

Efficient Frontier

If we introduce one risk-free asset with risk-free return $\mu_{\mathrm{rf}} < \mu_{\mathrm{mv}}$ then the *efficient frontier* becomes the tangent half-line given by

$$\mu = \mu_{\mathrm{rf}} + \nu_{\mathrm{tg}}\,\sigma \qquad \text{for } \sigma \geq 0\,, \tag{2.3a}$$

where the slope is

$$\nu_{\mathrm{tg}} = \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}}} = \nu_{\mathrm{as}}\sqrt{1 + \left(\frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\nu_{\mathrm{as}}\,\sigma_{\mathrm{mv}}}\right)^2}\,. \tag{2.3b}$$

This slope is the so-called *Sharpe ratio* of the efficient frontier.

Remark. The *Sharpe ratio* of any portfolio with return mean μ and volatility σ is defined as

$$\frac{\mu - \mu_{\mathrm{rf}}}{\sigma}\,.$$

Clearly ν_{tg} is the Sharpe ratio of every portfolio on the efficient frontier (2.3a). Moreover, ν_{tg} is the largest possible Sharpe ratio for any portfolio.

Efficient Frontier

The efficient frontier (2.3a) is tangent to the frontier hyperbola (2.2a) at the point $(\sigma_{\rm tg}, \mu_{\rm tg})$ where

$$\sigma_{\rm tg} = \sigma_{\rm mv} \sqrt{1 + \left(\frac{\nu_{\rm as}\, \sigma_{\rm mv}}{\mu_{\rm mv} - \mu_{\rm rf}} \right)^2} \,, \qquad \mu_{\rm tg} = \mu_{\rm mv} + \frac{\nu_{\rm as}^2\, \sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} \,.$$

The unique *tangency portfolio* associated with this point has allocation

$$\mathbf{f}_{\rm tg} = \frac{\sigma_{\rm mv}^2}{\mu_{\rm mv} - \mu_{\rm rf}} \, \mathbf{V}^{-1} \widetilde{\mathbf{m}} \,. \tag{2.4}$$

Every portfolio on the efficient frontier (2.3a) can be viewed as holding a position in this tangency portfolio and a position in a risk-free asset.

Efficient Frontier

We can select a particular portfolio on this efficient frontier by identifying an objective function to be maximized. In subsequent sections we derive and analyze explicit formulas for the maximizers for each family member of the parabolic, quadratic, and reasonable objectives given in (1.1).

Parabolic Objectives

First we consider the maximization problem

$$\mathbf{f}_* = \arg\max\{\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N\}\,, \tag{3.5a}$$

where $\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f})$ is the family of parabolic objectives parametrized by $\chi \geq 0$ and given by

$$\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) = \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,. \tag{3.5b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) = \widetilde{\mathbf{m}} - \mathbf{V}\mathbf{f} - \frac{\chi}{\sigma}\,\mathbf{V}\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} > 0$.

Parabolic Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = \widetilde{\mathbf{m}} - \frac{\sigma_* + \chi}{\sigma_*} \mathbf{V}\mathbf{f}_* \,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \mathbf{V} \mathbf{f}_*} > 0$.

Upon solving this equation for $\mathbf{f}_*$ we obtain

$$\mathbf{f}_* = \frac{\sigma_*}{\sigma_* + \chi} \mathbf{V}^{-1} \widetilde{\mathbf{m}} \,. \tag{3.6}$$

All that remains is to determine σ_*.

Parabolic Objectives

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \mathbf{V} \mathbf{f}_*}$ we have

$$\sigma_*^{\,2} = \mathbf{f}_*^{\mathrm{T}} \mathbf{V} \mathbf{f}_* = \frac{\sigma_*^{\,2}}{(\sigma_* + \chi)^2} \, \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{V}^{-1} \widetilde{\mathbf{m}} = \frac{\sigma_*^{\,2}}{(\sigma_* + \chi)^2} \, \nu_{\mathrm{tg}}^2 \,,$$

we conclude that σ_* satisfies

$$(\sigma_* + \chi)^2 = \nu_{\mathrm{tg}}^2 \,.$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < \nu_{\mathrm{tg}} \,, \tag{3.7}$$

and that σ_* is determined by

$$\sigma_* + \chi = \nu_{\mathrm{tg}} \,.$$

Parabolic Objectives

Then the maximizer $\mathbf{f}_*$ given by (3.6) becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\text{tg}}}\right) \mathbf{V}^{-1}\widetilde{\mathbf{m}} \,. \tag{3.8}$$

Remark. Kelly investors take $\chi = 0$, in which case (3.8) reduces to

$$\mathbf{f}_* = \mathbf{V}^{-1}\widetilde{\mathbf{m}} \,. \tag{3.9}$$

Formula (3.9) is often called *fortune's formula* in the belief that it is a good approximation to the Kelly strategy. In this view formula (3.8) gives an explicit fractional Kelly strategy for every $\chi \in (0, \nu_{\text{tg}})$. However, we will see that formula (3.9) gives an allocation that can be far from the Kelly strategy, and generally leads to overbetting.

Parabolic Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\mathrm{tg}}$. To treat that case we will use the *Cauchy inequality* in the form

$$\left|\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right| \leq \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,. \tag{3.10}$$

When $\chi \geq \nu_{\mathrm{tg}}$ the positive definiteness of $\mathbf{V}$, the fact $\chi \geq \nu_{\mathrm{tg}}$, the *Sharpe ratio* formula (2.3b), and the above Cauchy inequality imply

$$\begin{aligned}
\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) &= \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \nu_{\mathrm{tg}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&= \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathrm{rf}} = \Gamma_{\mathrm{p}}^{\chi}(\mathbf{0})\,.
\end{aligned}$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq \nu_{\mathrm{tg}}$.

Parabolic Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (3.5) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \mathbf{V}^{-1}\widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}}\,. \end{cases} \tag{3.11}$$

This solution lies on the efficient frontier (2.3a). It allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ given by (2.4) and $1 - f_{\mathrm{tg}}^{\chi}$ times the portfolio value in a risk-free asset, where

$$f_{\mathrm{tg}}^{\chi} = \left(1 - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}^2}\,. \tag{3.12}$$

Quadratic Objectives

Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\{\Gamma_q^\chi(\mathbf{f}) : \mathbf{f} \in \mathbb{R}^N\}, \tag{4.13a}$$

where $\Gamma_q^\chi(\mathbf{f})$ is the family of quadratic objectives parametrized by $\chi \geq 0$ and given by

$$\Gamma_q^\chi(\mathbf{f}) = \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,. \tag{4.13b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\Gamma_q^\chi(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\Gamma_q^\chi(\mathbf{f}) = (1 - \mu_{\mathrm{rf}})\widetilde{\mathbf{m}} - \widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \mathbf{V}\mathbf{f} - \frac{\chi}{\sigma}\,\mathbf{V}\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} > 0$.

Quadratic Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = (1 - \mu_{\mathrm{rf}})\widetilde{\mathbf{m}} - \widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* - \frac{\sigma_* + \chi}{\sigma_*}\,\mathbf{V}\mathbf{f}_*\,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}}\mathbf{V}\mathbf{f}_*} > 0$.

After multiplying this relation by $\mathbf{V}^{-1}$ and bringing the terms involving $\mathbf{f}_*$ to the left-hand side, we obtain

$$\frac{\sigma_* + \chi}{\sigma_*}\,\mathbf{f}_* + \mathbf{V}^{-1}\widetilde{\mathbf{m}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* = (1 - \mu_{\mathrm{rf}})\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,. \tag{4.14}$$

Quadratic Objectives

Now multiply this by $\sigma_* \, \widetilde{\mathbf{m}}^{\mathrm{T}}$ and use the *Sharpe ratio* formula (2.3b), $\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}} = \nu_{\mathrm{tg}}^2$, to obtain

$$(\sigma_* + \chi + \nu_{\mathrm{tg}}^2\, \sigma_*)\, \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* = (1 - \mu_{\mathrm{rf}})\, \nu_{\mathrm{tg}}^2\, \sigma_* \,,$$

which implies that

$$\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_* = (1 - \mu_{\mathrm{rf}})\, \frac{\nu_{\mathrm{tg}}^2\, \sigma_*}{\sigma_* + \chi + \nu_{\mathrm{tg}}^2\, \sigma_*} \,.$$

When this expression is placed into (4.14) we can solve for $\mathbf{f}_*$ to find

$$\mathbf{f}_* = (1 - \mu_{\mathrm{rf}})\, \frac{\sigma_*}{\sigma_* + \chi + \nu_{\mathrm{tg}}^2\, \sigma_*}\, \mathbf{V}^{-1}\widetilde{\mathbf{m}} \,. \tag{4.15}$$

All that remains is to determine σ_*.

Quadratic Objectives

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}} \mathbf{V} \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^{\mathrm{T}} \mathbf{V} \mathbf{f}_* = \frac{(1-\mu_{\mathrm{rf}})^2 \, \sigma_*^2}{\left(\left(1+\nu_{\mathrm{tg}}^2\right)\sigma_* + \chi\right)^2} \, \widetilde{\mathbf{m}}^{\mathrm{T}} \mathbf{V}^{-1} \widetilde{\mathbf{m}}$$
$$= \frac{(1-\mu_{\mathrm{rf}})^2 \, \sigma_*^2}{\left(\left(1+\nu_{\mathrm{tg}}^2\right)\sigma_* + \chi\right)^2} \, \nu_{\mathrm{tg}}^2 \,,$$

we conclude that σ_* satisfies

$$\left(\left(1+\nu_{\mathrm{tg}}^2\right)\sigma_* + \chi\right)^2 = (1-\mu_{\mathrm{rf}})^2 \, \nu_{\mathrm{tg}}^2 \,.$$

Quadratic Objectives

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < (1 - \mu_{\mathrm{rf}})\, \nu_{\mathrm{tg}}\,, \tag{4.16}$$

and that σ_* is determined by

$$(1 + \nu_{\mathrm{tg}}^2)\, \sigma_* + \chi = (1 - \mu_{\mathrm{rf}})\, \nu_{\mathrm{tg}}\,.$$

Then the maximizer $\mathbf{f}_*$ given by (4.15) becomes

$$\mathbf{f}_* = \left(1 - \mu_{\mathrm{rf}} - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{1}{1 + \nu_{\mathrm{tg}}^2}\, \mathbf{V}^{-1} \widetilde{\mathbf{m}}\,. \tag{4.17}$$

Quadratic Objectives

Remark. Kelly investors take $\chi = 0$, in which case (4.17) reduces to

$$\mathbf{f}_* = \frac{1 - \mu_{\mathrm{rf}}}{1 + \nu_{\mathrm{tg}}^2} \, \mathbf{V}^{-1} \widetilde{\mathbf{m}} \, . \tag{4.18}$$

Formula (4.18) differs significantly from formula (3.9) whenever the Sharpe ratio ν_{tg} is not small. Sharpe ratios are often near 1 and sometimes can be as large as 3. So which of these should be called *fortune's formula*? Certainly not formula (3.9)! To see why, set $\mathbf{f} = \mathbf{V}^{-1}\widetilde{\mathbf{m}}$ into the quadratic objective (4.13b) with $\chi = 0$ to obtain

$$\Gamma_{\mathrm{q}}^{0}(\mathbf{V}^{-1}\widetilde{\mathbf{m}}) = \mu_{\mathrm{rf}} + \tfrac{1}{2}\,\nu_{\mathrm{tg}}^2 - \tfrac{1}{2}\left(\mu_{\mathrm{rf}} + \nu_{\mathrm{tg}}^2\right)^2 ,$$

which can be negative when ν_{tg} is near 1. So formula (3.9) can overbet!

Quadratic Objectives

The foregoing analysis did not yield a maximzier when $\chi \geq (1-\mu_{\mathrm{rf}})\,\nu_{\mathrm{tg}}$. In that case the positive definiteness of $\mathbf{V}$, the fact $\chi \geq (1-\mu_{\mathrm{rf}})\,\nu_{\mathrm{tg}}$, the *Sharpe ratio* formula (2.3b), and the *Cauchy inequality* (3.10) imply

$$\begin{aligned}
\Gamma_{\mathrm{q}}^{\chi}(\mathbf{f}) &= \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathrm{rf}}^2 + 2\mu_{\mathrm{rf}}\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&= \mu_{\mathrm{rf}} - \tfrac{1}{2}\,\mu_{\mathrm{rf}}^2 + (1-\mu_{\mathrm{rf}})\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathrm{rf}} - \tfrac{1}{2}\,\mu_{\mathrm{rf}}^2 + (1-\mu_{\mathrm{rf}})\,\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - (1-\mu_{\mathrm{rf}})\,\nu_{\mathrm{tg}}\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&= \mu_{\mathrm{rf}} - \tfrac{1}{2}\,\mu_{\mathrm{rf}}^2 + (1-\mu_{\mathrm{rf}})\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}}}\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\right) \\
&\leq \mu_{\mathrm{rf}} - \tfrac{1}{2}\,\mu_{\mathrm{rf}}^2 = \Gamma_{\mathrm{q}}^{\chi}(\mathbf{0})\,.
\end{aligned}$$

Therefore $\mathbf{f}_* = \mathbf{0}$ when $\chi \geq (1-\mu_{\mathrm{rf}})\,\nu_{\mathrm{tg}}$.

Quadratic Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (4.13) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \mu_{\mathrm{rf}} - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{1 + \nu_{\mathrm{tg}}^2} & \text{if } \chi < (1 - \mu_{\mathrm{rf}})\, \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq (1 - \mu_{\mathrm{rf}})\, \nu_{\mathrm{tg}}\,. \end{cases} \tag{4.19}$$

This solution lies on the efficient frontier (2.3a). It allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ given by (2.4) and $1 - f_{\mathrm{tg}}^{\chi}$ times the portfolio value in a risk-free asset, where

$$f_{\mathrm{tg}}^{\chi} = \left(1 - \mu_{\mathrm{rf}} - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{1}{1 + \nu_{\mathrm{tg}}^2} \, \frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}^2}\,. \tag{4.20}$$

Reasonable Objectives

Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\{\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N\,,\, 1 + \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} > 0\}\,, \tag{5.21a}$$

where $\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f})$ is the family of reasonable objectives parametrized by $\chi \geq 0$ and given by

$$\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,. \tag{5.21b}$$

Because $\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f}) \to -\infty$ as $\mathbf{f}$ approaches the boundary of the domain being considered in (5.21a), the maximizer $\mathbf{f}_*$ must lie in the interior of the domain. If $\mathbf{f} \neq 0$ then the gradient of $\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f}) = \frac{1}{1+\mu}\,\widetilde{\mathbf{m}} - \mathbf{V}\mathbf{f} - \frac{\chi}{\sigma}\,\mathbf{V}\mathbf{f}\,,$$

where $\mu = \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}$ and $\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} > 0$.

Reasonable Objectives

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{f}_* = \frac{1}{1+\mu_*} \frac{\sigma_*}{\sigma_* + \chi} \mathbf{V}^{-1}\widetilde{\mathbf{m}}\,, \tag{5.22}$$

where $\mu_* = \mu_{\mathrm{rf}} + \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}_*$ and $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}}\mathbf{V}\mathbf{f}_*} > 0$.

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathrm{T}}\mathbf{V}\mathbf{f}_*}$ we have

$$\begin{aligned}\sigma_*^{\,2} = \mathbf{f}_*^{\mathrm{T}}\mathbf{V}\mathbf{f}_* &= \frac{1}{(1+\mu_*)^2} \frac{\sigma_*^{\,2}}{(\sigma_* + \chi)^2} \widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}} \\ &= \frac{1}{(1+\mu_*)^2} \frac{\sigma_*^{\,2}}{(\sigma_* + \chi)^2} \nu_{\mathrm{tg}}^2\,.\end{aligned}$$

Reasonable Objectives

From this we conclude that μ_* and σ_* satisfy

$$(\sigma_* + \chi)^2 = \frac{\nu_{\mathrm{tg}}^2}{(1+\mu_*)^2} .$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < \frac{\nu_{\mathrm{tg}}}{1+\mu_*} , \tag{5.23}$$

and that we can determine σ_* in terms of μ_* from

$$\sigma_* + \chi = \frac{\nu_{\mathrm{tg}}}{1+\mu_*} .$$

Then the maximizer $\mathbf{f}_*$ given by (5.22) becomes

$$\mathbf{f}_* = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathrm{tg}}} \right) \mathbf{V}^{-1} \widetilde{\mathbf{m}} , \tag{5.24}$$

Reasonable Objectives

Because $\mu_* = \mu_{\rm rf} + \widetilde{\mathbf{m}}^{\rm T}\mathbf{f}_*$, by the *Sharpe ratio* formula (2.3b) we have

$$\begin{aligned} \mu_* = \mu_{\rm rf} + \widetilde{\mathbf{m}}^{\rm T}\mathbf{f}_* &= \mu_{\rm rf} + \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\rm tg}}\right)\widetilde{\mathbf{m}}^{\rm T}\mathbf{V}^{-1}\widetilde{\mathbf{m}} \\ &= \mu_{\rm rf} + \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\rm tg}}\right)\nu_{\rm tg}^2\,. \end{aligned}$$

This can be reduced to the quadratic equation

$$\left(\frac{\nu_{\rm tg}}{1+\mu_*}\right)^2 + \left(\frac{1+\mu_{\rm rf}}{\nu_{\rm tg}} - \chi\right)\frac{\nu_{\rm tg}}{1+\mu_*} = 1\,,$$

which has the unique positive root

$$\frac{\nu_{\rm tg}}{1+\mu_*} = -\frac{1}{2}\left(\frac{1+\mu_{\rm rf}}{\nu_{\rm tg}} - \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1+\mu_{\rm rf}}{\nu_{\rm tg}} - \chi\right)^2}\,. \qquad (5.25)$$

Reasonable Objectives

Then condition (5.23) is satisfied if and only if

$$0 < \frac{\nu_{\text{tg}}}{1+\mu_*} - \chi = -\frac{1}{2}\left(\frac{1+\mu_{\text{rf}}}{\nu_{\text{tg}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1+\mu_{\text{rf}}}{\nu_{\text{tg}}} - \chi\right)^2} \,.$$

This inequality holds if and only if

$$0 < 1 + \frac{1}{4}\left(\frac{1+\mu_{\text{rf}}}{\nu_{\text{tg}}} - \chi\right)^2 - \frac{1}{4}\left(\frac{1+\mu_{\text{rf}}}{\nu_{\text{tg}}} + \chi\right)^2 = 1 - \frac{1+\mu_{\text{rf}}}{\nu_{\text{tg}}}\,\chi \,.$$

This holds if and only if χ satisfies the bounds

$$0 \leq \chi < \frac{\nu_{\text{tg}}}{1+\mu_{\text{rf}}} \,. \tag{5.26}$$

Reasonable Objectives

By using (5.25) to eliminate μ_* from the maximizer $\mathbf{f}_*$ given by (5.24) we find

$$\mathbf{f}_* = \left[-\frac{1}{2}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right)^2}\,\right] \frac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{\nu_{\mathrm{tg}}} .$$

This becomes

$$\mathbf{f}_* = \left(\frac{1}{1+\mu_{\mathrm{rf}}} - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{1}{D\left(\chi, \dfrac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}\right)} \mathbf{V}^{-1}\widetilde{\mathbf{m}} , \tag{5.27a}$$

where

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2} . \tag{5.27b}$$

Reasonable Objectives

Remark. Kelly investors take $\chi = 0$, in which case (5.27) reduces to

$$\mathbf{f}_* = \frac{1}{\frac{1}{2}(1+\mu_{\mathrm{rf}}) + \frac{1}{2}\sqrt{(1+\mu_{\mathrm{rf}})^2 + 4\nu_{\mathrm{tg}}^2}}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,. \tag{5.28}$$

This candidate for *fortune's formula* will be compared with the others later.

Reasonable Objectives

The foregoing analysis did not yield a maximzier when $(1+\mu_{\mathrm{rf}})\,\chi \geq \nu_{\mathrm{tg}}$. The definiteness of $\mathbf{V}$, the concavity of $\log(x)$, the fact $(1+\mu_{\mathrm{rf}})\,\chi \geq \nu_{\mathrm{tg}}$, the *Sharpe ratio* formula (2.3b), and *Cauchy inequality* (3.10) imply

$$
\begin{aligned}
\Gamma_{\mathrm{r}}^{\chi}(\mathbf{f}) &= \log\left(1+\mu_{\mathrm{rf}}+\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \log(1+\mu_{\mathrm{rf}}) + \frac{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}}{1+\mu_{\mathrm{rf}}} - \chi\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&\leq \log(1+\mu_{\mathrm{rf}}) + \frac{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f}}{1+\mu_{\mathrm{rf}}} - \frac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}} \\
&= \log(1+\mu_{\mathrm{rf}}) + \frac{1}{1+\mu_{\mathrm{rf}}}\left(\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{f} - \sqrt{\widetilde{\mathbf{m}}^{\mathrm{T}}\mathbf{V}^{-1}\widetilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\right) \\
&\leq \log(1+\mu_{\mathrm{rf}}) = \Gamma_{\mathrm{r}}^{\chi}(\mathbf{0})\,.
\end{aligned}
$$

Therefore $\mathbf{f}_{*} = \mathbf{0}$ when $(1+\mu_{\mathrm{rf}})\,\chi \geq \nu_{\mathrm{tg}}$.

Reasonable Objectives

Therefore the solution $\mathbf{f}_*$ of the maximization problem (5.21) is

$$\mathbf{f}_* = \begin{cases} \left(\dfrac{1}{1+\mu_{\mathrm{rf}}} - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{D\left(\chi, \dfrac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}\right)} & \text{if } \chi < \dfrac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}, \\ \mathbf{0} & \text{if } \chi \geq \dfrac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}, \end{cases} \tag{5.29}$$

where $D(\chi, y)$ was defined by (5.27b).

This solution lies on the efficient frontier (2.3a). It allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathrm{tg}}$ given by (2.4) and $1 - f_{\mathrm{tg}}^{\chi}$ times the portfolio value in a risk-free asset, where

$$f_{\mathrm{tg}}^{\chi} = \left(\frac{1}{1+\mu_{\mathrm{rf}}} - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \frac{1}{D\left(\chi, \dfrac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}\right)} \frac{\mu_{\mathrm{mv}} - \mu_{\mathrm{rf}}}{\sigma_{\mathrm{mv}}^2}. \tag{5.30}$$

Comparisons

The maximizers for the parabolic, quadratic, and reasonable objectives are given by (3.11), (4.19), and (5.29) respectively. They are

$$\mathbf{f}_*^{\mathrm{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \mathbf{V}^{-1}\widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}}\,, \end{cases} \qquad (6.31a)$$

$$\mathbf{f}_*^{\mathrm{q}} = \begin{cases} \left(1 - \mu_{\mathrm{rf}} - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{1 + \nu_{\mathrm{tg}}^2} & \text{if } \chi < (1 - \mu_{\mathrm{rf}})\,\nu_{\mathrm{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq (1 - \mu_{\mathrm{rf}})\,\nu_{\mathrm{tg}}\,, \end{cases} \qquad (6.31b)$$

$$\mathbf{f}_*^{\mathrm{r}} = \begin{cases} \left(\dfrac{1}{1 + \mu_{\mathrm{rf}}} - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{D\left(\chi, \dfrac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}}\right)} & \text{if } \chi < \dfrac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}}\,, \\ \mathbf{0} & \text{if } \chi \geq \dfrac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}}\,. \end{cases} \qquad (6.31c)$$

Comparisons

Fact 1. If $\mu_{\mathrm{rf}} \in [0,1)$ then $\mathbf{f}_*^{\mathrm{q}}$ is the most conservative of these allocations and $\mathbf{f}_*^{\mathrm{p}}$ is the most agressive.

Proof. First observe that because $\mu_{\mathrm{rf}} \in [0,1)$ we have

$$1 - \mu_{\mathrm{rf}} \leq \frac{1}{1 + \mu_{\mathrm{rf}}} \leq 1 \,.$$

These inequalities imply that

$$(1 - \mu_{\mathrm{rf}})\, \nu_{\mathrm{tg}} \leq \frac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}} \leq \nu_{\mathrm{tg}} \,, \qquad (6.32)$$

and that

$$1 - \mu_{\mathrm{rf}} - \frac{\chi}{\nu_{\mathrm{tg}}} \leq \frac{1}{1 + \mu_{\mathrm{rf}}} - \frac{\chi}{\nu_{\mathrm{tg}}} \leq 1 - \frac{\chi}{\nu_{\mathrm{tg}}} \,. \qquad (6.33)$$

Each of these inequalities is strict when $\mu_{\mathrm{rf}} \in (0,1)$.

Comparisons

Recall from (5.27b) that

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2}\,. \tag{6.34}$$

For every $y > 0$ we have

$$\partial_\chi D(\chi, y) = \tfrac{1}{2} y \left(1 - \frac{1 - \chi\, y}{\sqrt{(1 - \chi\, y)^2 + 4y^2}} \right) > 0\,,$$

whereby $D(\chi, y)$ is an strictly increasing function of χ. Hence, for every $\chi \in [0, y)$ we have

$$1 < D(0, y) \leq D(\chi, y) < D(y, y) = 1 + y^2\,. \tag{6.35}$$

Therefore when $\mu_{\mathrm{rf}} \geq 0$ we have

$$1 < D\!\left(\chi, \frac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}}\right) < 1 + \frac{\nu_{\mathrm{tg}}^2}{(1 + \mu_{\mathrm{rf}})^2} \leq 1 + \nu_{\mathrm{tg}}^2 \quad \text{if } \chi < \frac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}}\,. \tag{6.36}$$

Comparisons

The inequalities (6.32) imply that $\mathbf{f}_*^{\mathrm{q}}$ given by (6.31b) has the smallest critical value of χ at which it becomes $\mathbf{0}$, and that $\mathbf{f}_*^{\mathrm{p}}$ given by (6.31a) has the largest critical value of χ at which it becomes $\mathbf{0}$.

The inequalities (6.33) and (6.36) imply that the factor multiplying $\mathbf{V}^{-1}\widetilde{\mathbf{m}}$ in the expression for $\mathbf{f}_*^{\mathrm{q}}$ given by (6.31b) is smaller than the factor multiplying $\mathbf{V}^{-1}\widetilde{\mathbf{m}}$ in the expression for $\mathbf{f}_*^{\mathrm{r}}$ given by (6.31c), which is smaller than the factor multiplying $\mathbf{V}^{-1}\widetilde{\mathbf{m}}$ in the expression for $\mathbf{f}_*^{\mathrm{p}}$ given by (6.31a). Hence, $\mathbf{f}_*^{\mathrm{q}}$ is more conservative than $\mathbf{f}_*^{\mathrm{r}}$, which is more conservative than $\mathbf{f}_*^{\mathrm{p}}$. □

Remark. The risk-free return μ_{rf} is usually much smaller than the Sharpe ratio ν_{tg}. This means that the main differences between the maximizers given by formulas (6.31) arise due to their dependence upon ν_{tg}.

Comparisons

In practice μ_{rf} is often small enough that it can be neglected except in $\widetilde{\mathbf{m}}$. By setting $\mu_{\mathrm{rf}} = 0$ in (6.31) we get

$$\mathbf{f}_*^{\mathrm{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \mathbf{V}^{-1}\widetilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathrm{tg}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}} \,, \end{cases} \tag{6.37a}$$

$$\mathbf{f}_*^{\mathrm{q}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{1 + \nu_{\mathrm{tg}}^2} & \text{if } \chi < \nu_{\mathrm{tg}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}} \,, \end{cases} \tag{6.37b}$$

$$\mathbf{f}_*^{\mathrm{r}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathrm{tg}}}\right) \dfrac{\mathbf{V}^{-1}\widetilde{\mathbf{m}}}{D(\chi, \nu_{\mathrm{tg}})} & \text{if } \chi < \nu_{\mathrm{tg}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathrm{tg}} \,, \end{cases} \tag{6.37c}$$

where $D(\chi, y)$ is given by (6.34). These all are nonzero for $\chi < \nu_{\mathrm{tg}}$ and all vanish for $\chi \geq \nu_{\mathrm{tg}}$. We see from (6.35) that $1 < D(\chi, \nu_{\mathrm{tg}}) < 1 + \nu_{\mathrm{tg}}^2$.

Comparisons

We now use formulas (6.37b) and (6.37c) to isolate the dependence of the maximizers $\mathbf{f}_*^{\mathrm{q}}$ and $\mathbf{f}_*^{\mathrm{r}}$ upon ν_{tg}.

Fact 2. For every $\chi \in [0, \nu_{\mathrm{tg}})$ we have

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}}{1 + \nu_{\mathrm{tg}}^2} \leq \frac{D(\chi, \nu_{\mathrm{tg}})}{1 + \nu_{\mathrm{tg}}^2} < 1\,, \tag{6.38}$$

where the left-hand side is a strictly decreasing function of ν_{tg}.

Proof. By (6.35) we have

$$1 + \nu_{\mathrm{tg}}^2 > D(\chi, \nu_{\mathrm{tg}}) \geq D(0, \nu_{\mathrm{tg}}) = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}\,.$$

The inequalities (6.38) follow. The task of proving the left-hand side of (6.38) is a strictly decreasing function of ν_{tg} is left as an exercise. □

Comparisons

We now use **Fact 2** to show that $\mathbf{f}_*^{\mathrm{q}}$ and $\mathbf{f}_*^{\mathrm{r}}$ are close when $\nu_{\mathrm{tg}} \leq \frac{2}{3}$.

Fact 3. If $\nu_{\mathrm{tg}} \leq \frac{2}{3}$ then for every $\chi \in [0, \nu_{\mathrm{tg}})$ we have

$$\frac{12}{13} \leq \frac{D(\chi, \nu_{\mathrm{tg}})}{1 + \nu_{\mathrm{tg}}^2} < 1\,. \tag{6.39}$$

Proof. By the monotonicity asserted in **Fact 2** if $\nu_{\mathrm{tg}} \leq \frac{2}{3}$ then

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}}{1 + \nu_{\mathrm{tg}}^2} \geq \frac{\frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3}}{1 + \frac{4}{9}} = \frac{\frac{4}{3}}{\frac{13}{9}} = \frac{12}{13}\,.$$

Then (6.39) follows from inequality (6.38) of **Fact 2**. □

Comparisons

Remark. A Kelly investor would set $\chi = 0$, in which case (6.31) gives

$$\mathbf{f}_*^{\mathrm{p}} = \mathbf{V}^{-1}\widetilde{\mathbf{m}}\,, \tag{6.40a}$$

$$\mathbf{f}_*^{\mathrm{q}} = \frac{1-\mu_{\mathrm{rf}}}{1+\nu_{\mathrm{tg}}^2}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,, \tag{6.40b}$$

$$\mathbf{f}_*^{\mathrm{r}} = \frac{1}{\frac{1}{2}(1+\mu_{\mathrm{rf}}) + \frac{1}{2}\sqrt{(1+\mu_{\mathrm{rf}})^2 + 4\nu_{\mathrm{tg}}^2}}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,. \tag{6.40c}$$

This is the case for which the difference between $\mathbf{f}_*^{\mathrm{q}}$ and $\mathbf{f}_*^{\mathrm{r}}$ is greatest. To get a feel for this difference, when $\mu_{\mathrm{rf}} = 0$ and $\nu_{\mathrm{tg}} = \sqrt{2}$ these become

$$\mathbf{f}_*^{\mathrm{q}} = \tfrac{1}{3}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,, \qquad \mathbf{f}_*^{\mathrm{r}} = \tfrac{1}{2}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,,$$

while when $\mu_{\mathrm{rf}} = 0$ and $\nu_{\mathrm{tg}} = \sqrt{6}$ these become

$$\mathbf{f}_*^{\mathrm{q}} = \tfrac{1}{7}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,, \qquad \mathbf{f}_*^{\mathrm{r}} = \tfrac{1}{3}\,\mathbf{V}^{-1}\widetilde{\mathbf{m}}\,.$$

Seven Lessons Learned

Here are some insights that we have gained.

1. The Sharpe ratio ν_{tg} and the caution coefficient χ play a large role in determining the optimal allocation. In particular, when $\chi \geq \nu_{\mathrm{tg}}$ the optimal allocation is entirely in risk-free assets.

2. The risk-free return μ_{rf} plays a role in determining the optimal allocation mainly through $\widetilde{\mathbf{m}}$.

3. For any choice of χ the maximizer for the quadratic objective is more conservative than the maximizer for the reasonable objective, which is more conservative than the maximizer for the parabolic objective.

4. The maximizer for a parabolic objective is agressive and will overbet when the Sharpe ratio ν_{tg} is not small.

Seven Lessons Learned

5. The maximizers for quadratic and reasonable objectives are close when the Sharpe ratio ν_{tg} is not large. As χ approaches ν_{tg}, the maximizers for the quadratic and reasonable objectives will become closer.

6. We will have greater confidence in the computed Sharpe ratio ν_{tg} when the tangency portfolio lies towards the "nose" of the efficient frontier. This translates into greater confidence in the maximizers for the quadratic and reasonable objectives.

7. Analyzing the maximizers for both the quadratic and reasonable objectives gave greater insights than analyzing each of them separately. Together they are *fortune's formulas*.