Portfolios that Contain Risky Assets 12: Growth Rate Mean and Variance Estimators

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Portfolios that Contain Risky Assets Part II: Stochastic Models

Growth Rate Mean and Variance Estimators

Introduction

The idea now is to treat the Markowitz portfolio associated with $\mathbf{f}$ *as a single risky asset that can be modeled by the IID process associated with the growth rate probability density* $p_{\mathbf{f}}(X)$ *given by*

$$p_{\mathbf{f}}(X) = q_{\mathbf{f}}\left(e^X - 1\right)\, e^X \,.$$

The mean γ and variance θ of X are given by

$$\gamma = \int X\, p_{\mathbf{f}}(X)\,\mathrm{d}X\,, \qquad \theta = \int (X-\gamma)^2 p_{\mathbf{f}}(X)\,\mathrm{d}X\,.$$

We know from our study of one risky asset that γ is a good proxy for reward, while $\sqrt{\theta}$ is a good proxy for risk. Therefore we would like to estimate γ and θ in terms of the estimators $\hat{\mu}$ and $\hat{\xi}$ that we studied earlier.

Moment and Cumulant Generating Functions

Estimators for γ and θ will be constructed from the positive function

$$M(\tau) = \mathrm{Ex}\Big(e^{\tau X}\Big) = \int e^{\tau X}\, p_{\mathsf{f}}(X)\, \mathrm{d}X\,.$$

We will assume $M(\tau)$ is defined for every τ in an open interval $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ that contains the interval $[0, 2]$. It can then be shown that $M(\tau)$ is infinitely differentiable over $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ with

$$M^{(m)}(\tau) = \mathrm{Ex}\Big(X^m\, e^{\tau X}\Big) = \int X^m\, e^{\tau X}\, p_{\mathsf{f}}(X)\, \mathrm{d}X\,.$$

We call $M(\tau)$ the *moment generating function* for X because, by setting $\tau = 0$ in the above expression, we see that the *moments* $\{\mathrm{Ex}(X^m)\}_{m=1}^{\infty}$ are generated from $M(\tau)$ by the formula

$$\mathrm{Ex}(X^m) = \int X^m p_{\mathsf{f}}(X)\, \mathrm{d}X = M^{(m)}(0)\,.$$

Moment and Cumulant Generating Functions

A related inifinitely differentiable function over $(\tau_{\rm mn}, \tau_{\rm mx})$ is

$$K(\tau) = \log(M(\tau)) = \log\Big(\mathrm{Ex}\Big(e^{\tau X}\Big)\Big) \,.$$

We call $K(\tau)$ the *cumulant generating function* because the *cumulants* $\{\kappa_m\}_{m=1}^{\infty}$ of X are generated by the formula $\kappa_m = K^{(m)}(0)$. We see that

$$K'(\tau) = \frac{\mathrm{Ex}\big(X\, e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} \,,$$

$$K''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^2 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} \,,$$

$$K'''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^3 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} \,,$$

$$K''''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^4 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} - 3K''(\tau)^2 \,.$$

Moment and Cumulant Generating Functions

By evaluating these at $\tau = 0$ we see that the first four cumulants of X are

$$\begin{aligned}
\kappa_1 &= K'(0) = \mathrm{Ex}(X) = \gamma\,,\\
\kappa_2 &= K''(0) = \mathrm{Ex}\big((X-\gamma)^2\big) = \theta\,,\\
\kappa_3 &= K'''(0) = \mathrm{Ex}\big((X-\gamma)^3\big)\,,\\
\kappa_4 &= K''''(0) = \mathrm{Ex}\big((X-\gamma)^4\big) - 3\theta^2\,.
\end{aligned}$$

These are respectively the mean, variance, skewness, and kurtosis.

Skewness measures an asymmetry in the tails of the distribution. It is positive or negative depending on whether the fatter tail is to the right or to the left respectively.

Kurtosis measures a balance between the tails and the center of the distribution. It is larger for distributions with greater weight in the tails than in the center.

Moment and Cumulant Generating Functions

Remark. The formulas

$$K'(\tau) = \frac{\mathrm{Ex}\big(X\, e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)}\,,$$

$$K''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^2 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)}\,,$$

$$K'''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^3 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)}\,,$$

$$K''''(\tau) = \frac{\mathrm{Ex}\big((X - K'(\tau))^4 e^{\tau X}\big)}{\mathrm{Ex}\big(e^{\tau X}\big)} - 3K''(\tau)^2\,,$$

show that $K'(\tau)$, $K''(\tau)$, $K'''(\tau)$, and $K''''(\tau)$ are the mean, variance, skewness, and kurtosis for the probability density $e^{\tau X} p_{\mathbf{f}}(X)/\mathrm{Ex}(e^{\tau X})$.

Moment and Cumulant Generating Functions

Remark. If X is normally distributed with mean γ and variance θ then

$$p_{\mathsf{f}}(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

A direct calculation then shows that

$$\begin{aligned} \mathrm{Ex}\left(e^{\tau X}\right) &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma)^2}{2\theta} + \tau X\right) \mathrm{d}X \\ &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma-\tau\theta)^2}{2\theta} + \tau\gamma + \tfrac{1}{2}\tau^2\theta\right) \mathrm{d}X \\ &= \exp\left(\tau\gamma + \tfrac{1}{2}\tau^2\theta\right) , \end{aligned}$$

whereby $K(\tau) = \log(\mathrm{Ex}(e^{\tau X})) = \tau\gamma + \frac{1}{2}\tau^2\theta$. *Hence, when X is normally distributed the skewness, kurtosis, and all higher-order cumulants vanish. Conversely, if all of these cumulants vanish then X is normally distributed.*

Moment and Cumulant Generating Functions

Remark. The cumulant generating function $K(\tau)$ is *strictly convex* over the interval $(\tau_{\rm mn}, \tau_{\rm mx})$ because $K''(\tau) > 0$.

Remark. We can also see that $K(\tau)$ is convex over $(\tau_{\rm mn}, \tau_{\rm mx})$ as follows. Let τ_0, $\tau_1 \in (\tau_{\rm mn}, \tau_{\rm mx})$. By applying the *Hölder inequality* with $p = \frac{1}{1-s}$ and $p^* = \frac{1}{s}$, we see that for every $s \in (0,1)$ we have

$$
\begin{aligned}
M((1-s)\tau_0 + s\tau_1)) &= \int e^{(1-s)\tau_0 X}\, e^{s\tau_1 X}\, p_{\mathbf{f}}(X)\,\mathrm{d}X \\
&\leq \left(\int e^{\tau_0 X} p_{\mathbf{f}}(X)\,\mathrm{d}X\right)^{1-s} \left(\int e^{\tau_1 X} p_{\mathbf{f}}(X)\,\mathrm{d}X\right)^{s} \\
&= M(\tau_0)^{1-s} M(\tau_1)^s \,.
\end{aligned}
$$

By taking the logarithm of this inequality we obtain

$$
K((1-s)\tau_0 + s\tau_1) \leq (1-s)K(\tau_0) + sK(\tau_1) \quad \text{for every } s \in (0,1)\,.
$$

Therefore $K(\tau)$ is a convex function over $(\tau_{\rm mn}, \tau_{\rm mx})$.

Estimators from Moment Generating Functions

We will now construct estimators for γ and θ by using the moment generating function

$$M(\tau) = \mathrm{Ex}\Big(e^{\tau X}\Big) .$$

Because $R = e^X - 1$ and $\mathrm{Ex}(e^X) = M(1)$, we have

$$\mu = \mathrm{Ex}(R) = M(1) - 1 .$$

Because $R - \mu = e^X - M(1)$ and $\mathrm{Ex}(e^{2X}) = M(2)$, we have

$$\xi = \mathrm{Ex}\Big((R-\mu)^2\Big) = M(2) - M(1)^2 .$$

These equations can be solved for $M(1)$ and $M(2)$ as

$$M(1) = 1 + \mu , \qquad M(2) = (1+\mu)^2 + \xi .$$

Therefore knowing μ and ξ is equivalent to knowing $M(1)$ and $M(2)$.

Estimators from Moment Generating Functions

Because $\mathrm{Ex}(X) = M'(0)$ and $\mathrm{Ex}(X^2) = M''(0)$, we see that

$$\begin{aligned}\gamma &= \mathrm{Ex}(X) = M'(0)\,,\\ \theta &= \mathrm{Ex}\big((X-\gamma)^2\big)\\ &= \mathrm{Ex}(X^2) - \gamma^2 = M''(0) - M'(0)^2\,.\end{aligned}$$

Because $M(0) = 1$, we construct an estimator of $M(\tau)$ by interpolating the values $M(0)$, $M(1)$, and $M(2)$ with a quadratic polynomial as

$$\begin{aligned}\widehat{M}(\tau) &= 1 + \tau\big(M(1) - 1\big) + \tau(\tau-1)\tfrac{1}{2}\big(M(2) - 2M(1) + 1\big)\\ &= 1 + \tau\mu + \tfrac{1}{2}\tau(\tau-1)\left(\mu^2 + \xi\right)\,.\end{aligned}$$

By direct calculation we see that

$$\widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,, \qquad \widehat{M}''(0) = \mu^2 + \xi\,.$$

Estimators from Moment Generating Functions

The idea is to now construct estimators for γ and θ by using

$$\widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,, \qquad \widehat{M}''(0) = \mu^2 + \xi\,, \tag{3.1}$$

as estimators for $M'(0)$ and $M''(0)$ in the formulas

$$\gamma = M'(0)\,, \qquad \theta = M''(0) - M'(0)^2\,.$$

We thereby construct estimators $\hat{\gamma}$ and $\hat{\theta}$ as functions of μ and ξ by

$$\hat{\gamma} = \widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,,$$
$$\hat{\theta} = \widehat{M}''(0) - \widehat{M}'(0)^2 = \mu^2 + \xi - \left(\mu - \tfrac{1}{2}(\mu^2 + \xi)\right)^2\,.$$

Estimators from Moment Generating Functions

By replacing the μ and ξ that appear in the foregoing estimators with the estimators

$$\hat{\mu} = \mu_{\rm rf}(1 - \mathbf{1}^{\rm T}\mathbf{f}) + \mathbf{m}^{\rm T}\mathbf{f}\,, \qquad \hat{\xi} = \frac{1}{1-\bar{w}}\,\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}\,. \tag{3.2a}$$

we obtain the estimators

$$\begin{aligned} \hat{\gamma} &= \hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right)\,, \\ \hat{\theta} &= \hat{\mu}^2 + \hat{\xi} - \left(\hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right)\right)^2\,, \end{aligned} \tag{3.2b}$$

The variance θ is generally positive, but the estimator $\hat{\theta}$ given above is not intrinsically positive.

Estimators from Moment Generating Functions

Expanding the above expression for $\hat{\theta}$ in powers of $\hat{\mu}$ and $\hat{\xi}$ yields

$$\hat{\theta} = \hat{\xi} + \hat{\mu}\left(\hat{\mu}^2 + \hat{\xi}\right) - \tfrac{1}{4}\left(\hat{\mu}^2 + \hat{\xi}\right)^2 .$$

The only term in this expansion that is intrinsically positive is the first one. *Therefore we make the smallness assumptions*

$$|\hat{\mu}| \ll 1 , \qquad \hat{\xi} \ll 1 , \qquad |\hat{\mu}|^3 \ll \xi ,$$

and keep only through quadratic statistics — i.e. through quadratic in $\hat{\mu}$ and linear in $\hat{\xi}$. We thereby arrive at the *quadratic estimators*

$$\hat{\gamma} = \hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right) , \qquad \hat{\theta} = \hat{\xi} , \tag{3.3}$$

where $\hat{\mu}$ and $\hat{\xi}$ are given by (3.2a).

Remark. These smallness assumptions are very easy to check.

Estimators from Moment Generating Functions

Remark. The quadratic estimators $\hat{\gamma}$ and $\hat{\theta}$ given by (3.3) have at least three potential sources of error:

- the estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$ used in (3.1) to approximate γ and θ as functions of μ and ξ,
- the estimators $\hat{\mu}$ and $\hat{\xi}$ used in (3.2a) to approximate μ and ξ,
- the smallness assumptions that lead to (3.3).

The derivation of the first estimators assumes that the returns for each Markowitz portfolio are described by a density $q_{\mathbf{f}}(\mathbf{R})$ that is narrow enough for some moment beyond the second to exist. All of these approximations should be examined carefully, especially when markets are highly volatile.

Estimators from Cumulant Generating Functions

We will now give an alternative derivation of quadratic estimators (3.3) that uses the cumulent generating function $K(\tau) = \log(M(\tau))$ and is based on the fact that $\gamma = K'(0)$ and $\theta = K''(0)$. It begins by observing that

$$K(1) = \log\big(M(1)\big) = \log(1+\mu)\,,$$
$$K(2) = \log\big(M(2)\big) = \log\Big((1+\mu)^2 + \xi\Big)\,.$$

Therefore knowing μ and ξ is equivalent to knowing $K(1)$ and $K(2)$.

Because $K(0) = 0$, we construct an estimator of $K(\tau)$ by interpolating the values $K(0)$, $K(1)$, and $K(2)$ with a quadratic polynomial as

$$\begin{aligned}\hat{K}(\tau) &= \tau K(1) + \tau(\tau-1)\tfrac{1}{2}\big(K(2) - 2K(1)\big)\\ &= \tau\log(1+\mu) + \tau(\tau-1)\tfrac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right)\,.\end{aligned}$$

Estimators from Cumulant Generating Functions

This yields the estimators

$$\hat{\gamma} = \hat{K}'(0) = \log(1+\mu) - \frac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right) ,$$

$$\hat{\theta} = \hat{K}''(0) = \log\left(1 + \frac{\xi}{(1+\mu)^2}\right) .$$

By replacing the μ and ξ that appear above with the estimators $\hat{\mu}$ and $\hat{\xi}$ given by (3.2a), we obtain the new estimators

$$\hat{\gamma} = \log(1+\hat{\mu}) - \frac{1}{2}\log\left(1 + \frac{\hat{\xi}}{(1+\hat{\mu})^2}\right) ,$$

$$\hat{\theta} = \log\left(1 + \frac{\hat{\xi}}{(1+\hat{\mu})^2}\right) .$$

So long as $1+\hat{\mu} > 0$ these estimators are well defined and $\hat{\theta}$ is positive.

Estimators from Cumulant Generating Functions

If $1 + \hat{\mu} > 0$ and we make the smallness assumption

$$\frac{\hat{\xi}}{(1+\hat{\mu})^2} \ll 1\,,$$

then we obtain the estimators

$$\hat{\gamma} = \log(1+\hat{\mu}) - \tfrac{1}{2}\frac{\hat{\xi}}{(1+\hat{\mu})^2}\,, \qquad \hat{\theta} = \frac{\hat{\xi}}{(1+\hat{\mu})^2}\,. \tag{4.4}$$

Finally, if we make the additional smallness assumptions

$$|\hat{\mu}| \ll 1\,, \qquad |\hat{\mu}|^3 \ll \hat{\xi}\,,$$

use the fact

$$\log(1+\hat{\mu}) = \hat{\mu} - \tfrac{1}{2}\hat{\mu}^2 + \tfrac{1}{3}\hat{\mu}^3 + \cdots\,,$$

and keep only through quadratic statistics then we obtain the *quadratic estimators* (3.3) derived earlier.

Estimators from Cumulant Generating Functions

Remark. *The fact that both derivations lead to the same estimators gives us greater confidence in the validity the quadratic estimators.*

Remark. If the Markowitz portfolio specified by $\mathbf{f}$ has growth rates X that are normally distributed with mean γ and variance θ then we have seen that $K(\tau) = \tau\gamma + \frac{1}{2}\tau^2\theta$. In this case we have $\hat{K}(\tau) = K(\tau)$, so the estimators $\hat{\gamma} = \hat{K}'(0)$ and $\hat{\theta} = \hat{K}''(0)$ are exact.

Remark. The biggest uncertainty associated with these estimators for $\hat{\gamma}$ and $\hat{\theta}$ is usually the uncertainty inherited from the estimators for $\hat{\mu}$ and $\hat{\xi}$.

Estimators from Cumulant Generating Functions

Exercise. When the quadratic estimators $\hat{\gamma}$ and $\hat{\theta}$ are applied to a single risky asset, they reduce to

$$\hat{\gamma} = \hat{\mu} - \tfrac{1}{2}\left(\hat{\mu}^2 + \hat{\xi}\right), \qquad \hat{\theta} = \hat{\xi}.$$

Use these to estimate γ and θ for each of the following assets given the share price history $\{s(d)\}_{d=0}^{D}$. How do these $\hat{\gamma}$ and $\hat{\theta}$ compare with the unbiased estimators for γ and θ that you obtained in the previous problem?

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Exercise. Compute $\hat{\gamma}$ and $\hat{\theta}$ based on daily data for the Markowitz portfolio with value equally distributed among the assets in each of the groups given in the previous exercise.

Interpolation Errors

Here we examine the errors of the interpolation-based estimators given by

$$\widehat{M}'(0) = 2\big(M(1) - 1\big) - \tfrac{1}{2}\big(M(2) - 1\big)\,,$$
$$\widehat{M}''(0) = M(2) - 2M(1) + 1\,.$$

Let $M(\tau)$ be any thrice continuously differentiable function over $[0,2]$ that satisfies $M(0) = 1$. The Cauchy form of the Taylor remainder then yields

$$M(1) = 1 + M'(0) + \tfrac{1}{2}M''(0) + \tfrac{1}{2}\int_0^1 (1-s)^2 M'''(s)\,\mathrm{d}s\,,$$
$$M(2) = 1 + 2M'(0) + 2M''(0) + \tfrac{1}{2}\int_0^2 (2-s)^2 M'''(s)\,\mathrm{d}s\,.$$

By placing these into the above formulas for $\widehat{M}'(0)$ and $\widehat{M}''(0)$ we obtain

$$\widehat{M}'(0) = M'(0) + E_1\,, \qquad \widehat{M}''(0) = M''(0) + E_2\,,$$

Interpolation Errors

where the errors E_1 and E_2 are given by

$$
\begin{aligned}
E_1 &= \left[\int_0^1 (1-s)^2 M'''(s)\,\mathrm{d}s - \tfrac{1}{4}\int_0^2 (2-s)^2 M'''(s)\,\mathrm{d}s \right] \\
&= -\left[\int_0^1 \left(s - \tfrac{3}{4}s^2\right) M'''(s)\,\mathrm{d}s + \tfrac{1}{4}\int_1^2 (2-s)^2 M'''(s)\,\mathrm{d}s \right], \\
E_2 &= \left[\tfrac{1}{2}\int_0^2 (2-s)^2 M'''(s)\,\mathrm{d}s - \int_0^1 (1-s)^2 M'''(s)\,\mathrm{d}s \right] \\
&= \left[\tfrac{1}{2}\int_1^2 (2-s)^2 M'''(s)\,\mathrm{d}s + \int_0^1 \left(1 - \tfrac{1}{2}s^2\right) M'''(s)\,\mathrm{d}s \right].
\end{aligned}
$$

Here the integrals seen in the second expression for each error are written so that the factor multiplying $M'''(s)$ inside each integral is nonnegative. This shows that if $M'''(s) \geq 0$ over $[0,2]$ then $E_1 < 0$ and $E_2 > 0$, while if $M'''(s) \leq 0$ over $[0,2]$ then $E_1 > 0$ and $E_2 < 0$.

Interpolation Errors

The errors E_1 and E_2 may be bounded in terms of

$$\|M'''\|_\infty = \max\left\{|M'''(\tau)| \;:\; \tau \in [0,2]\right\}.$$

Specifically, because

$$\int_0^1 \left(s - \tfrac{3}{4}s^2\right) \mathrm{d}s = \tfrac{1}{4}\,, \qquad \int_1^2 (2-s)^2 \,\mathrm{d}s = \tfrac{1}{3}\,,$$
$$\int_0^1 \left(1 - \tfrac{1}{2}s^2\right) \mathrm{d}s = \tfrac{5}{6}\,,$$

we obtain the bounds

$$|E_1| \leq \tfrac{1}{3}\|M'''\|_\infty\,, \qquad |E_2| \leq \|M'''\|_\infty\,.$$

Interpolation Errors

If we want to use these error bounds then we must find either a bound of or an approximation to $\|M'''\|_\infty$. From the definition of $M(\tau)$ we see that

$$M'''(\tau) = \mathrm{Ex}\big(X^3 e^{\tau X}\big) = \int X^3 e^{\tau X} p_{\mathsf{f}}(X)\,\mathrm{d}X\,.$$

Because

$$M''''(\tau) = \mathrm{Ex}\big(X^4 e^{\tau X}\big) = \int X^4 e^{\tau X} p_{\mathsf{f}}(X)\,\mathrm{d}X > 0\,,$$

we see that $M'''(\tau)$ is a strictly increasing function of τ.

Interpolation Errors

Because $M'''(\tau)$ is a strictly increasing function of τ we have

$$\|M'''\|_\infty = \max\{\, -M'''(0)\,,\; M'''(2)\}\,,$$

where the quantities $M'''(0)$ and $M'''(2)$ can be expressed in terms of the return density as

$$M'''(0) = \int_{-1}^{\infty} \left(\log(1+R)\right)^3 q_{\mathsf{f}}(R)\,\mathrm{d}R\,,$$
$$M'''(2) = \int_{-1}^{\infty} \left(\log(1+R)\right)^3 (1+R)^2 q_{\mathsf{f}}(R)\,\mathrm{d}R\,.$$

Interpolation Errors

These quantities can be approximated by the sample means

$$\widetilde{M'''}(0) = \sum_{d=1}^{D} w(d) \left(\log\left(1 + r(d)\right)\right)^3 ,$$

$$\widetilde{M'''}(2) = \sum_{d=1}^{D} w(d) \left(\log\left(1 + r(d)\right)\right)^3 \left(1 + r(d)\right)^2 ,$$

where $\{r(d)\}_{d=1}^{D}$ is the portfolio return history given by

$$r(d) = (1 - \mathbf{1}^{\mathrm{T}}\mathbf{f})\mu_{\mathrm{rf}} + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d) .$$

By arguing as we did for $M'''(\tau)$, we can show that $\widetilde{M'''}(0) < \widetilde{M'''}(2)$. Therefore we can approximate $\|M'''\|_\infty$ by

$$\|M'''\|_\infty \approx \max\{ -\widetilde{M'''}(0) , \widetilde{M'''}(2) \} .$$

Sample Uncertainties

We now turn our attention to the errors associated with the sample return mean and variance unbiased estimators

$$\hat{\mu} = \sum_{d=1}^{D} w(d) R(d) \,, \qquad \hat{\xi} = \sum_{d=1}^{D} \frac{w(d)}{1 - \bar{w}} \big(R(d) - \hat{\mu}\big)^2 \,.$$

Eariler we estimated how close $\hat{\mu}$ is to μ by computing its variance. We found that

$$\mathrm{Var}(\hat{\mu}) = \mathrm{Ex}\Big((\hat{\mu} - \mu)^2 \Big) = \bar{w}\, \xi \,, \qquad \text{where} \quad \bar{w} = \sum_{d=1}^{D} w(d)^2 \,.$$

This showed that $\hat{\mu}$ converges to μ like $\sqrt{\bar{w}}$ as $\bar{w} \to 0$.

Sample Uncertainties

Remark. The Cauchy inequality implies that

$$1 = \left(\sum_{d=1}^{D} 1 \cdot w(d) \right)^2 \leq \left(\sum_{d=1}^{D} 1^2 \right) \left(\sum_{d=1}^{D} w(d)^2 \right) = D\, \bar{w}\,.$$

This shows that for any weighting we have $\bar{w} \geq 1/D$. Therefore the variance is smallest for uniform weights when we have $w(d) = 1/D$.

Remark. For uniform weights the formula for $\mathrm{Var}(\hat{\mu})$ reduces to

$$\mathrm{Var}(\hat{\mu}) = \frac{1}{D}\, \xi\,.$$

Therefore $\hat{\mu}$ converges to μ like $1/\sqrt{D}$ as $D \to \infty$ for uniform weights.

Sample Uncertainties

The above considerations suggest that the uncertainties associated with the unbiased estimator $\hat{\mu}$ can be measured by

$$\left(\bar{w}\,\hat{\xi}\right)^{\frac{1}{2}} .$$

We can also estimate how close $\hat{\xi}$ is to ξ by computing its variance. To do this we will assume that the probability density $q_{\mathsf{f}}(R)$ has a finite fourth moment. Let ξ_4 be the centered fourth moment, which is given by

$$\xi_4 = \mathrm{Ex}\Big((R-\mu)^4\Big) = \int_{-D}^{\infty} (R-\mu)^4 q_{\mathsf{f}}(R)\,\mathrm{d}R < \infty .$$

Observe that by the strict Cauchy inequality we have

$$\xi_4 = \int_{-D}^{\infty} (R-\mu)^4 q_{\mathsf{f}}(R)\,\mathrm{d}R > \left(\int_{-D}^{\infty} (R-\mu)^2 q_{\mathsf{f}}(R)\,\mathrm{d}R\right)^2 = \xi^2 .$$

Sample Uncertainties

The first step is to let $\tilde{R}(d) = R(d) - \mu$ and express $\hat{\xi}$ as

$$\begin{aligned}\hat{\xi} &= \frac{1}{1-\bar{w}}\left(\sum_{d=1}^{D} w(d)\tilde{R}(d)^2 - (\hat{\mu}-\mu)^2\right) \\ &= \frac{1}{1-\bar{w}}\left(\sum_{d=1}^{D} w(d)\tilde{R}(d)^2 - \sum_{d_1=1}^{D}\sum_{d_2=1}^{D} w(d_1)w(d_2)\tilde{R}(d_1)\tilde{R}(d_2)\right) .\end{aligned}$$

Sample Uncertainties

By squaring this expression and relabeling some indices we obtain

$$
\begin{aligned}
\hat{\xi}^2 = & \sum_{d=1}^{D} \sum_{d'=1}^{D} \frac{w(d)w(d')}{(1-\bar{w})^2} \tilde{R}(d)^2 \tilde{R}(d')^2 \\
& - 2 \sum_{d=1}^{D} \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} \frac{w(d)w(d_1)w(d_2)}{(1-\bar{w})^2} \tilde{R}(d)^2 \tilde{R}(d_1) \tilde{R}(d_2) \\
& + \sum_{d_1=1}^{D} \sum_{d_2=1}^{D} \sum_{d_3=1}^{D} \sum_{d_4=1}^{D} \bigg(\frac{w(d_1)w(d_2)w(d_3)w(d_4)}{(1-\bar{w})^2} \\
& \qquad \cdot \tilde{R}(d_1)\tilde{R}(d_2)\tilde{R}(d_3)\tilde{R}(d_4) \bigg) .
\end{aligned}
$$

Sample Uncertainties

The next step is to compute $\mathrm{Ex}\left(\hat{\xi}^2\right)$, which requires us to compute

$$\mathrm{Ex}\left(\tilde{R}(d)^2\tilde{R}(d')^2\right), \qquad \mathrm{Ex}\left(\tilde{R}(d)^2\tilde{R}(d_1)\tilde{R}(d_2)\right),$$
$$\mathrm{Ex}\left(\tilde{R}(d_1)\tilde{R}(d_2)\tilde{R}(d_3)\tilde{R}(d_4)\right).$$

Let $\delta_{dd'}$ denote the Kronecker delta, which is defined by

$$\delta_{dd'} = \begin{cases} 1 & \text{if } d = d', \\ 0 & \text{if } d \neq d'. \end{cases}$$

Sample Uncertainties

Because $\tilde{R}(d)$ and $\tilde{R}(d')$ are independent when $d \neq d'$, and because $\mathrm{Ex}\Big(\tilde{R}(d)\Big) = 0$, $\mathrm{Ex}\Big(\tilde{R}(d)^2\Big) = \xi$, and $\mathrm{Ex}\Big(\tilde{R}(d)^4\Big) = \xi_4$, we see that

$$\begin{aligned}
\mathrm{Ex}\Big(\tilde{R}(d)^2\tilde{R}(d')^2\Big) &= \delta_{dd'}\,\xi_4 + (1-\delta_{dd'})\,\xi^2\,,\\
\mathrm{Ex}\Big(\tilde{R}(d)^2\tilde{R}(d_1)\tilde{R}(d_2)\Big) &= \delta_{d_1d_2}\Big(\delta_{dd_1}\,\xi_4 + (1-\delta_{dd_1})\,\xi^2\Big)\,,\\
\mathrm{Ex}\Big(\tilde{R}(d_1)\tilde{R}(d_2)\tilde{R}(d_3)\tilde{R}(d_4)\Big) &= \delta_{d_1d_2}\,\delta_{d_2d_3}\,\delta_{d_3d_4}\,\xi_4\\
&\quad + \delta_{d_1d_2}\,\delta_{d_3d_4}\,(1-\delta_{d_1d_3})\,\xi^2\\
&\quad + \delta_{d_1d_3}\,\delta_{d_4d_2}\,(1-\delta_{d_1d_4})\,\xi^2\\
&\quad + \delta_{d_1d_4}\,\delta_{d_2d_3}\,(1-\delta_{d_1d_2})\,\xi^2\,.
\end{aligned}$$

Sample Uncertainties

Then the expected value of the quantity $\hat{\xi}^2$ given three slides back is

$$\mathrm{Ex}\left(\hat{\xi}^2\right) = \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\xi_4 + \frac{1 - 3\bar{w} + 2\overline{w^2} + 3\bar{w}^2 - 3\overline{w^3}}{(1-\bar{w})^2}\,\xi^2\,,$$

where $\bar{w}$, $\overline{w^2}$, and $\overline{w^3}$ are given by

$$\bar{w} = \sum_{d=1}^{D} w(d)^2\,, \qquad \overline{w^2} = \sum_{d=1}^{D} w(d)^3\,, \qquad \overline{w^3} = \sum_{d=1}^{D} w(d)^4\,.$$

Sample Uncertainties

Therefore the variance of $\hat{\xi}$ is

$$
\begin{aligned}
\mathrm{Var}\big(\hat{\xi}\big) &= \mathrm{Ex}\Big((\hat{\xi}-\xi)^2\Big) = \mathrm{Ex}\Big(\hat{\xi}^2\Big) - \xi^2 \\
&= \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\xi_4 + \frac{-\bar{w} + 2\overline{w^2} + 2\bar{w}^2 - 3\overline{w^3}}{(1-\bar{w})^2}\,\xi^2 \\
&= \frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2}\,\big(\xi_4 - \xi^2\big) + 2\,\frac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2}\,\xi^2\,.
\end{aligned}
$$

Sample Uncertainties

Remark. For uniform weights this formula for $\mathrm{Var}(\hat{\xi})$ reduces to

$$\mathrm{Var}\left(\hat{\xi}\right) = \frac{1}{D}\left(\xi_4 - \xi^2\right) + \frac{2}{D(D-1)}\,\xi^2\,.$$

Therefore $\hat{\xi}$ converges to ξ like $1/\sqrt{D}$ as $D \to \infty$ for uniform weights. The coefficient in front of $(\xi_4 - \xi^2)$ above is the smallest possible because, by the Cauchy inequality, the general coefficient of $(\xi_4 - \xi^2)$ satisfies

$$\begin{aligned}
\frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2} &= \frac{1}{(1-\bar{w})^2}\sum_{d=1}^{D}\big(1 - w(d)\big)^2 w(d)^2 \\
&\geq \frac{1}{(1-\bar{w})^2}\,\frac{1}{D}\left(\sum_{d=1}^{D}\big(1 - w(d)\big)w(d)\right)^2 \\
&= \frac{1}{(1-\bar{w})^2}\,\frac{1}{D}\,(1-\bar{w})^2 = \frac{1}{D}\,.
\end{aligned}$$

Sample Uncertainties

In order to treat cases when the weights are not uniform it is useful to derive an upper bound for $\mathrm{Var}\left(\hat{\xi}\right)$ in which the coefficients of $(\xi_4 - \xi^2)$ and ξ^2 depend on $\bar{w}$ but not on $\overline{w^2}$ and $\overline{w^3}$.

Because the Jensen inequality implies that $\bar{w}^3 \leq \overline{w^3}$, the coefficient of ξ^2 can be bounded as

$$\frac{\bar{w}^2 - \overline{w^3}}{(1-\bar{w})^2} \leq \frac{\bar{w}^2 - \bar{w}^3}{(1-\bar{w})^2} = \frac{\bar{w}^2}{1-\bar{w}} \,.$$

Sample Uncertainties

The coefficient of $(\xi_4 - \xi^2)$ requires more work. It can be checked that $f(z) = z - 2z^2 + z^3$ is concave over $[0, \frac{2}{3}]$. Hence, when the weights $\{w(d)\}_{d=1}^{D}$ all lie in $[0, \frac{2}{3}]$ the Jensen inequality with $z(d) = w(d)$ yields

$$\overline{w - 2w^2 + w^3} = \overline{f(w)} \leq f(\bar{w}) = \bar{w} - 2\bar{w}^2 + \bar{w}^3 \,.$$

In that case the coefficient of $(\xi_4 - \xi^2)$ can be bounded as

$$\frac{\bar{w} - 2\overline{w^2} + \overline{w^3}}{(1-\bar{w})^2} \leq \frac{\bar{w} - 2\bar{w}^2 + \bar{w}^3}{(1-\bar{w})^2} = \bar{w} \,.$$

Therefore if every $w(d) \leq \frac{2}{3}$ then we obtain the upper bound

$$\mathrm{Var}\Big(\hat{\xi}\Big) \leq \bar{w}\,(\xi_4 - \xi^2) + \frac{2\bar{w}^2}{1-\bar{w}}\,\xi^2 \,.$$

This shows that $\hat{\xi}$ converges to ξ like $\sqrt{\bar{w}}$ as $\bar{w} \to 0$ for arbitrary weights. Moreover, the above inequality is an equality for uniform weights.

Sample Uncertainties

The foregoing considerations suggest that the uncertainties associated with the unbiased estimator $\hat{\xi}$ can be measured by

$$\left(\bar{w} \left(\hat{\xi}_4 - \hat{\xi}^2 \right) + \frac{2\bar{w}^2}{1 - \bar{w}} \hat{\xi}^2 \right)^{\frac{1}{2}} ,$$

where we choose to use the (biased) estimator of ξ_4 given by

$$\hat{\xi}_4 = \frac{1}{(1 - \bar{w})^2} \sum_{d=1}^{D} w(d) \big(R(d) - \hat{\mu} \big)^4 .$$

Comparing Uncertainties

Recall that the quadratic estimators $\hat{\gamma}$ and $\hat{\theta}$ given by (3.3) have at least three potential sources of error:

- the estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$ used in (3.1) to approximate γ and θ as functions of μ and ξ,
- the estimators $\hat{\mu}$ and $\hat{\xi}$ used in (3.2a) to approximate μ and ξ,
- the smallness assumptions that lead to (3.3).

Here we summarize how to assess these uncertainies.

Comparing Uncertainties

First, we just saw that the uncertainties associated with approximating $M'(0)$ and $M''(0)$ by $\widehat{M}'(0)$ and $\widehat{M}''(0)$ can be measured respectively by

$$\tfrac{1}{3}\max\{\,-\widetilde{M'''}(0)\,,\,\widetilde{M'''}(2)\}\,,\qquad \max\{\,-\widetilde{M'''}(0)\,,\,\widetilde{M'''}(2)\}\,,$$

where $\widetilde{M'''}(0)$ and $\widetilde{M'''}(2)$ are given by the sample means

$$\widetilde{M'''}(0) = \sum_{d=1}^{D} w(d)\,\big(\log(1+r(d))\big)^3\,,$$

$$\widetilde{M'''}(2) = \sum_{d=1}^{D} w(d)\,\big(\log(1+r(d))\big)^3\,\big(1+r(d)\big)^2\,.$$

Comparing Uncertainties

Second, earlier we saw that the uncertainties associated with approximating μ and ξ by $\hat{\mu}$ and $\hat{\xi}$ can be measured respectively by

$$\left(\bar{w}\,\hat{\xi}\right)^{\frac{1}{2}}\,, \qquad \left(\bar{w}\left(\hat{\xi}_4 - \hat{\xi}^2\right) + \frac{2\bar{w}^2}{1-\bar{w}}\,\hat{\xi}^2\right)^{\frac{1}{2}}\,,$$

where the estimator $\hat{\xi}_4$ is given by

$$\hat{\xi}_4 = \frac{1}{(1-\bar{w})^2}\sum_{d=1}^{D_h} w(d)\big(r(d) - \hat{\mu}\big)^4\,.$$

Comparing Uncertainties

Finally, the uncertainties associated with the smallness assumptions can be measured by

$$|\hat{\mu}| \,, \qquad \frac{|\hat{\mu}|^3}{\hat{\xi}} \,, \qquad \hat{\xi} \,.$$

While it is unclear which of these uncertainty measures will dominate for a given Markowitz portfolio, some general relationships are clear.