# Portfolios that Contain Risky Assets 10: Limited Portfolios with Risk-Free Assets

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# Portfolios that Contain Risky Assets
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# Limited Portfolios with Risk-Free Assets

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# Long Portfolios with a Safe Investment

We now consider investors who will not hold a short position in *any asset*. Such an investor will not borrow to invest in a risky asset, so *the safe investment is the only risky-free asset that we need to consider.*

We will use the capital allocation line construction to obtain the *efficient long frontier* for long portfolios that might include the safe investment. We assume that the long frontier for the risky assets has already been constructed, and is given by $\sigma = \sigma_{\mathrm{lf}}(\mu)$ for $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$, where

$$\mu_{\mathrm{mn}} = \min\{m_i \,:\, i = 1, \cdots, N\}\,,$$
$$\mu_{\mathrm{mx}} = \max\{m_i \,:\, i = 1, \cdots, N\}\,.$$

We will assume that $\mu_{\mathrm{si}} < \mu_{\mathrm{mx}}$, because otherwise the safe investment is more efficient than any portfolio of risky assets.

# Long Portfolios with a Safe Investment

The capital allocation line between the safe investment and the portfolio on the long frontier with return $\mu$ is the line segment in the $\sigma\mu$-plane between the points $(0, \mu_{\rm si})$ and $(\sigma_{\rm lf}(\mu), \mu)$, the slope of which is

$$\nu_{\rm ca}(\mu) = \frac{\mu - \mu_{\rm si}}{\sigma_{\rm lf}(\mu)} \,.$$

The efficient long frontier is obtained by first finding the capital allocation line with the greatest slope. In other words, we solve

$$\mu_{\rm st} = \arg\max\{\nu_{\rm ca}(\mu) \,:\, \mu \in [\mu_{\rm mn}, \mu_{\rm mx}]\} \,. \tag{1.1}$$

We set $\nu_{\rm st} = \nu_{\rm ca}(\mu_{\rm st})$ and $\sigma_{\rm st} = \sigma_{\rm lf}(\mu_{\rm st})$.

# Long Portfolios with a Safe Investment

Let us consider the maximization problem given in (1.1):

$$\mu_{\text{st}} = \arg\max\{\nu_{\text{ca}}(\mu) \;:\; \mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]\},$$

where

$$\nu_{\text{ca}}(\mu) = \frac{\mu - \mu_{\text{si}}}{\sigma_{\text{lf}}(\mu)}.$$

Recall that the function $\mu \mapsto \sigma_{\text{lf}}(\mu)$ is positive and continuous over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$. This implies that the function $\mu \mapsto \nu_{\text{ca}}(\mu)$ is continuous over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$, which implies that it has a maximum over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$. Because $\mu_{\text{si}} < \mu_{\text{mx}}$ we see that

$$\nu_{\text{ca}}(\mu_{\text{mx}}) = \frac{\mu_{\text{mx}} - \mu_{\text{si}}}{\sigma_{\text{lf}}(\mu_{\text{mx}})} > 0,$$

which implies that the maximum must be positive.

# Long Portfolios with a Safe Investment

Because the function $\mu \mapsto \sigma_{\text{lf}}(\mu)$ is strictly convex over $[\mu_{\text{mn}}, \mu_{\text{mx}}]$, the maximizer $\mu_{\text{st}}$ must be unique.

If $\mu_{\text{st}} < \mu_{\text{mx}}$ then the function $\mu \mapsto \sigma_{\text{lf}}(\mu)$ is increasing over $[\mu_{\text{st}}, \mu_{\text{mx}}]$ that maps onto $[\sigma_{\text{st}}, \sigma_{\text{mx}}]$, where $\sigma_{\text{mx}} = \sigma_{\text{lf}}(\mu_{\text{mx}})$.

Let $\sigma \mapsto \sigma_{\text{lf}}^{-1}(\sigma)$ denote the inverse function of $\mu \mapsto \sigma_{\text{lf}}(\mu)$ over $[\sigma_{\text{st}}, \sigma_{\text{mx}}]$. Then the efficient long frontier is then given by $\mu = \mu_{\text{ef}}(\sigma)$ where

$$\mu_{\text{ef}}(\sigma) = \begin{cases} \mu_{\text{si}} + \nu_{\text{st}}\sigma & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \sigma_{\text{lf}}^{-1}(\sigma) & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{lf}}(\mu_{\text{mx}})]\,. \end{cases} \tag{1.2}$$

# Long Portfolios with a Safe Investment

We will suppose that $\sigma_{\rm lf}'(\mu_{\rm mx}) > 0$ and that $\sigma_{\rm lf}'(\mu_{\rm mn}) \leq 0$, which is a common situation.

**Efficient Long Frontier.** The tangent line to the curve $\sigma = \sigma_{\rm lf}(\mu)$ at the point $(\sigma_{\rm mx}, \mu_{\rm mx})$ will intersect the $\mu$-axis at $\mu = \eta_{\rm mx}$ where

$$\eta_{\rm mx} = \mu_{\rm mx} - \frac{\sigma_{\rm lf}(\mu_{\rm mx})}{\sigma_{\rm lf}'(\mu_{\rm mx})} .$$

We will consider the cases $\mu_{\rm si} \geq \eta_{\rm mx}$ and $\mu_{\rm si} < \eta_{\rm mx}$ separately.

# Long Portfolios with a Safe Investment

For the case when $\mu_{\text{si}} \geq \eta_{\text{mx}}$ we will make the additional assumption that $\mu_{\text{si}} < \mu_{\text{mx}}$. Then the efficient long frontier is simply given by

$$\mu_{\text{ef}}(\sigma) = \mu_{\text{si}} + \frac{\mu_{\text{mx}} - \mu_{\text{si}}}{\sigma_{\text{mx}}}\,\sigma \qquad \text{for } \sigma \in [0, \sigma_{\text{mx}}]\,.$$

Our additional assumption states that there is at least one risky asset that has a return mean greater than the return for the safe investment. This is usually the case. If it is not, the formula for $\mu_{\text{ef}}(\sigma)$ can be modified by appealing to the capital allocation line construction.

**Remark.** Notice that $\mu_{\text{ef}}(\sigma)$ given above is increasing over $\sigma \in [0, \sigma_{\text{mx}}]$. When $\mu_{\text{si}} = \mu_{\text{mx}}$ the capital allocation line construction would produce an expression for $\mu_{\text{ef}}(\sigma)$ that is constant, but might be defined over an interval larger than $[0, \sigma_{\text{mx}}]$. When $\mu_{\text{si}} > \mu_{\text{mx}}$ the capital allocation line construction would produce an expression for $\mu_{\text{ef}}(\sigma)$ that is decreasing over an interval larger than $[0, \sigma_{\text{mx}}]$.

# Long Portfolios with a Safe Investment

For the case when $\mu_{\text{si}} < \eta_{\text{mx}}$ there is a frontier portfolio $(\sigma_{\text{st}}, \mu_{\text{st}})$ such that the capital allocation between it and $(0, \mu_{\text{si}})$ lies above the efficient long frontier. This means that $\mu_{\text{st}} > \mu_{\text{si}}$ and

$$\frac{\mu - \mu_{\text{si}}}{\mu_{\text{st}} - \mu_{\text{si}}}\,\sigma_{\text{st}} \leq \sigma_{\text{lf}}(\mu) \quad \text{for every } \mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]\,.$$

Because $\sigma_{\text{lf}}(\mu)$ is an increasing, continuous function over $[\mu_{\text{st}}, \mu_{\text{mx}}]$ with image $[\sigma_{\text{st}}, \sigma_{\text{mx}}]$, it has an increasing, continuous inverse function $\sigma_{\text{lf}}^{-1}(\sigma)$ over $[\sigma_{\text{st}}, \sigma_{\text{mx}}]$ with image $[\mu_{\text{st}}, \mu_{\text{mx}}]$. The efficient long frontier is then given by

$$\mu_{\text{ef}}(\sigma) = \begin{cases} \mu_{\text{si}} + \dfrac{\mu_{\text{st}} - \mu_{\text{si}}}{\sigma_{\text{st}}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \sigma_{\text{lf}}^{-1}(\sigma) & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{mx}}]\,. \end{cases}$$

# Long Portfolios with a Safe Investment

**Remark.** If we had also added a credit line to the portolio then we would have had to find the credit tangency portfolio and added the appropriate capital allocation line to the efficient long frontier. Typically there are two kinds of credit lines an investor might consider.

One available from your broker usually requires that some of your risky assets be held as collateral. A downside of using this kind of credit line is that when the market goes down then your broker can force you either to add assets to your collateral or to sell assets in a low market to pay off your loan.

# Long Portfolios with a Safe Investment

Another kind of credit line might use real estate as collateral. Of course, if the price of real estate falls then you again might be forced to sell assets in a low market to pay off your loan.

For investors who hold short positions in risky assets, these risks are hedged because they will also make money when markets go down. *Investors who hold only long positions in risky assets and use a credit line can find themselves highly exposed to large losses in a market downturn.* It is not a wise position to take — yet many do in a bubble.

# General Portfolio with Two Risky Assets

Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Here we will assume that $m_1 < m_2$, so that $\mu_{\mathrm{mn}} = m_1$ and $\mu_{\mathrm{mx}} = m_2$.

The minimum volatility portfolio is

$$\mathbf{f}_{\mathrm{mv}} = \frac{1}{v_{11} + v_{22} - 2v_{12}} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix}.$$

This will be a long portfolio if and only if $v_{12} \leq v_{11}$ and $v_{12} \leq v_{22}$.

# General Portfolio with Two Risky Assets

The long frontier is given by

$$\sigma_{\mathrm{lf}}(\mu) = \sqrt{\sigma_{\mathrm{mv}}^2 + \left(\frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}}\right)^2} \qquad \text{for } \mu \in [m_1, m_2]\,,$$

where the frontier parameters are

$$\sigma_{\mathrm{mv}} = \sqrt{\frac{v_{11}v_{22} - v_{12}^2}{v_{11} + v_{22} - 2v_{12}}}\,, \qquad \nu_{\mathrm{as}} = \sqrt{\frac{(m_2 - m_1)^2}{v_{11} + v_{22} - 2v_{12}}}\,,$$

$$\mu_{\mathrm{mv}} = \frac{(v_{22} - v_{12})m_1 + (v_{11} - v_{12})m_2}{v_{11} + v_{22} - 2v_{12}}\,.$$

Recall that $v_{11} + v_{22} - 2v_{12} > 0$ because $\mathbf{V}$ is positive definite.

## General Portfolio with Two Risky Assets

It follows that

$$\mu_{\rm mv} - m_1 = \frac{(v_{11} - v_{12})(m_2 - m_1)}{v_{11} + v_{22} - 2v_{12}},$$

$$m_2 - \mu_{\rm mv} = \frac{(v_{22} - v_{12})(m_2 - m_1)}{v_{11} + v_{22} - 2v_{12}}.$$

There are three cases to consider.

**Case 1.** If $v_{12} > v_{11}$ then $v_{12} < v_{22}$ and

$$\mu_{\rm mv} < m_1 < m_2 .$$

In this case the efficient long frontier is given by

$$(\sigma_{\rm lf}(\mu), \mu) \quad \text{where} \quad \mu \in [m_1, m_2] .$$

In other words, the entire long frontier is efficient.

# General Portfolio with Two Risky Assets

**Case 2.** If $v_{12} \le v_{11}$ and $v_{12} \le v_{22}$ then $\mathbf{f}_{\mathrm{mv}} \ge \mathbf{0}$ and

$$m_1 \le \mu_{\mathrm{mv}} \le m_2 \,.$$

This will be the case when $v_{12} \le 0$, which is how two-asset portfolios are often built. In this case the efficient long frontier is given by

$$(\sigma_{\mathrm{lf}}(\mu), \mu) \quad \text{where} \quad \mu \in [\mu_{\mathrm{mv}}, m_2] \,.$$

In other words, only part of the long frontier is efficient.

**Case 3.** If $v_{12} > v_{22}$ then $v_{12} < v_{11}$ and

$$m_1 < m_2 < \mu_{\mathrm{mv}} \,.$$

In this case the efficient long frontier is the single point $(\sigma_2, m_2)$.

# General Portfolio with Two Risky Assets

Now we show how Case 2 is modified by the inclusion of a safe investment. The $\mu$-intercept of the tangent line through $(\sigma_{\rm mx}, \mu_{\rm mx}) = (\sigma_2, m_2)$ is

$$\begin{aligned}\eta_{\rm mx} &= \mu_{\rm mx} - \frac{\sigma_{\rm lf}(\mu_{\rm mx})}{\sigma_{\rm lf}'(\mu_{\rm mx})} \\ &= m_2 - \frac{\nu_{\rm as}^2\,\sigma_2^2}{m_2 - \mu_{\rm mv}} = \frac{v_{22}m_1 - v_{12}m_2}{v_{22} - v_{12}}\,.\end{aligned}$$

We will present the two cases that arise in order of increasing complexity: $\eta_{\rm mx} \leq \mu_{\rm si}$ and $\mu_{\rm si} < \eta_{\rm mx}$.

When $\eta_{\rm mx} \leq \mu_{\rm si}$ the efficient long frontier is determined by

$$\mu_{\rm ef}(\sigma) = \mu_{\rm si} + \frac{m_2 - \mu_{\rm si}}{\sigma_2}\,\sigma \quad \text{for } \sigma \in [0, \sigma_2]\,.$$

# General Portfolio with Two Risky Assets

When $\mu_{\rm si} < \eta_{\rm mx}$ the tangency portfolio parameters are given by

$$\nu_{\rm st} = \nu_{\rm mv} \sqrt{1 + \left(\frac{\mu_{\rm mv} - \mu_{\rm si}}{\nu_{\rm as}\,\sigma_{\rm mv}}\right)^2}\,, \qquad \sigma_{\rm st} = \sigma_{\rm mv} \sqrt{1 + \left(\frac{\nu_{\rm as}\,\sigma_{\rm mv}}{\mu_{\rm mv} - \mu_{\rm si}}\right)^2}\,,$$

and the efficient long frontier is determined by

$$\mu_{\rm ef}(\sigma) = \begin{cases} \mu_{\rm si} + \nu_{\rm st}\,\sigma & \text{for } \sigma \in [0, \sigma_{\rm st}]\,, \\ \mu_{\rm mv} + \nu_{\rm as}\sqrt{\sigma^2 - \sigma_{\rm mv}^2} & \text{for } \sigma \in [\sigma_{\rm st}, \sigma_2]\,. \end{cases}$$

# Simple Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m-d \\ m \\ m+d \end{pmatrix}, \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

The efficient long frontier associated with just these three risky assets is given by $(\sigma_{\mathrm{lf}}(\mu), \mu)$ where $\mu \in [m, m+d]$ and

$$\sigma_{\mathrm{lf}}(\mu) = \begin{cases} s\sqrt{\dfrac{1+2r}{3} + \dfrac{1-r}{2}\left(\dfrac{\mu-m}{d}\right)^2} & \text{for } \mu \in [m, m+\frac{2}{3}d], \\ s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\dfrac{\mu-m-\frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m+\frac{2}{3}d, m+d]. \end{cases}$$

We now show how this is modified by including a safe investment.

# Simple Portfolio with Three Risky Assets

In the construction of $\sigma_{\mathrm{lf}}(\mu)$ we found that

$$\bar{\mu}_0 = m\,, \qquad \bar{\mu}_1 = m + \tfrac{2}{3}d\,, \qquad \bar{\mu}_2 = \mu_{\mathrm{mx}} = m + d\,,$$

$$\bar{\sigma}_0 = s\sqrt{\frac{1+2r}{3}}\,, \qquad \bar{\sigma}_1 = s\sqrt{\frac{5+4r}{9}}\,, \qquad \bar{\sigma}_2 = \sigma_{\mathrm{mx}} = s\,.$$

The frontier parameters for $\sigma_{\mathrm{f}_0}(\mu)$ were

$$\sigma_{\mathrm{mv}_0} = s\sqrt{\frac{1+2r}{3}}\,, \qquad \mu_{\mathrm{mv}_0} = m\,, \qquad \nu_{\mathrm{as}_0} = \frac{d}{s}\sqrt{\frac{2}{1-r}}\,,$$

while those for $\sigma_{\mathrm{f}_1}(\mu)$ were

$$\sigma_{\mathrm{mv}_1} = s\sqrt{\frac{1+r}{2}}\,, \qquad \mu_{\mathrm{mv}_1} = m + \tfrac{1}{2}d\,, \qquad \nu_{\mathrm{as}_1} = \frac{d}{2s}\sqrt{\frac{2}{1-r}}\,.$$

# Simple Portfolio with Three Risky Assets

Because

$$\sigma_{\rm lf}(\mu)\sigma_{\rm lf}'(\mu) = s^2\frac{1-r}{2}\begin{cases} \dfrac{\mu - m}{d^2}\,, & \text{for } \mu \in [m, m + \frac{2}{3}d]\,, \\ \dfrac{\mu - m - \frac{1}{2}d}{\frac{1}{4}d^2}\,, & \text{for } \mu \in [m + \frac{2}{3}d, m + d]\,, \end{cases}$$

we can see that the $\mu$-intercepts of the tangent lines through the points $(\overline{\sigma}_1, \overline{\mu}_1)$ and $(\overline{\sigma}_2, \overline{\mu}_2) = (\sigma_{\rm mx}, \mu_{\rm mx})$ are respectively

$$\overline{\eta}_1 = \overline{\mu}_1 - \frac{\sigma_{\rm lf}(\overline{\mu}_1)}{\sigma_{\rm lf}'(\overline{\mu}_1)} = m + \tfrac{2}{3}d - \frac{5+4r}{3-3r}\,d = m - \frac{1+2r}{1-r}\,d\,,$$

$$\eta_{\rm mx} = \mu_{\rm mx} - \frac{\sigma_{\rm lf}(\mu_{\rm mx})}{\sigma_{\rm lf}'(\mu_{\rm mx})} = m + d - \frac{1}{1-r}\,d = m - \frac{r}{1-r}\,d\,.$$

We will present the three cases that arise in order of increasing complexity: $\eta_{\rm mx} \leq \mu_{\rm si}$, $\overline{\eta}_1 \leq \mu_{\rm si} < \eta_{\rm mx}$, and $\mu_{\rm si} < \overline{\eta}_1$.

# Simple Portfolio with Three Risky Assets

When $\eta_{\text{mx}} \leq \mu_{\text{si}}$ the efficient long frontier is determined by

$$\mu_{\text{ef}}(\sigma) = \mu_{\text{si}} + \frac{\mu_{\text{mx}} - \mu_{\text{si}}}{\sigma_{\text{mx}}}\,\sigma \quad \text{for } \sigma \in [0, \sigma_{\text{mx}}]\,.$$

When $\overline{\eta}_1 \leq \mu_{\text{si}} < \eta_{\text{mx}}$ the tangency portfolio parameters are given by

$$\nu_{\text{st}} = \nu_{\text{as}_1}\sqrt{1 + \left(\frac{\mu_{\text{mv}_1} - \mu_{\text{si}}}{\nu_{\text{as}_1}\,\sigma_{\text{mv}_1}}\right)^2}\,, \quad \sigma_{\text{st}} = \sigma_{\text{mv}_1}\sqrt{1 + \left(\frac{\nu_{\text{as}_1}\,\sigma_{\text{mv}_1}}{\mu_{\text{mv}_1} - \mu_{\text{si}}}\right)^2}\,,$$

and the efficient long frontier is determined by

$$\mu_{\text{ef}}(\sigma) = \begin{cases} \mu_{\text{si}} + \nu_{\text{st}}\,\sigma & \text{for } \sigma \in [0, \sigma_{\text{st}}]\,, \\ \mu_{\text{mv}_1} + \nu_{\text{as}_1}\sqrt{\sigma^2 - \sigma_{\text{mv}_1}^2} & \text{for } \sigma \in [\sigma_{\text{st}}, \sigma_{\text{mx}}]\,. \end{cases}$$

# Simple Portfolio with Three Risky Assets

When $\mu_{\rm si} < \overline{\eta}_1$ the tangency portfolio parameters are given by

$$\nu_{\rm st} = \nu_{\rm as_0} \sqrt{1 + \left( \frac{\mu_{\rm mv_0} - \mu_{\rm si}}{\nu_{\rm as_0}\, \sigma_{\rm mv_0}} \right)^2} \,, \qquad \sigma_{\rm st} = \sigma_{\rm mv_0} \sqrt{1 + \left( \frac{\nu_{\rm as_0}\, \sigma_{\rm mv_0}}{\mu_{\rm mv_0} - \mu_{\rm si}} \right)^2} \,,$$

and the efficient long frontier is determined by

$$\mu_{\rm ef}(\sigma) = \begin{cases} \mu_{\rm si} + \nu_{\rm st}\, \sigma & \text{for } \sigma \in [0, \sigma_{\rm st}] \,, \\ \mu_{\rm mv_0} + \nu_{\rm as_0} \sqrt{\sigma^2 - \sigma_{\rm mv_0}^2} & \text{for } \sigma \in [\sigma_{\rm st}, \overline{\sigma}_1] \,, \\ \mu_{\rm mv_1} + \nu_{\rm as_1} \sqrt{\sigma^2 - \sigma_{\rm mv_1}^2} & \text{for } \sigma \in [\overline{\sigma}_1, \sigma_{\rm mx}] \,. \end{cases}$$

**Remark.** The above formulas for $\mu_{\rm ef}(\sigma)$ can be made more explicit by replacing $\sigma_{\rm mv_0}$, $\mu_{\rm mv_0}$, $\nu_{\rm as_0}$, $\sigma_{\rm mv_1}$, $\mu_{\rm mv_1}$, $\nu_{\rm as_1}$, $\sigma_{\rm mx}$, $\mu_{\rm mx}$, and $\overline{\sigma}_1$ with their explicit expressions in terms of $m$, $d$, $s$ and $r$.