Portfolios that Contain Risky Assets 8: Markowitz Frontiers for Limited Portfolios

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Portfolios that Contain Risky Assets
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Markowitz Frontiers for Limited Portfolios

Limited-Leverage Constraints

Recall that Π_ℓ is the set of all limited-leverage portfolio allocations with leverage limit $\ell \geq 0$ and that $\Pi_\ell(\mu)$ is the set of all such allocations with return mean μ. These sets are given by

$$\Pi_\ell = \left\{ \, \mathbf{f} \in \mathbb{R}^N \; : \; \|\mathbf{f}\|_1 \leq 1 + 2\ell \, , \; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \, \right\},$$
$$\Pi_\ell(\mu) = \left\{ \, \mathbf{f} \in \Pi_\ell \; : \; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \, \right\}.$$

Clearly $\Pi_\ell(\mu) \subset \Pi_\ell$ for every $\mu \in \mathbb{R}$.

The set Π_ℓ is a convex polytope of dimension $N-1$ that is contained in the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \; : \; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\}$. The set $\Pi_\ell(\mu)$ is the intersection of Π_ℓ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \; : \; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. Because we have assumed that $\mathbf{m}$ and $\mathbf{1}$ are not proportional, the intersection of these hyperplanes is a set of dimension $N-2$. Therefore the set $\Pi_\ell(\mu)$ is a convex polytope of dimension at most $N-2$, but it might be empty.

Limited-Leverage Constraints

We start by charactering those μ for which $\Pi_\ell(\mu)$ is nonempty. Recall that

$$\mu_{\rm mn} = \min\{m_i \,:\, i = 1, \cdots, N\}\,, \quad \mu_{\rm mx} = \max\{m_i \,:\, i = 1, \cdots, N\}\,.$$

We expect that the ℓ-limited leverage portfolio with the highest mean return would have a long allocation of $1 + \ell$ in an asset with mean return $\mu_{\rm mx}$ and short allocation of $-\ell$ in an asset with mean return $\mu_{\rm mn}$. The mean return of such a portfolio is

$$\mu^\ell_{\rm mx} = (1 + \ell)\mu_{\rm mx} - \ell\mu_{\rm mn} = \mu_{\rm mx} + \ell(\mu_{\rm mx} - \mu_{\rm mn})\,.$$

Similarly, we expect that the ℓ-limited leverage portfolio with the lowest mean return would have a long allocation of $1 + \ell$ in an asset with mean return $\mu_{\rm mn}$ and short allocation of $-\ell$ in an asset with mean return $\mu_{\rm mx}$. The mean return of such a portfolio is

$$\mu^\ell_{\rm mn} = (1 + \ell)\mu_{\rm mn} - \ell\mu_{\rm mx} = \mu_{\rm mn} - \ell(\mu_{\rm mx} - \mu_{\rm mn})\,.$$

Limited-Leverage Constraints

Indeed, we will prove the following.

Fact. *For every* $\ell \geq 0$ *the set* $\Pi_\ell(\mu)$ *is nonempty if and only if* $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$, *where*

$$\mu_{\mathrm{mn}}^{\ell} = \mu_{\mathrm{mn}} - \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,, \qquad \mu_{\mathrm{mx}}^{\ell} = \mu_{\mathrm{mx}} + \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\,.$$

Remark. Because we have assumed that $\mathbf{m}$ and $\mathbf{1}$ are not proportional, the mean returns $\{m_i\}_{i=1}^{N}$ are not identical. This implies that $\mu_{\mathrm{mn}} < \mu_{\mathrm{mx}}$, which implies that the interval $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ does not reduce to a point. Indeed, when $\ell_2 > \ell_1 \geq 0$ we have

$$\mu_{\mathrm{mn}}^{\ell_2} < \mu_{\mathrm{mn}}^{\ell_1} < \mu_{\mathrm{mx}}^{\ell_1} < \mu_{\mathrm{mx}}^{\ell_2}\,.$$

Limited-Leverage Constraints

Proof. Let $\Pi_\ell(\mu)$ be nonempty for some $\mu \in \mathbb{R}$. Let $\mathbf{f} \in \Pi_\ell(\mu)$ and let $\mathbf{f} = \mathbf{f}_+ - \mathbf{f}_-$ be the long-short decomposistion of $\mathbf{f}$. Because $\mu_{\rm mn}\mathbf{1} \leq \mathbf{m} \leq \mu_{\rm mx}\mathbf{1}$, and because $\mathbf{f}_\pm \geq \mathbf{0}$, we have

$$\mu_{\rm mn}\mathbf{1}^{\rm T}\mathbf{f}_\pm \leq \mathbf{m}^{\rm T}\mathbf{f}_\pm \leq \mu_{\rm mx}\mathbf{1}^{\rm T}\mathbf{f}_\pm \,.$$

Because $\mathbf{1}^{\rm T}\mathbf{f} = 1$ we have

$$\mathbf{1}^{\rm T}\mathbf{f}_+ = \mathbf{1}^{\rm T}(\mathbf{f} + \mathbf{f}_-) = \mathbf{1}^{\rm T}\mathbf{f} + \mathbf{1}^{\rm T}\mathbf{f}_- = 1 + \mathbf{1}^{\rm T}\mathbf{f}_- \,.$$

Because $\mathbf{f} \in \Pi_\ell$ we have $\mathbf{1}^{\rm T}\mathbf{f}_- \leq \ell$. These facts combine to give

$$\begin{aligned} \mu = \mathbf{m}^{\rm T}\mathbf{f} = \mathbf{m}^{\rm T}\mathbf{f}_+ - \mathbf{m}^{\rm T}\mathbf{f}_- &\leq \mu_{\rm mx}\mathbf{1}^{\rm T}\mathbf{f}_+ - \mu_{\rm mn}\mathbf{1}^{\rm T}\mathbf{f}_- \\ &= \mu_{\rm mx} + (\mu_{\rm mx} - \mu_{\rm mn})\mathbf{1}^{\rm T}\mathbf{f}_- \\ &\leq \mu_{\rm mx} + \ell(\mu_{\rm mx} - \mu_{\rm mn}) = \mu_{\rm mx}^\ell \,. \end{aligned}$$

Limited-Leverage Constraints

Similarly,

$$\begin{aligned}\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mathbf{m}^{\mathrm{T}}\mathbf{f}_{+} - \mathbf{m}^{\mathrm{T}}\mathbf{f}_{-} &\geq \mu_{\mathrm{mn}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_{+} - \mu_{\mathrm{mx}}\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \\ &= \mu_{\mathrm{mn}} - (\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})\mathbf{1}^{\mathrm{T}}\mathbf{f}_{-} \\ &\geq \mu_{\mathrm{mn}} - \ell(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}) = \mu_{\mathrm{mn}}^{\ell} \,.\end{aligned}$$

Therefore $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$. Because $\mathbf{f} \in \Pi_{\ell}(\mu)$ was arbitrary, we conclude that *if $\Pi_{\ell}(\mu)$ is nonempty then $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$.*

Conversely, first choose $\mathbf{e}_{\mathrm{mn}}$ and $\mathbf{e}_{\mathrm{mx}}$ so that

$$\begin{aligned}\mathbf{e}_{\mathrm{mn}} &= \mathbf{e}_i \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\mathrm{mn}} \,, \\ \mathbf{e}_{\mathrm{mx}} &= \mathbf{e}_j \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\mathrm{mx}} \,.\end{aligned}$$

Limited-Leverage Constraints

Now let $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$ and set

$$\mathbf{f} = \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mathbf{e}_{\mathrm{mx}} \,.$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} = \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} = 1$, $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} = \mu_{\mathrm{mn}}$, and $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} = \mu_{\mathrm{mx}}$, we see

$$\begin{aligned}
\mathbf{1}^{\mathrm{T}}\mathbf{f} &= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} \\
&= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}} = 1 \,, \\
\mathbf{m}^{\mathrm{T}}\mathbf{f} &= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mathbf{m}^{\mathrm{T}}\mathbf{e}_{\mathrm{mx}} \\
&= \frac{\mu_{\mathrm{mx}} - \mu}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mu_{\mathrm{mn}} + \frac{\mu - \mu_{\mathrm{mn}}}{\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}}}\, \mu_{\mathrm{mx}} = \mu \,.
\end{aligned}$$

Hence, $\mathbf{f} \in \Pi_{\infty}(\mu)$. We still need to show that $\mathbf{f} \in \Pi_{\ell}(\mu)$.

Limited-Leverage Constraints

Because $\mu \in [\mu_{\rm mn}^{\ell}, \mu_{\rm mx}^{\ell}]$ the allocations of $\mathbf{f}$ are bounded by

$$-\ell = \frac{\mu_{\rm mx} - \mu_{\rm mx}^{\ell}}{\mu_{\rm mx} - \mu_{\rm mn}} \leq \frac{\mu_{\rm mx} - \mu}{\mu_{\rm mx} - \mu_{\rm mn}} \leq \frac{\mu_{\rm mx} - \mu_{\rm mn}^{\ell}}{\mu_{\rm mx} - \mu_{\rm mn}} = 1 + \ell \,,$$

$$-\ell = \frac{\mu_{\rm mn}^{\ell} - \mu_{\rm mn}}{\mu_{\rm mx} - \mu_{\rm mn}} \leq \frac{\mu - \mu_{\rm mn}}{\mu_{\rm mx} - \mu_{\rm mn}} \leq \frac{\mu_{\rm mx}^{\ell} - \mu_{\rm mn}}{\mu_{\rm mx} - \mu_{\rm mn}} = 1 + \ell \,.$$

Because they sum to 1, at most one of them is negative. Hence,

$$\mathbf{1}^{\rm T}\mathbf{f}_{-} \leq \max\left\{ -\frac{\mu_{\rm mx} - \mu}{\mu_{\rm mx} - \mu_{\rm mn}} \,,\, -\frac{\mu - \mu_{\rm mn}}{\mu_{\rm mx} - \mu_{\rm mn}} \right\} \leq \ell \,.$$

Hence, $\mathbf{f} \in \Pi_{\ell}(\mu)$. *Therefore if $\mu \in [\mu_{\rm mn}^{\ell}, \mu_{\rm mx}^{\ell}]$ then $\Pi_{\ell}(\mu)$ is nonempty.* □

Limited-Leverage Constraints

Remark. For every $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$ the set $\Pi_{\ell}(\mu)$ *is a nonempty, closed, bounded, convex polytope of dimension at most* $N - 2$.

- When $N = 2$ it is a point.
- When $N = 3$ it is either a point or a line segment.
- When $N = 4$ it is either a point, a line segment, or a convex polygon.

Limited-Leverage Frontiers

The set Π_ℓ in $\mathbb{R}^N$ of all portfolio allocations with leverage limit ℓ is associated with the set $\Sigma(\Pi_\ell)$ in the $\sigma\mu$-plane of volatilities and return means given by

$$\Sigma(\Pi_\ell) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \,:\, \sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,,\; \mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,,\; \mathbf{f} \in \Pi_\ell \right\} .$$

The set $\Sigma(\Pi_\ell)$ is the image in $\mathbb{R}^2$ of the polytope Π_ℓ in $\mathbb{R}^N$ under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set Π_ℓ is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set $\Sigma(\Pi_\ell)$ is compact.

Limited-Leverage Frontiers

We have seen that the set $\Pi_\ell(\mu)$ of all ℓ-limited portfolio allocations with return mean μ is nonempty if and only if $\mu \in [\mu^\ell_{\rm mn}, \mu^\ell_{\rm mx}]$. Hence, $\Sigma(\Pi_\ell)$ can be expressed as

$$\Sigma(\Pi_\ell) = \left\{ \left(\sqrt{\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}}\,,\ \mu \right) \,:\, \mu \in [\mu^\ell_{\rm mn}, \mu^\ell_{\rm mx}]\,,\ \mathbf{f} \in \Pi_\ell(\mu) \right\}.$$

The points on the boundary of $\Sigma(\Pi_\ell)$ that correspond to those ℓ-limited portfolios that have less volatility than every other ℓ-limited portfolio with the same return mean is called the *ℓ-limited frontier.*

Limited-Leverage Frontiers

The ℓ-limited frontier is the curve in the $\sigma\mu$-plane given by the equation

$$\sigma = \sigma_{\mathrm{f}}^{\ell}(\mu) \quad \text{over} \quad \mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]\,,$$

where the value of $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is obtained for each $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ by solving the constrained minimization problem

$$\sigma_{\mathrm{f}}^{\ell}(\mu)^2 = \min\left\{ \sigma^2 \;:\; (\sigma,\mu) \in \Sigma(\Pi_\ell) \right\} = \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Pi_\ell(\mu) \right\}.$$

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is continuous over the compact set $\Pi_\ell(\mu)$, *a minimizer exists.*

Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Pi_\ell(\mu)$, whereby *the minimizer is unique.*

Limited-Leverage Frontiers

If we denote this unique minimizer by $\mathbf{f}_{\rm f}^{\ell}(\mu)$ then for every $\mu \in [\mu_{\rm mn}^{\ell}, \mu_{\rm mx}^{\ell}]$ the function $\sigma_{\rm f}^{\ell}(\mu)$ is given by

$$\sigma_{\rm f}^{\ell}(\mu) = \sqrt{\mathbf{f}_{\rm f}^{\ell}(\mu)^{\rm T}\mathbf{V}\mathbf{f}_{\rm f}^{\ell}(\mu)}\,,$$

where $\mathbf{f}_{\rm f}^{\ell}(\mu)$ is

$$\mathbf{f}_{\rm f}^{\ell}(\mu) = \arg\min\left\{\ \tfrac{1}{2}\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}\ :\ \mathbf{f} \in \Pi_{\ell}(\mu)\ \right\}.$$

Here $\arg\min$ is read *"the argument that minimizes".* It means that $\mathbf{f}_{\rm f}^{\ell}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2}\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}$ subject to the given constraints.

Remark. This problem cannot be solved by Lagrange multipliers because the set $\Pi_{\ell}(\mu)$ is defined by inequality constraints. It is harder to solve analytically than the analogous minimization problem for portfolios with unlimited leverage. Therefore we will first present a numerical approach that can generally be applied.

Quadratic Programming

Because the function being minimized is quadratic in $\mathbf{f}$ while the constraints are linear in $\mathbf{f}$, this is called a *quadratic programming problem*. It can be solved for a particular $\mathbf{V}$, $\mathbf{m}$, and μ by using either the Matlab command "quadprog" or an equivalent command in some other language.

The Matlab command quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\text{eq}}, \mathbf{d}_{\text{eq}})$ returns the solution of a quadratic programming problem in the standard form

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} \;:\; \mathbf{x} \in \mathbb{R}^{M},\ \mathbf{C}\mathbf{x} \leq \mathbf{d},\ \mathbf{C}_{\text{eq}}\mathbf{x} = \mathbf{d}_{\text{eq}} \right\},$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^{M}$, $\mathbf{C} \in \mathbb{R}^{K\times M}$, $\mathbf{d} \in \mathbb{R}^{K}$, $\mathbf{C}_{\text{eq}} \in \mathbb{R}^{K_{\text{eq}}\times M}$, and $\mathbf{d}_{\text{eq}} \in \mathbb{R}^{K_{\text{eq}}}$. Here K and K_{eq} are the number of inequality and equality constraints respectively.

Quadratic Programming

Given **V**, **m**, and $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$, the problem that we want to solve to obtain $\mathbf{f}^{\ell}_{\mathrm{f}}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N ,\; \|\mathbf{f}\|_1 \leq 1 + 2\ell ,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 ,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\} .$$

By comparing this with the standard quadratic programming problem on the previous slide we see that if we set $\mathbf{x} = \mathbf{f}$ then $M = N$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \mathbf{V} , \quad \mathbf{b} = \mathbf{0} , \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix} , \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} .$$

However, it is not as clear how to express the inequality constraint $\|\mathbf{f}\|_1 \leq 1 + 2\ell$ in the standard form $\mathbf{C}\mathbf{f} \leq \mathbf{d}$.

Quadratic Programming

The inequality $\|\mathbf{f}\|_1 \leq 1 + 2\ell$ can be expressed as the inequality constraints

$$\pm f_1 \pm f_2 \pm \cdots \pm f_{N-1} \pm f_N \leq 1 + 2\ell \,,$$

where there are 2^N choices of $\pm$ signs. When the $\pm$ are chosen to be the same sign then the inequality constraint is always satisfied because of the the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$. That leaves $2^N - 2$ inequality constraints that still need to be imposed.

The number $2^N - 2$ grows too fast with N for this approach to be useful for all but small values of N. For example, when $N = 9$ we have $2^N - 2 = 510$. With this many inequality constraints quadprog could suffer numerical difficulties. This raises the following question.

Are all of these $2^N - 2$ inequality constraints needed?

Quadratic Programming

The answer is yes if we insist on setting $\mathbf{x} = \mathbf{f}$. However, the answer is no if we enlarge the dimension of $\mathbf{x}$.

To understand why the answer is yes if we insist on setting $\mathbf{x} = \mathbf{f}$, consider any of these inequality constraints written along with the equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ as

$$\begin{aligned} \pm f_1 \pm f_2 \pm \cdots \pm f_{N-1} \pm f_N &\leq 1 + 2\ell \,, \\ f_1 + f_2 + \cdots + f_{N-1} + f_N &= 1 \,. \end{aligned}$$

By adding these and dividing by 2 we obtain

$$\sum_{i \in S} f_i \leq 1 + \ell \,,$$

where S is the subset of indices i with a plus in the inequality constraint.

Quadratic Programming

For every $S \subset \{1, 2, \cdots, N\}$ define the i^{th} entry of $\mathbf{1}_S \in \mathbb{R}^N$ by

$$\mathrm{ent}_i(\mathbf{1}_S) = \begin{cases} 1 & \text{if } i \in S\,, \\ 0 & \text{if } i \notin S\,. \end{cases}$$

Then the $2^N - 2$ inequality conatraints can be expressed as

$$\mathbf{1}_S^{\mathrm{T}}\mathbf{f} \leq 1 + \ell \quad \text{for every nonempty, proper } S \subset \{1, 2, \cdots, N\}\,.$$

The equality constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ can be used to show that these $2^N - 2$ inequality conatraints can also be expressed as

$$-\ell \leq \mathbf{1}_S^{\mathrm{T}}\mathbf{f} \quad \text{for every nonempty, proper } S \subset \{1, 2, \cdots, N\}\,.$$

Quadratic Programming

To understand why the answer is no if we enlarge the dimension of $\mathbf{x}$, consider the following equivalences.

$$\begin{aligned}\Pi_\ell &= \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \mathbf{s} \in \mathbb{R}^N\,,\; \mathbf{s} \geq \mathbf{0}\,,\; (\mathbf{f} + \mathbf{s}) \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{s} \leq \ell \right\} \\ &= \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\; \mathbf{g} \in \mathbb{R}^N\,,\; (\mathbf{g} \pm \mathbf{f}) \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{g} \leq 1 + 2\ell \right\}.\end{aligned}$$

The two sets on the right-hand side above are equal by the relations

$$\mathbf{s} = \tfrac{1}{2}(\mathbf{g} - \mathbf{f})\,, \qquad \mathbf{g} = \mathbf{f} + 2\mathbf{s}\,.$$

We must show that they are also equal to Π_ℓ. This is left as an exercise.

Quadratic Programming

If we use the first equivalence then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{\ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\ :\ \mathbf{s}\geq\mathbf{0}\,,\ (\mathbf{f}+\mathbf{s})\geq\mathbf{0}\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{s}\leq\ell\,,\ \mathbf{1}^{\mathrm{T}}\mathbf{f}=1\,,\ \mathbf{m}^{\mathrm{T}}\mathbf{f}=\mu\ \right\}.$$

By comparing this with the standard quadratic programming problem we see that if we set $\mathbf{x}=(\mathbf{f}\ \mathbf{s})^{\mathrm{T}}$ then $M=2N$, $K=2N+1$, $K_{\mathrm{eq}}=2$, and

$$\mathbf{A}=\begin{pmatrix}\mathbf{V} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix}-\mathbf{I} & -\mathbf{I}\\ \mathbf{O} & -\mathbf{I}\\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}}\end{pmatrix},\quad \mathbf{d}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ \ell\end{pmatrix},\quad \mathbf{C}_{\mathrm{eq}}=\begin{pmatrix}\mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\end{pmatrix},\quad \mathbf{d}_{\mathrm{eq}}=\begin{pmatrix}1\\ \mu\end{pmatrix},$$

where $\mathbf{O}$ and $\mathbf{I}$ are the $N\times N$ zero and identity matrices.

Quadratic Programming

If we use the second equivalence then the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; (\mathbf{g}\pm\mathbf{f})\geq\mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{g}\leq 1+2\ell\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{f}=1\,,\; \mathbf{m}^{\mathrm{T}}\mathbf{f}=\mu \right\}.$$

By comparing this with the standard quadratic programming problem we see that if we set $\mathbf{x}=(\mathbf{f}\ \mathbf{g})^{\mathrm{T}}$ then $M=2N$, $K=2N+1$, $K_{\mathrm{eq}}=2$, and

$$\mathbf{A}=\begin{pmatrix}\mathbf{V} & \mathbf{O}\\ \mathbf{O} & \mathbf{O}\end{pmatrix},\quad \mathbf{b}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix},$$

$$\mathbf{C}=\begin{pmatrix}-\mathbf{I} & -\mathbf{I}\\ \mathbf{I} & -\mathbf{I}\\ \mathbf{0}^{\mathrm{T}} & \mathbf{1}^{\mathrm{T}}\end{pmatrix},\quad \mathbf{d}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\\ 1+2\ell\end{pmatrix},\quad \mathbf{C}_{\mathrm{eq}}=\begin{pmatrix}\mathbf{1}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\\ \mathbf{m}^{\mathrm{T}} & \mathbf{0}^{\mathrm{T}}\end{pmatrix},\quad \mathbf{d}_{\mathrm{eq}}=\begin{pmatrix}1\\ \mu\end{pmatrix},$$

where $\mathbf{O}$ and $\mathbf{I}$ are the $N\times N$ zero and identity matrices.

Quadratic Programming

In either case $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$ can be obtained as the first N entries of the output x of a quadprog command that is formated as

$$\mathrm{x} = \text{quadprog}(\mathrm{A}, \mathrm{b}, \mathrm{C}, \mathrm{d}, \mathrm{Ceq}, \mathrm{deq}),$$

where the matrices A, C, and Ceq, and the vectors b, d, and deq are given on the previous slides.

Remark. By doubling the dimension of the vector $\mathbf{x}$ from N to $2N$ we have reduced the number of inequality constraints from $2^N - 2$ to $2N + 1$. When $N = 9$ this is a reduction from 510 to 19!

Remark. There are other ways to use quadprog to obtain $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu)$. Documentation for this command is easy to find on the web.

Computing Limited-Leverage Frontiers

When computing an ℓ-limited frontier, it helps to know some general properties of the function $\sigma_{\mathrm{f}}^{\ell}(\mu)$. These include:

- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is *continuous* over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is *strictly convex* over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$;
- $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is *piecewise hyperbolic* over $[\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$.

This means that $\sigma_{\mathrm{f}}^{\ell}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell})$ where $\sigma_{\mathrm{f}}^{\ell}(\mu)$ has either *a jump discontinuity in its first derivative* or *a jump discontinuity in its second derivative.*

Guided by these facts we now show how *an ℓ-limited frontier can be approximated numerically with the Matlab command quadprog.*

Computing Limited-Leverage Frontiers

First, partition the interval $[\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ as

$$\mu^\ell_{\mathrm{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu^\ell_{\mathrm{mx}} .$$

For example, set $\mu_k = \mu^\ell_{\mathrm{mn}} + k(\mu^\ell_{\mathrm{mx}} - \mu^\ell_{\mathrm{mn}})/n$ for a uniform partition. Pick n large enough to resolve all the features of the ℓ-limited frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

Second, for every $k = 0, \cdots, n$ use quadprog to compute $\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)\}_{k=0}^n$ should be saved.

Third, for every $k = 0, \cdots, n$ compute σ_k by

$$\sigma_k = \sigma^\ell_{\mathrm{f}}(\mu_k) = \sqrt{\mathbf{f}^\ell_{\mathrm{f}}(\mu_k)^{\mathrm{T}} \mathbf{V} \mathbf{f}^\ell_{\mathrm{f}}(\mu_k)} .$$

Computing Limited-Leverage Frontiers

Remark. There is typically a unique m_i such that $\mu^\ell_{\mathrm{mn}} = m_i$, in which case we have

$$\mathbf{f}^\ell_{\mathrm{f}}(\mu_0) = \mathbf{e}_i \,, \qquad \sigma_0 = \sqrt{v_{ii}} \,.$$

Similarly, there is typically a unique m_j such that $\mu^\ell_{\mathrm{mx}} = m_j$, in which case we have

$$\mathbf{f}^\ell_{\mathrm{f}}(\mu_n) = \mathbf{e}_j \,, \qquad \sigma_n = \sqrt{v_{jj}} \,.$$

Finally, we "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ to build an approximation to the ℓ-limited frontier in the $\sigma\mu$-plane. This can be done by linear interpolation. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\sigma}^\ell_{\mathrm{f}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\,\sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\,\sigma_k \,.$$

Computing Limited-Leverage Frontiers

A better way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k)\,,$$

and then set

$$\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu)^{\mathrm{T}}\mathbf{V}\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu)}\,.$$

Remark. This will be a very good approximation if n is large enough. Over each interval (μ_{k-1}, μ_k) it approximates $\sigma_{\mathrm{f}}^{\ell}(\mu)$ with a hyperbola rather than with a line.

Computing Limited-Leverage Frontiers

Remark. Because $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_k) \in \Pi_\ell(\mu_k)$ and $\mathbf{f}_{\mathrm{f}}^{\ell}(\mu_{k-1}) \in \Pi_\ell(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_{\mathrm{f}}^{\ell}(\mu) \in \Pi_\ell(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k)\,.$$

Therefore $\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu)$ gives an approximation to the ℓ-limited frontier that lies on or to the right of the ℓ-limited frontier in the $\sigma\mu$-plane.

Remark. When there are no nodes in the interval (μ_{k-1}, μ_k) then we can use the two-fund property to show that $\tilde{\sigma}_{\mathrm{f}}^{\ell}(\mu) = \sigma_{\mathrm{f}}^{\ell}(\mu)$.

General Portfolio with Two Risky Assets

Recall the portfolio of two risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\mathrm{mn}} = m_1$, $\mu_{\mathrm{mx}} = m_2$ and

$$\mu^{\ell}_{\mathrm{mn}} = m_1 - \ell(m_2 - m_1), \qquad \mu^{\ell}_{\mathrm{mn}} = m_2 + \ell(m_2 - m_1).$$

Recall that for every $\mu \in \mathbb{R}$ the unique portfolio allocation that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu) \in \Pi_\ell$ if and only if $\mu \in [\mu^{\ell}_{\mathrm{mn}}, \mu^{\ell}_{\mathrm{mx}}]$.

General Portfolio with Two Risky Assets

Therefore the set $\Pi_\ell(\mu)$ is given by

$$\Pi_\ell = \left\{ \mathbf{f}(\mu) \, : \, \mu \in [\mu_{\rm mn}^\ell, \mu_{\rm mx}^\ell] \right\}.$$

In other words, the set Π_ℓ is the line segment in $\mathbb{R}^2$ that is the image of the interval $[\mu_{\rm mn}^\ell, \mu_{\rm mx}^\ell]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$.

Because for every $\mu \in [\mu_{\rm mn}^\ell, \mu_{\rm mx}^\ell]$ the set $\Pi_\ell(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^{\rm T}\mathbf{V}\mathbf{f}$ over $\Pi_\ell(\mu)$ is $\mathbf{f}(\mu)$. Therefore the ℓ-limited frontier portfolios are

$$\mathbf{f}_{\rm f}^\ell(\mu) = \mathbf{f}(\mu) \qquad \text{for } \mu \in [\mu_{\rm mn}^\ell, \mu_{\rm mx}^\ell],$$

and the ℓ-limited frontier is given by

$$\sigma = \sigma_{\rm f}^\ell(\mu) = \sqrt{\mathbf{f}(\mu)^{\rm T}\mathbf{V}\,\mathbf{f}(\mu)} \qquad \text{for } \mu \in [\mu_{\rm mn}^\ell, \mu_{\rm mx}^\ell].$$

Hence, the ℓ-limited frontier is simply a segment of the frontier hyperbola. It has no nodes.

General Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \leq m_2 \leq m_3 \,, \qquad m_1 < m_3 \,.$$

Then $\mu_{\mathrm{mn}} = m_1$, $\mu_{\mathrm{mx}} = m_3$ and

$$\mu_{\mathrm{mn}}^{\ell} = m_1 - \ell(m_3 - m_1)\,, \qquad \mu_{\mathrm{mn}}^{\ell} = m_3 + \ell(m_3 - m_1)\,.$$

General Portfolio with Three Risky Assets

Recall that for every $\mu \in \mathbb{R}$ the portfolios that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\,\mathbf{n}\,, \qquad \text{for some } \phi \in \mathbb{R}\,,$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1}\begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1}\begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

It can be shown that $\mathbf{f}(\mu, \phi) \in \Pi_\ell$ if and only if $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$, $\phi \in [-\ell, 1 + \ell]$, and

$$-\ell \leq \frac{m_3 - \mu}{m_3 - m_1} - \phi\,\frac{m_3 - m_2}{m_3 - m_1} \leq 1 + \ell\,,$$

$$-\ell \leq \frac{\mu - m_1}{m_3 - m_1} - \phi\,\frac{m_2 - m_1}{m_3 - m_1} \leq 1 + \ell\,.$$

General Portfolio with Three Risky Assets

This region can be expressed as

$$\phi^{\ell}_{\rm mn}(\mu) \leq \phi \leq \phi^{\ell}_{\rm mx}(\mu)\,,$$

where

$$\phi^{\ell}_{\rm mn}(\mu) = -\min\left\{ \frac{\mu - \mu^{\ell}_{\rm mn}}{m_3 - m_2}\,,\ \ell\,,\ \frac{\mu^{\ell}_{\rm mx} - \mu}{m_2 - m_1} \right\}\,,$$

$$\phi^{\ell}_{\rm mx}(\mu) = \min\left\{ \frac{\mu - \mu^{\ell}_{\rm mn}}{m_2 - m_1}\,,\ 1+\ell\,,\ \frac{\mu^{\ell}_{\rm mx} - \mu}{m_3 - m_2} \right\}\,.$$

When $\ell > 0$ it is the hexagon $\mathcal{H}_\ell$ in the $\mu\phi$-plane whose vertices are the six distinct points

$$\big(\mu^{\ell}_{\rm mn}, 0\big)\,, \quad \big(m_1 - \ell(m_2 - m_1), -\ell\big)\,, \quad \big(m_2 - \ell(m_3 - m_2), 1+\ell\big)\,,$$

$$\big(\mu^{\ell}_{\rm mx}, 0\big)\,, \quad \big(m_3 + \ell(m_3 - m_2), -\ell\big)\,, \quad \big(m_2 + \ell(m_2 - m_1), 1+\ell\big)\,.$$

General Portfolio with Three Risky Assets

Therefore the set Π_ℓ is given by

$$\Pi_\ell = \left\{ \mathbf{f}(\mu, \phi) \,:\, (\mu, \phi) \in \mathcal{H}_\ell \right\}.$$

In other words, the set Π_ℓ is the hexagon in $\mathbb{R}^3$ that is the image of the hexagon $\mathcal{H}_\ell$ under the affine mapping $(\mu, \phi) \mapsto \mathbf{f}(\mu, \phi)$.

Because for every $\mu \in [\mu^\ell_{\mathrm{mn}}, \mu^\ell_{\mathrm{mx}}]$ the set $\Pi_\ell(\mu)$ is the intersection of the hexagon Π_ℓ with the plane $\{\mathbf{f} \in \mathbb{R}^3 \,:\, \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. This is a line segment that might be a single point. It is given by

$$\Pi_\ell(\mu) = \left\{ \mathbf{f}(\mu, \phi) \,:\, \phi^\ell_{\mathrm{mn}}(\mu) \leq \phi \leq \phi^\ell_{\mathrm{mx}}(\mu) \right\}.$$

In other words, the line segment $\Pi_\ell(\mu)$ in $\mathbb{R}^3$ is the image of the interval $[\phi^\ell_{\mathrm{mn}}(\mu), \phi^\ell_{\mathrm{mx}}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu, \phi)$.

General Portfolio with Three Risky Assets

Hence, the point on the ℓ-limited frontier associated with $\mu \in [\mu_{\mathrm{mn}}^{\ell}, \mu_{\mathrm{mx}}^{\ell}]$ is $(\sigma_{\mathrm{f}}^{\ell}(\mu), \mu)$ where $\sigma_{\mathrm{f}}^{\ell}(\mu)$ solves the constrained minimization problem

$$\begin{aligned} \sigma_{\mathrm{f}}^{\ell}(\mu)^2 &= \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Pi_{\ell}(\mu) \right\} \\ &= \min\left\{ \mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) \;:\; \phi_{\mathrm{mn}}^{\ell}(\mu) \leq \phi \leq \phi_{\mathrm{mx}}^{\ell}(\mu) \right\} . \end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\,\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}$$

is a quadratic in ϕ, we see that it has a unique global minimizer at

$$\phi = \phi_{\mathrm{f}}(\mu) = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} .$$

This global minimizer corresponds to the frontier. It will be the minimizer of our constrained minimization problem for the ℓ-limited frontier if and only if $\phi_{\mathrm{mn}}^{\ell} \leq \phi_{\mathrm{f}}(\mu) \leq \phi_{\mathrm{mx}}^{\ell}(\mu)$.

General Portfolio with Three Risky Assets

If $\phi_{\mathrm{f}}(\mu) < \phi^{\ell}_{\mathrm{mn}}(\mu)$ then the objective function is increasing over $[\phi^{\ell}_{\mathrm{mn}}(\mu), \phi^{\ell}_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi^{\ell}_{\mathrm{mn}}(\mu)$.

If $\phi^{\ell}_{\mathrm{mx}}(\mu) < \phi_{\mathrm{f}}(\mu)$ then the objective function is decreasing over $[\phi^{\ell}_{\mathrm{mn}}(\mu), \phi^{\ell}_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi^{\ell}_{\mathrm{mx}}(\mu)$.

Hence, the minimizer $\phi^{\ell}_{\mathrm{f}}(\mu)$ of our constrained minimization problem is

$$
\begin{aligned}
\phi^{\ell}_{\mathrm{f}}(\mu) &= \begin{cases} \phi^{\ell}_{\mathrm{mn}}(\mu) & \text{if } \phi_{\mathrm{f}}(\mu) < \phi^{\ell}_{\mathrm{mn}}(\mu) \\ \phi_{\mathrm{f}}(\mu) & \text{if } \phi^{\ell}_{\mathrm{mn}}(\mu) \leq \phi_{\mathrm{f}}(\mu) \leq \phi^{\ell}_{\mathrm{mx}}(\mu) \\ \phi^{\ell}_{\mathrm{mx}}(\mu) & \text{if } \phi^{\ell}_{\mathrm{mx}}(\mu) < \phi_{\mathrm{f}}(\mu) \end{cases} \\
&= \max\left\{\phi^{\ell}_{\mathrm{mn}}(\mu)\,,\ \min\left\{\phi_{\mathrm{f}}(\mu)\,,\ \phi^{\ell}_{\mathrm{mx}}(\mu)\right\}\right\} \\
&= \min\left\{\max\left\{\phi^{\ell}_{\mathrm{mn}}(\mu)\,,\ \phi_{\mathrm{f}}(\mu)\right\}\,,\ \phi^{\ell}_{\mathrm{mx}}(\mu)\right\}\,.
\end{aligned}
$$

Therefore $\sigma^{\ell}_{\mathrm{f}}(\mu)^2 = \mathbf{f}(\mu, \phi^{\ell}_{\mathrm{f}}(\mu))^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu, \phi^{\ell}_{\mathrm{f}}(\mu))$.