Portfolios that Contain Risky Assets 7: Markowitz Frontiers for Long Portfolios

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Portfolios that Contain Risky Assets
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Markowitz Frontiers for Long Portfolios

Long Constraints

Because the value of any portfolio with short positions can become negative, many investors will not hold a short position in any risky asset. These investors consider only portfolios that hold either a long or neutral position in each risky asset, so-called *long portfolios.* The allocation $\mathbf{f}$ of a *long Markowitz portfolio* satisfies the constraints $\mathbf{f} \geq \mathbf{0}$.

Let Λ be the set of all long portfolio allocations and $\Lambda(\mu)$ be the set of all long portfolio allocations with return mean μ. These sets are given by

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \;:\; \mathbf{f} \geq \mathbf{0}\,,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 \right\},$$
$$\Lambda(\mu) = \left\{ \mathbf{f} \in \Lambda \;:\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}.$$

Clearly $\Lambda(\mu) \subset \Lambda$ for every $\mu \in \mathbb{R}$.

Long Constraints

We first consider the set Λ. Let $\mathbf{e}_i$ denote the vector whose i^{th} entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where $f_i \geq 0$ for every $i = 1, \cdots, N$ and

$$\sum_{i=1}^{N} f_i = \mathbf{1}^{\mathrm{T}} \mathbf{f} = 1 \,.$$

Therefore Λ is simply all convex combinations of the vectors $\{\mathbf{e}_i\}_{i=1}^{N}$. For small N we can visualize Λ. When $N = 2$ it is the line segment in $\mathbb{R}^2$ that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \,, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \,.$$

Long Constraints

When $N = 3$ it is the triangle in $\mathbb{R}^3$ that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron in $\mathbb{R}^4$ that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general N it is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$.

Long Constraints

Remark. When $N = 4$ it is easy to check that the tetrahedron $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ given by

$$\mathcal{T} = \left\{ \mathbf{z} \in \mathbb{R}^3 \ : \ \mathbf{w}_k \cdot \mathbf{z} \leq 1 \ \text{ for } k = 1, 2, 3, 4 \right\} ,$$

where

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} , \quad \mathbf{w}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} , \quad \mathbf{w}_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} ,$$

under the one-to-one affine mapping $\mathbf{\Phi} : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\mathbf{\Phi}(\mathbf{z}) = \frac{1}{4} \begin{pmatrix} 1 - \mathbf{w}_1 \cdot \mathbf{z} \\ 1 - \mathbf{w}_2 \cdot \mathbf{z} \\ 1 - \mathbf{w}_3 \cdot \mathbf{z} \\ 1 - \mathbf{w}_4 \cdot \mathbf{z} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -z_1 \\ -z_2 \\ -z_3 \end{pmatrix} .$$

Long Constraints

Because Λ is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$, it is a nonempty, convex, and bounded set. In addition, *Λ is a closed set.*

Proof. For any $\mathbf{f}$ in the closure of Λ there exists a sequence $\{\mathbf{f}_n\}_{n\in\mathbb{N}} \subset \Lambda$ such that

$$\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_n .$$

Because $\mathbf{f}_n \geq \mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}\mathbf{f}_n = 1$ for every $n \in \mathbb{N}$, we see that

$$\mathbf{f} = \lim_{n\to\infty} \mathbf{f}_n \geq \mathbf{0}, \qquad \mathbf{1}^{\mathrm{T}}\mathbf{f} = \lim_{n\to\infty} \mathbf{1}^{\mathrm{T}}\mathbf{f}_n = 1 .$$

Hence, $\mathbf{f} \in \Lambda$. Therefore Λ is a closed set. □

Therefore Λ is a nonempty, closed, bounded, convex set.

Long Constraints

Recall that for every $\mu \in \mathbb{R}$ the set $\Lambda(\mu)$ was defined to be the intersection of the simplex Λ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. We begin with a characterization of those μ for which $\Lambda(\mu)$ is nonempty. Let

$$\mu_{\mathrm{mn}} = \min\{m_i \,:\, i = 1, \cdots, N\},$$
$$\mu_{\mathrm{mx}} = \max\{m_i \,:\, i = 1, \cdots, N\}.$$

We will prove the following.

Fact. *The set $\Lambda(\mu)$ is nonempty if and only if $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$.*

Remark. Because we have assumed that $\mathbf{m}$ is not proportional to $\mathbf{1}$, the return means $\{m_i\}_{i=1}^N$ are not identical. This implies that $\mu_{\mathrm{mn}} < \mu_{\mathrm{mx}}$, which implies that the interval $[\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ does not reduce to a point.

Long Constraints

Proof. Because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda(\mu)$ we have the inequalities

$$\mu_{\mathrm{mn}} = \mu_{\mathrm{mn}} \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mu_{\mathrm{mn}} \sum_{i=1}^{N} f_i \leq \sum_{i=1}^{N} m_i f_i = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \,,$$

$$\mu = \mathbf{m}^{\mathrm{T}}\mathbf{f} = \sum_{i=1}^{N} m_i f_i \leq \mu_{\mathrm{mx}} \sum_{i=1}^{N} f_i = \mu_{\mathrm{mx}} \mathbf{1}^{\mathrm{T}}\mathbf{f} = \mu_{\mathrm{mx}} \,.$$

Therefore if $\Lambda(\mu)$ is nonempty then $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$.

Conversely, first choose $\mathbf{e}_{\mathrm{mn}}$ and $\mathbf{e}_{\mathrm{mx}}$ so that

$$\mathbf{e}_{\mathrm{mn}} = \mathbf{e}_i \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\mathrm{mn}} \,,$$
$$\mathbf{e}_{\mathrm{mx}} = \mathbf{e}_j \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\mathrm{mx}} \,.$$

Long Constraints

Now let $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ and set

$$\mathbf{f} = \frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mathbf{e}_{\text{mn}} + \frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mathbf{e}_{\text{mx}} \,.$$

Clearly $\mathbf{f} \geq \mathbf{0}$. Because $\mathbf{1}^{\mathrm{T}}\mathbf{e}_{\text{mn}} = \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\text{mx}} = 1$, $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\text{mn}} = \mu_{\text{mn}}$, and $\mathbf{m}^{\mathrm{T}}\mathbf{e}_{\text{mx}} = \mu_{\text{mx}}$, we see that

$$\begin{aligned}
\mathbf{1}^{\mathrm{T}}\mathbf{f} &= \frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\text{mn}} + \frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mathbf{1}^{\mathrm{T}}\mathbf{e}_{\text{mx}} \\
&= \frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}} + \frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}} = 1 \,, \\
\mathbf{m}^{\mathrm{T}}\mathbf{f} &= \frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mathbf{m}^{\mathrm{T}}\mathbf{e}_{\text{mn}} + \frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mathbf{m}^{\mathrm{T}}\mathbf{e}_{\text{mx}} \\
&= \frac{\mu_{\text{mx}} - \mu}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mu_{\text{mn}} + \frac{\mu - \mu_{\text{mn}}}{\mu_{\text{mx}} - \mu_{\text{mn}}}\, \mu_{\text{mx}} = \mu \,.
\end{aligned}$$

Hence, $\mathbf{f} \in \Lambda(\mu)$. *Therefore if $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ then $\Lambda(\mu)$ is nonempty.* $\square$

Long Constraints

For every $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ the set $\Lambda(\mu)$ is the nonempty intersection in $\mathbb{R}^N$ of the $N-1$ dimensional simplex Λ with the $N-1$ dimensional hyperplane $\{\mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\}$. *Therefore $\Lambda(\mu)$ will be a nonempty, closed, bounded, convex polytope of dimension at most $N-2$.*

Remark. When there are n assets with $m_i > \mu$ and $N-n$ assets with $m_i < \mu$ then $\Lambda(\mu)$ will have $n(N-n)$ vertices. This means that $\Lambda(\mu)$ can have at most $\frac{1}{4}N^2$ vertices when N is even and can have at most $\frac{1}{4}(N^2-1)$ vertices when N is odd.

Long Constraints

We can visualize the polytope $\Lambda(\mu)$ when N is small.

- When $N = 2$ it is a point because it is the intersection of the line segment Λ with a transverse line.
- When $N = 3$ it is either a point or line segment because it is the intersection of the triangle Λ with a transverse plane.
- When $N = 4$ it is either a point, line segment, triangle, or convex quadralateral because it is the intersection of the tetrahedron Λ with a transverse hyperplane.

Long Constraints

Remark. Recall from our last remark that when $N = 4$ the set $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ under the one-to-one affine mapping $\mathbf{\Phi} : \mathbb{R}^3 \to \mathbb{R}^4$ given there. The set $\Lambda(\mu) \subset \mathbb{R}^4$ is thereby the image under $\mathbf{\Phi}$ of the intersection of $\mathcal{T}$ with the hyperplane H_μ given by

$$H_\mu = \left\{ \mathbf{z} \in \mathbb{R}^3 \; ; \; \mathbf{m}^{\mathrm{T}} \mathbf{\Phi}(\mathbf{z}) = \mu \; \right\} .$$

Hence, the set $\Lambda(\mu)$ in $\mathbb{R}^4$ can be visualized in $\mathbb{R}^3$ as the set $\mathcal{T}_\mu = \mathcal{T} \cap H_\mu$. Because $\mathbf{\Phi}$ is one-to-one and $\mathbf{m}$ is arbitrary, H_μ can be any hyperplane in $\mathbb{R}^3$. Therefore $\mathcal{T}_\mu$ can be the intersection of the tetrahedron $\mathcal{T}$ with any hyperplane in $\mathbb{R}^3$.

Long Constraints

When such an intersection is nonempty it can be either

1. a *point* that is a vertex of $\mathcal{T}$,
2. a *line segment* that is an edge of $\mathcal{T}$,
3. a *triangle* with vertices on edges of $\mathcal{T}$,
4. a *convex quadrilateral* with vertices on edges of $\mathcal{T}$.

These are each convex polytopes of dimension at most 2.

Long Frontiers

The set Λ in $\mathbb{R}^N$ of all long portfolios is associated with the set $\Sigma(\Lambda)$ in the $\sigma\mu$-plane of volatilities and return means given by

$$\Sigma(\Lambda) = \left\{ (\sigma, \mu) \in \mathbb{R}^2 \ : \ \sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \ \mu = \mathbf{m}^{\mathrm{T}}\mathbf{f}\,, \ \mathbf{f} \in \Lambda \right\} .$$

The set $\Sigma(\Lambda)$ is the image in $\mathbb{R}^2$ of the simplex Λ in $\mathbb{R}^N$ under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set Λ is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set $\Sigma(\Lambda)$ is compact.

We have seen that the set $\Lambda(\mu)$ of all long portfolios with return mean μ is nonempty if and only if $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$. Hence, $\Sigma(\Lambda)$ can be expressed as

$$\Sigma(\Lambda) = \left\{ \left(\sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}\,, \ \mu \right) \ : \ \mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]\,, \ \mathbf{f} \in \Lambda(\mu) \right\} .$$

The points on the boundary of $\Sigma(\Lambda)$ that correspond to those long portfolios that have less volatility than every other long portfolio with the same return mean is called the *long frontier.*

Long Frontiers

The long frontier is the curve in the $\sigma\mu$-plane given by the equation

$$\sigma = \sigma_{\mathrm{lf}}(\mu) \quad \text{over} \quad \mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]\,,$$

where the value of $\sigma_{\mathrm{lf}}(\mu)$ is obtained for each $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ by solving the constrained minimization problem

$$\sigma_{\mathrm{lf}}(\mu)^2 = \min\Big\{\; \sigma^2 \;:\; (\sigma,\mu) \in \Sigma \;\Big\} = \min\Big\{\; \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Lambda(\mu) \;\Big\}\,.$$

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is continuous over the compact set $\Lambda(\mu)$, *a minimizer exists.*

Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Lambda(\mu)$, whereby *the minimizer is unique.*

Long Frontiers

If we denote this unique minimizer by $\mathbf{f}_{\mathrm{lf}}(\mu)$ then for every $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ the function $\sigma_{\mathrm{lf}}(\mu)$ is given by

$$\sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}_{\mathrm{lf}}(\mu)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{lf}}(\mu)}\,,$$

where $\mathbf{f}_{\mathrm{lf}}(\mu)$ can be expressed as

$$\mathbf{f}_{\mathrm{lf}}(\mu) = \arg\min \left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N,\; \mathbf{f} \geq \mathbf{0},\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}.$$

Here $\arg\min$ is read *"the argument that minimizes"*. It means that $\mathbf{f}_{\mathrm{lf}}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ subject to the given constraints.

Remark. This problem can not be solved by Lagrange multipliers because of the inequality constraints $\mathbf{f} \geq \mathbf{0}$ associated with the set $\Lambda(\mu)$. It is harder to solve analytically than the analogous minimization problem for portfolios with unlimited leverage. Therefore we will first present a numerical approach that can generally be applied.

Long Frontiers

Because the function being minimized is quadratic in $\mathbf{f}$ while the constraints are linear in $\mathbf{f}$, this is called a *quadratic programming problem*. It can be solved for a particular $\mathbf{V}$, $\mathbf{m}$, and μ by using either the Matlab command "quadprog" or an equivalent command in some other language.

The Matlab command $\text{quadprog}(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\text{eq}}, \mathbf{d}_{\text{eq}})$ returns the solution of a quadratic programming problem in the standard form

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{x} + \mathbf{b}^{\mathrm{T}}\mathbf{x} \;:\; \mathbf{x} \in \mathbb{R}^M,\; \mathbf{C}\mathbf{x} \leq \mathbf{d},\; \mathbf{C}_{\text{eq}}\mathbf{x} = \mathbf{d}_{\text{eq}} \right\},$$

where $\mathbf{A} \in \mathbb{R}^{M\times M}$ is nonnegative definite, $\mathbf{b} \in \mathbb{R}^M$, $\mathbf{C} \in \mathbb{R}^{K\times M}$, $\mathbf{d} \in \mathbb{R}^K$, $\mathbf{C}_{\text{eq}} \in \mathbb{R}^{K_{\text{eq}}\times M}$, and $\mathbf{d}_{\text{eq}} \in \mathbb{R}^{K_{\text{eq}}}$. Here K and K_{eq} are the number of inequality and equality constraints respectively.

Long Frontiers

Given $\mathbf{V}$, $\mathbf{m}$, and $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$, the problem that we want to solve to obtain $\mathbf{f}_{\mathrm{lf}}(\mu)$ is

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \mathbb{R}^N ,\; \mathbf{f} \geq \mathbf{0} ,\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1 ,\; \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\} .$$

By comparing the standard quadratic programming problem given on the previous slide we see that we can set $\mathbf{x} = \mathbf{f}$ then $M = N$, $K = N$, $K_{\mathrm{eq}} = 2$, and

$$\mathbf{A} = \mathbf{V} , \quad \mathbf{b} = \mathbf{0} , \quad \mathbf{C} = -\mathbf{I} , \quad \mathbf{d} = \mathbf{0} , \quad \mathbf{C}_{\mathrm{eq}} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix} , \quad \mathbf{d}_{\mathrm{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} ,$$

where $\mathbf{I}$ is the $N \times N$ identity. Notice that

- $M = N$ because $\mathbf{x} = \mathbf{f} \in \mathbb{R}^N$,
- $K = N$ because $\mathbf{f} \geq \mathbf{0}$ gives N inequality constraints,
- $K_{\mathrm{eq}} = 2$ because $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are two equality constraints.

Long Frontiers

Therefore $\mathbf{f}_{\mathrm{lf}}(\mu)$ can be obtained as the output f of a quadprog command that is formated as

$$\mathrm{f} = \mathsf{quadprog}(\mathrm{V}, \mathrm{z}, -\mathrm{I}, \mathrm{z}, \mathrm{Ceq}, \mathrm{deq})\,,$$

where the matrices V, I, and Ceq, and vectors z and deq are given by

$$\mathrm{V} = \mathbf{V}\,, \quad \mathrm{z} = \mathbf{0}\,, \quad \mathrm{I} = \mathbf{I}\,, \quad \mathrm{Ceq} = \begin{pmatrix} \mathbf{1}^{\mathrm{T}} \\ \mathbf{m}^{\mathrm{T}} \end{pmatrix}\,, \quad \mathrm{deq} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}\,.$$

Remark. There are other ways to use quadprog to obtain $\mathbf{f}_{\mathrm{lf}}(\mu)$. Documentation for this command is easy to find on the web.

Long Frontiers

When computing a long frontier, it helps to know some general properties of the function $\sigma_{\rm lf}(\mu)$. These include:

- $\sigma_{\rm lf}(\mu)$ is *continuous* over $[\mu_{\rm mn}, \mu_{\rm mx}]$;
- $\sigma_{\rm lf}(\mu)$ is *strictly convex* over $[\mu_{\rm mn}, \mu_{\rm mx}]$;
- $\sigma_{\rm lf}(\mu)$ is *piecewise hyperbolic* over $[\mu_{\rm mn}, \mu_{\rm mx}]$.

This means that $\sigma_{\rm lf}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\rm mn}, \mu_{\rm mx})$ where $\sigma_{\rm lf}(\mu)$ has either *a jump discontinuity in its first derivative* or *a jump discontinuity in its second derivative.*

Guided by these facts we now show how *a long frontier can be approximated numerically with the Matlab command quadprog.*

Long Frontiers

First, partition the interval $[\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ as

$$\mu_{\mathrm{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathrm{mx}} .$$

For example, set $\mu_k = \mu_{\mathrm{mn}} + k(\mu_{\mathrm{mx}} - \mu_{\mathrm{mn}})/n$ for a uniform partition. Pick n large enough to resolve all the features of the long frontier. There should be at most one node in each subinterval $[\mu_{k-1}, \mu_k]$.

Second, for every $k = 0, \cdots, n$ use quadprog to compute $\mathbf{f}_{\mathrm{lf}}(\mu_k)$. (This computation will not be exact, but we will speak as if it is.) The allocations $\{\mathbf{f}_{\mathrm{lf}}(\mu_k)\}_{k=0}^{n}$ should be saved.

Third, for every $k = 0, \cdots, n$ compute σ_k by

$$\sigma_k = \sigma_{\mathrm{lf}}(\mu_k) = \sqrt{\mathbf{f}_{\mathrm{lf}}(\mu_k)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{\mathrm{lf}}(\mu_k)} .$$

Long Frontiers

Remark. There is typically a unique m_i such that $\mu_{\mathrm{mn}} = m_i$, in which case we have

$$\mathbf{f}_{\mathrm{lf}}(\mu_0) = \mathbf{e}_i\,, \qquad \sigma_0 = \sqrt{v_{ii}}\,.$$

Similarly, there is typically a unique m_j such that $\mu_{\mathrm{mx}} = m_j$, in which case we have

$$\mathbf{f}_{\mathrm{lf}}(\mu_n) = \mathbf{e}_j\,, \qquad \sigma_n = \sqrt{v_{jj}}\,.$$

Finally, we "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the long frontier in the $\sigma\mu$-plane. This can be done by linear interpolation. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\sigma}_{\mathrm{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\,\sigma_{k-1} + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\,\sigma_k\,.$$

Long Frontiers

A better way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}} \, \mathbf{f}_{\mathrm{lf}}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}} \, \mathbf{f}_{\mathrm{lf}}(\mu_k) \,,$$

and then set

$$\tilde{\sigma}_{\mathrm{lf}}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu)^{\mathrm{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathrm{lf}}(\mu)} \,.$$

Remark. This will be a very good approximation if n is large enough. Over each interval (μ_{k-1}, μ_k) it approximates $\sigma_{\mathrm{f}}^{\ell}(\mu)$ with a hyperbola rather than with a line.

Long Frontiers

Remark. Because $\mathbf{f}_{\mathrm{lf}}(\mu_k) \in \Lambda(\mu_k)$ and $\mathbf{f}_{\mathrm{lf}}(\mu_{k-1}) \in \Lambda(\mu_{k-1})$, we can show that

$$\tilde{\mathbf{f}}_{\mathrm{lf}}(\mu) \in \Lambda(\mu) \quad \text{for every } \mu \in (\mu_{k-1}, \mu_k)\,.$$

Therefore $\tilde{\sigma}_{\mathrm{lf}}(\mu)$ gives an approximation to the long frontier that lies on or to the right of the long frontier in the $\sigma\mu$-plane.

Remark. When there are no nodes in the interval (μ_{k-1}, μ_k) then we can use the two-fund property to show that $\tilde{\sigma}_{\mathrm{lf}}(\mu) = \sigma_{\mathrm{lf}}(\mu)$.

General Portfolio with Two Risky Assets

Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\mathrm{mn}} = m_1$ and $\mu_{\mathrm{mx}} = m_2$. Recall that for every $\mu \in \mathbb{R}$ the unique portfolio that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_2] = [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$. Therefore the set Λ of long portfolios is given by

$$\Lambda = \left\{ \mathbf{f}(\mu) \,:\, \mu \in [m_1, m_2] \right\}.$$

General Portfolio with Two Risky Assets

In other words, the line segment Λ in $\mathbb{R}^2$ is the image of the interval $[m_1, m_2]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$.

Because for every $\mu \in [m_1, m_2]$ the set $\Lambda(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ over $\Lambda(\mu)$ is $\mathbf{f}(\mu)$. Therefore the long frontier portfolios are

$$\mathbf{f}_{\mathrm{lf}}(\mu) = \mathbf{f}(\mu) \qquad \text{for } \mu \in [m_1, m_2]\,,$$

and the long frontier is given by

$$\sigma = \sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\,\mathbf{f}(\mu)} \qquad \text{for } \mu \in [m_1, m_2]\,.$$

Hence, the long frontier is simply a segment of the frontier hyperbola. It has no nodes.

General Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \leq m_2 \leq m_3 , \qquad m_1 < m_3 .$$

Then $\mu_{\mathrm{mn}} = m_1$ and $\mu_{\mathrm{mx}} = m_3$.

General Portfolio with Three Risky Assets

Recall that for every $\mu \in \mathbb{R}$ the portfolios that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\mathbf{n}\,, \qquad \text{for some } \phi \in \mathbb{R}\,,$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1}\begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1}\begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu, \phi) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_3] = [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ and

$$0 \leq \phi \leq \min\left\{\frac{m_3 - \mu}{m_3 - m_2}, \frac{\mu - m_1}{m_2 - m_1}\right\}.$$

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For every $\mu \in [m_1, m_3]$ we define

$$\phi_{\mathrm{mx}}(\mu) = \min\left\{ \frac{m_3 - \mu}{m_3 - m_2}, \frac{\mu - m_1}{m_2 - m_1} \right\}.$$

Then the set Λ of long portfolios is given by

$$\Lambda = \left\{ \mathbf{f}(\mu, \phi) \,:\, (\mu, \phi) \in \mathcal{T}_\Lambda \right\},$$

where $\mathcal{T}_\Lambda$ is the triangle in the $\mu\phi$-plane given by

$$\mathcal{T}_\Lambda = \left\{ (\mu, \phi) \in \mathbb{R}^2 \,:\, \mu \in [m_1, m_3],\, 0 \leq \phi \leq \phi_{\mathrm{mx}}(\mu) \right\}.$$

The base of this triangle is the interval $[m_1, m_3]$ on the μ-axis. Its peak is the point $(m_2, 1)$, so its height is 1.

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Therefore the sets Λ and $\Lambda(\mu)$ in $\mathbb{R}^3$ can be visualized as follows.

The set Λ is the triangle in $\mathbb{R}^3$ that is the image of the triangle $\mathcal{T}_\Lambda$ under the affine mapping $(\mu, \phi) \mapsto \mathbf{f}(\mu, \phi)$.

For every $\mu \in [m_1, m_3]$ the set $\Lambda(\mu)$ is given by

$$\Lambda(\mu) = \left\{ \mathbf{f}(\mu, \phi) \; : \; 0 \leq \phi \leq \phi_{\mathrm{mx}}(\mu) \right\} .$$

Therefore the set $\Lambda(\mu)$ is the line segment in $\mathbb{R}^3$ that is the image of the interval $[0, \phi_{\mathrm{mx}}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu, \phi)$.

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Hence, the point on the long frontier associated with $\mu \in [\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}]$ is $(\sigma_{\mathrm{lf}}(\mu), \mu)$ where $\sigma_{\mathrm{lf}}(\mu)$ solves the constrained minimization problem

$$\begin{aligned} \sigma_{\mathrm{lf}}(\mu)^2 &= \min\left\{ \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Lambda(\mu) \right\} \\ &= \min\left\{ \mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) \;:\; 0 \leq \phi \leq \phi_{\mathrm{mx}}(\mu) \right\}. \end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu,\phi)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu,\phi) = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\,\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}$$

is a quadratic in ϕ, we see that it has a unique global minimizer at

$$\phi = \phi_{\mathrm{f}}(\mu) = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}.$$

This global minimizer corresponds to the frontier. It will be the minimizer of our constrained minimization problem for the long frontier if and only if $0 \leq \phi_{\mathrm{f}}(\mu) \leq \phi_{\mathrm{mx}}(\mu)$.

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If $\phi_{\mathrm{f}}(\mu) < 0$ then the objective function is increasing over $[0, \phi_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = 0$.

If $\phi_{\mathrm{mx}}(\mu) < \phi_{\mathrm{f}}(\mu)$ then the objective function is decreasing over $[0, \phi_{\mathrm{mx}}(\mu)]$, whereby its minimizer is $\phi = \phi_{\mathrm{mx}}(\mu)$.

Hence, the minimizer $\phi_{\mathrm{lf}}(\mu)$ of our constrained minimization problem is

$$
\begin{aligned}
\phi_{\mathrm{lf}}(\mu) &= \begin{cases} 0 & \text{if } \phi_{\mathrm{f}}(\mu) < 0 \\ \phi_{\mathrm{f}}(\mu) & \text{if } 0 \leq \phi_{\mathrm{f}}(\mu) \leq \phi_{\mathrm{mx}}(\mu) \\ \phi_{\mathrm{mx}}(\mu) & \text{if } \phi_{\mathrm{mx}}(\mu) < \phi_{\mathrm{f}}(\mu) \end{cases} \\
&= \max\{0\,,\, \min\{\phi_{\mathrm{f}}(\mu)\,,\, \phi_{\mathrm{mx}}(\mu)\}\} \\
&= \min\{\max\{0\,,\, \phi_{\mathrm{f}}(\mu)\}\,,\, \phi_{\mathrm{mx}}(\mu)\}\,.
\end{aligned}
$$

Therefore $\sigma_{\mathrm{lf}}(\mu)^2 = \mathbf{f}(\mu, \phi_{\mathrm{lf}}(\mu))^{\mathrm{T}} \mathbf{V} \mathbf{f}(\mu, \phi_{\mathrm{lf}}(\mu))$.

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Understanding the long frontier thereby reduces to understanding $\phi_{\mathrm{lf}}(\mu)$. This can be done graphically in the $\mu\phi$-plane by considering the triangle $\mathcal{T}_\Lambda$ and the line $\mathcal{L}_{\mathrm{f}}$ given by

$$\phi = \phi_{\mathrm{f}}(\mu)\,.$$

Because

$$\mathbf{f}_{13}(m_1) = \mathbf{e}_1\,, \qquad \mathbf{f}_{13}(m_2) = -\mathbf{n} + \mathbf{e}_2\,, \quad \text{and} \quad \mathbf{f}_{13}(m_3) = \mathbf{e}_3\,,$$

we see that

$$\begin{aligned}
\phi_{\mathrm{f}}(m_1) &= -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(m_1)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{e}_1}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,,\\
\phi_{\mathrm{f}}(m_2) &= -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(m_2)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} = 1 - \frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{e}_2}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,,\\
\phi_{\mathrm{f}}(m_3) &= -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(m_3)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}} = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{e}_3}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,.
\end{aligned}$$

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This shows we can read off from the entries of **Vn** that:

$$
\begin{aligned}
&\mathcal{L}_{\mathrm{f}} \text{ lies below the vertex } (m_1, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \quad \mathbf{e}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0\,;\\
&\mathcal{L}_{\mathrm{f}} \text{ lies above the vertex } (m_1, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \quad \mathbf{e}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,;\\
&\mathcal{L}_{\mathrm{f}} \text{ lies below the vertex } (m_2, 1) \text{ of } \mathcal{T}_\Lambda \text{ iff } \quad \mathbf{e}_2^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0\,;\\
&\mathcal{L}_{\mathrm{f}} \text{ lies above the vertex } (m_2, 1) \text{ of } \mathcal{T}_\Lambda \text{ iff } \quad \mathbf{e}_2^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,;\\
&\mathcal{L}_{\mathrm{f}} \text{ lies below the vertex } (m_3, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \quad \mathbf{e}_3^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0\,;\\
&\mathcal{L}_{\mathrm{f}} \text{ lies above the vertex } (m_3, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \quad \mathbf{e}_3^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,.
\end{aligned}
$$

Below we consider three of the many different cases that can arise. For simplicity we will assume that $m_1 < m_2 < m_3$.

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Case 1. The line $\mathcal{L}_{\mathrm{f}}$ lies below the interior of $\mathcal{T}_{\Lambda}$ if and only if

$$\mathbf{e}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} \geq 0\,, \quad \text{and} \quad \mathbf{e}_3^{\mathrm{T}}\mathbf{V}\mathbf{n} \geq 0\,.$$

Then $\phi_{\mathrm{lf}}(\mu) = 0$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\mathrm{lf}}(\mu) = \sqrt{\mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}\,.$$

This is the long frontier built from assets 1 and 3.

General Portfolio with Three Risky Assets

Case 2. The line $\mathcal{L}_\mathrm{f}$ lies above the interior of $\mathcal{T}_\Lambda$ if and only if

$$\mathbf{e}_1^\mathrm{T}\mathbf{V}\mathbf{n} \leq 0\,, \quad \mathbf{e}_2^\mathrm{T}\mathbf{V}\mathbf{n} \leq 0\,, \quad \text{and} \quad \mathbf{e}_3^\mathrm{T}\mathbf{V}\mathbf{n} \leq 0\,.$$

Then $\phi_\mathrm{lf}(\mu) = \phi_\mathrm{mx}(\mu)$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_\mathrm{lf}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^\mathrm{T}\mathbf{V}\mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, m_2]\,, \\ \sqrt{\mathbf{f}_{23}(\mu)^\mathrm{T}\mathbf{V}\mathbf{f}_{23}(\mu)} & \text{for } \mu \in [m_2, m_3]\,. \end{cases}$$

This patches the long frontier built from assets 1 and 2 with the long frontier built from assets 2 and 3. It generally has a jump discontinuity in its first derivative at the node $\mu = m_2$.

General Portfolio with Three Risky Assets

Case 3. The line $\mathcal{L}_{\mathrm{f}}$ lies above the base of $\mathcal{T}_{\Lambda}$ but intersects the interior of $\mathcal{T}_{\Lambda}$ if and only if

$$\mathbf{e}_1^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,, \quad \mathbf{e}_2^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0\,, \quad \text{and} \quad \mathbf{e}_3^{\mathrm{T}}\mathbf{V}\mathbf{n} < 0\,.$$

Then there exists $\mu_1 \in [m_1, m_2]$ and $\mu_2 \in [m_2, m_3]$ such that

$$\phi_{\mathrm{lf}}(\mu) = \begin{cases} \dfrac{\mu - m_1}{m_2 - m_1} & \text{for } \mu \in [m_1, \mu_1]\,, \\ \phi_{\mathrm{f}}(\mu) & \text{for } \mu \in (\mu_1, \mu_2)\,, \\ \dfrac{m_3 - \mu}{m_3 - m_2} & \text{for } \mu \in [\mu_2, m_3]\,. \end{cases}$$

General Portfolio with Three Risky Assets

The long frontier is

$$\sigma = \sigma_{\mathrm{lf}}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, \mu_1]\,, \\ \sigma_{\mathrm{f}}(\mu) & \text{for } \mu \in (\mu_1, \mu_2)\,, \\ \sqrt{\mathbf{f}_{23}(\mu)^{\mathrm{T}} \mathbf{V} \mathbf{f}_{23}(\mu)} & \text{for } \mu \in [\mu_2, m_3]\,. \end{cases}$$

It generally has jump discontinuities in its second derivative at the nodes $\mu = \mu_1$ and $\mu = \mu_2$.

Simple Portfolio with Three Risky Assets

Recall the portfolio of three risky assets with mean vector **m** and covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$, where the last condition is equivalent to the condition that **V** is positive definite given $s > 0$.

Simple Portfolio with Three Risky Assets

Its frontier parameters are

$$\sigma_{\mathrm{mv}} = \sqrt{\frac{1}{a}} = s\sqrt{\frac{1+2r}{3}}\,, \qquad \mu_{\mathrm{mv}} = \frac{b}{a} = m\,,$$

$$\nu_{\mathrm{as}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s}\sqrt{\frac{2}{1-r}}\,.$$

Its minimum volatility portfolio is $\mathbf{f}_{\mathrm{mv}} = \frac{1}{3}\mathbf{1}$, whereby we can take $\mu_0 = m$. Clearly $[\mu_{\mathrm{mn}}, \mu_{\mathrm{mx}}] = [m-d, m+d]$. Its frontier is determined by

$$\sigma_{\mathrm{f}}(\mu) = s\sqrt{\frac{1+2r}{3} + \frac{1-r}{2}\left(\frac{\mu - m}{d}\right)^2} \qquad \text{for } \mu \in (-\infty, \infty)\,.$$

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The allocation of the frontier portfolio with return mean μ is

$$\mathbf{f}_{\mathrm{f}}(\mu) = \begin{pmatrix} \frac{1}{3} - \frac{\mu - m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu - m}{2d} \end{pmatrix} = \begin{pmatrix} \frac{m + \frac{2}{3}d - \mu}{2d} \\ \frac{1}{3} \\ \frac{\mu - m + \frac{2}{3}d}{2d} \end{pmatrix} .$$

The frontier portfolio holds long postitions when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. Therefore $[\underline{\mu}_1, \overline{\mu}_1] = [m - \frac{2}{3}d, m + \frac{2}{3}d]$ and the long frontier satisfies

$$\sigma_{\mathrm{lf}}(\mu) = \sigma_{\mathrm{f}}(\mu) \qquad \text{for } \mu \in [m - \tfrac{2}{3}d, m + \tfrac{2}{3}d] .$$

The allocation of first asset vanishes at the right endpoint while that of the third vanishes at the left endpoint.

Simple Portfolio with Three Risky Assets

In order to extend the long frontier beyond the right endpoint $\overline{\mu}_1 = m + \frac{2}{3}d$ to $\mu_{\mathrm{mx}} = m + d$ we reduce the portfolio by removing the first asset and set

$$\overline{\mathbf{m}}_1 = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m \\ m + d \end{pmatrix}, \qquad \overline{\mathbf{V}}_1 = s^2 \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

Then

$$\overline{\mathbf{V}}_1^{-1} = \frac{1}{s^2(1 - r^2)} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}, \qquad \overline{\mathbf{V}}_1^{-1}\mathbf{1} = \frac{1}{s^2(1 + r)}\mathbf{1},$$

whereby

$$\overline{a}_1 = \mathbf{1}^{\mathrm{T}}\overline{\mathbf{V}}_1^{-1}\mathbf{1} = \frac{2}{s^2(1 + r)}, \qquad \overline{b}_1 = \mathbf{1}^{\mathrm{T}}\overline{\mathbf{V}}_1^{-1}\overline{\mathbf{m}}_1 = \frac{2m + d}{s^2(1 + r)},$$

$$\overline{c}_1 = \overline{\mathbf{m}}_1^{\mathrm{T}}\overline{\mathbf{V}}_1^{-1}\overline{\mathbf{m}}_1 = \frac{2m(m + d)}{s^2(1 + r)} + \frac{d^2}{s^2(1 - r^2)}.$$

Simple Portfolio with Three Risky Assets

The associated frontier parameters are

$$\sigma_{\mathrm{mv}_1} = \sqrt{\frac{1}{\bar{a}_1}} = s\sqrt{\frac{1+r}{2}}\,, \qquad \mu_{\mathrm{mv}_1} = \frac{\bar{b}_1}{\bar{a}_1} = m + \tfrac{1}{2}d\,,$$

$$\nu_{\mathrm{as}_1} = \sqrt{\bar{c}_1 - \frac{\bar{b}_1^{\,2}}{\bar{a}_1}} = \frac{d}{2s}\sqrt{\frac{2}{1-r}}\,,$$

whereby the frontier of the reduced portfolio is given by

$$\sigma_{\bar{\mathrm{f}}_1}(\mu) = s\sqrt{\frac{1+r}{2} + \frac{1-r}{2}\left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d}\right)^2}\,.$$

Simple Portfolio with Three Risky Assets

Similarly, in order to extend the long frontier beyond the left endpoint $\underline{\mu}_1 = m - \frac{2}{3}d$ to $\mu_{\mathrm{mn}} = m - d$ we reduce the portfolio by removing the third asset. We find that the frontier of the reduced portfolio is given by

$$\sigma_{\underline{\mathrm{f}}_1}(\mu) = s\sqrt{\frac{1+r}{2} + \frac{1-r}{2}\left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d}\right)^2}\,.$$

Simple Portfolio with Three Risky Assets

By putting these pieces together we see that the long frontier is given by

$$\sigma_{\mathrm{lf}}(\mu) = \begin{cases} s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\dfrac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m-d, m-\tfrac{2}{3}d]\,, \\ s\sqrt{\dfrac{1+2r}{3} + \dfrac{1-r}{2}\left(\dfrac{\mu - m}{d}\right)^2} & \text{for } \mu \in [m-\tfrac{2}{3}d, m+\tfrac{2}{3}d]\,, \\ s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\dfrac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m+\tfrac{2}{3}d, m+d]\,. \end{cases}$$

This is strictly convex and continuously differentiable over $[m-d, m+d]$.

Simple Portfolio with Three Risky Assets

Its second derivative is defined and positive everywhere in $[m-d, m+d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where it has jump discontinuities. Thus,

$$\sigma_{\mathrm{lf}}(m \pm \tfrac{2}{3}d) = s\sqrt{\frac{5+4r}{9}}\,, \qquad \sigma_{\mathrm{lf}}(m \pm d) = s\,.$$

Simple Portfolio with Three Risky Assets

Finally, the long frontier allocations are given by

$$\mathbf{f}_{\mathrm{lf}}(\mu) = \begin{cases} \begin{pmatrix} \frac{m-\mu}{d} \\ \frac{\mu-m+d}{d} \\ 0 \end{pmatrix} & \text{for } \mu \in [m-d, m-\tfrac{2}{3}d]\,, \\ \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} & \text{for } \mu \in [m-\tfrac{2}{3}d, m+\tfrac{2}{3}d]\,, \\ \begin{pmatrix} 0 \\ \frac{m+d-\mu}{d} \\ \frac{\mu-m}{d} \end{pmatrix} & \text{for } \mu \in [m+\tfrac{2}{3}d, m+d]\,. \end{cases}$$

Notice that these allocations do not depend on either s or r.

Simple Portfolio with Three Risky Assets

Remark. These long frontier allocations are continuous and piecewise linear over $[m-d, m+d]$. Their first derivatives are defined everywhere in $[m-d, m+d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where they have jump discontinuities. The allocations at these nodes are

$$\mathbf{f}_{\mathrm{lf}}(m - \tfrac{2}{3}d) = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}, \qquad \mathbf{f}_{\mathrm{lf}}(m + \tfrac{2}{3}d) = \begin{pmatrix} 0 \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}.$$