Portfolios that Contain Risky Assets 6: Markowitz Frontiers for Unlimited Portfolios

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Markowitz Frontiers for Unlimited Portfolios

Markowitz Portfolio Theory

The 1952 paper by Markowitz initiated what was later called *Modern Portfolio Theory* (MPT). Because 1952 was long ago, and because some of the later additions to that theory are wrong, we will use the name *Markowitz Portfolio Theory* (still MPT), to distinguish the orginal work from what came later. (Markowitz simply called it portfolio theory, and often made fun of the name it aquired.)

Portfolio theories strive to maximize reward for a given risk — or what is related, mimimize risk for a given reward. They do this by quantifying the notions of reward and risk, and identifying a class of idealized portfolios for which an analysis is tractable. Here we present MPT, the first such theory. *Markowitz chose to use the return mean μ as the proxy for the reward of a portfolio, and the volatility $\sigma = \sqrt{v}$ as the proxy for its risk. He also chose to analyze the class that we have dubbed Markowitz portfolios.*

Markowitz Portfolio Theory

The simplest setting is to use the set Π_∞ of all Markowitz portfolios. Then for a portfolio of N risky assets characterized by $\mathbf{m}$ and $\mathbf{V}$ the problem of minimizing risk for a given reward becomes the problem of minimizing

$$\sigma = \sqrt{\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}}$$

over $\mathbf{f} \in \mathbb{R}^N$ subject to the constraints

$$\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,, \qquad \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu\,,$$

where μ is given. Here $\mathbf{1}$ is the N-vector that has every entry equal to 1.

Remark. Additional constraints can be added. We can restrict to the long portfolios Λ by adding the inequality constraints $\mathbf{f} \geq \mathbf{0}$. We can restrict to the limited leverage portfolios Π_ℓ by adding the inequality constraints $|f| \leq 1 + 2\ell$. These constraints are treated in the next two chapters.

Constrained Minimization Problem

Because $\sigma > 0$, minimizing σ is equivalent to minimizing σ^2. Because σ^2 is a quadratic function of $\mathbf{f}$, it is easier to minimize than σ. We therefore choose to solve the constrained minimization problem

$$\min_{\mathbf{f}\in\mathbb{R}^N} \left\{ \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\ \mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu \right\}. \tag{2.1}$$

Because there are two equality constraints, we introduce the *Lagrange multipliers* α and β, and define

$$\Phi(\mathbf{f},\alpha,\beta) = \tfrac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} - \alpha\left(\mathbf{1}^{\mathrm{T}}\mathbf{f} - 1\right) - \beta\left(\mathbf{m}^{\mathrm{T}}\mathbf{f} - \mu\right).$$

By setting the partial derivatives of $\Phi(\mathbf{f},\alpha,\beta)$ equal to zero we obtain

$$\begin{aligned}
\mathbf{0} &= \nabla_{\mathbf{f}}\Phi(\mathbf{f},\alpha,\beta) = \mathbf{V}\mathbf{f} - \alpha\mathbf{1} - \beta\mathbf{m}\,,\\
0 &= \partial_\alpha\Phi(\mathbf{f},\alpha,\beta) = -\mathbf{1}^{\mathrm{T}}\mathbf{f} + 1\,,\\
0 &= \partial_\beta\Phi(\mathbf{f},\alpha,\beta) = -\mathbf{m}^{\mathrm{T}}\mathbf{f} + \mu\,.
\end{aligned}$$

Constrained Minimization Problem

Because $\mathbf{V}$ is positive definite we may solve the first equation for $\mathbf{f}$ as

$$\mathbf{f} = \alpha\, \mathbf{V}^{-1}\mathbf{1} + \beta\, \mathbf{V}^{-1}\mathbf{m}\,.$$

By setting this into the second and third equations we obtain the system

$$\begin{aligned} \alpha\, \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} + \beta\, \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} &= 1\,, \\ \alpha\, \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} + \beta\, \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} &= \mu\,. \end{aligned}$$

If we introduce a, b, and c by

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,, \qquad c = \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,,$$

then the above system can be expressed as

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}\,.$$

Constrained Minimization Problem

The foregoing linear algebraic system has a unique solution if and only if

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \quad \text{is invertible}\,,$$

where

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,, \qquad c = \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m}\,.$$

For every $x, y \in \mathbb{R}$ we have

$$\begin{pmatrix} x \\ y \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x\mathbf{1} + y\mathbf{m})^{\mathrm{T}}\mathbf{V}^{-1}(x\mathbf{1} + y\mathbf{m})\,.$$

Because $\mathbf{V}^{-1}$ is positive definite, we see that the above 2×2 matrix is positive definite if and only if the vectors $\mathbf{1}$ and $\mathbf{m}$ are not co-linear ($\mathbf{m} \neq \mu\mathbf{1}$ for every μ). This is usually the case in practice.

Constrained Minimization Problem

When $\mathbf{1}$ and $\mathbf{m}$ are not co-linear, we find that α and β are

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{ac-b^2}\begin{pmatrix} c & -b\\ -b & a\end{pmatrix}\begin{pmatrix}1\\ \mu\end{pmatrix} = \frac{1}{ac-b^2}\begin{pmatrix} c-b\mu\\ a\mu-b\end{pmatrix}.$$

Hence, for each μ there is a unique minimizer given by

$$\mathbf{f}(\mu) = \frac{c-b\mu}{ac-b^2}\,\mathbf{V}^{-1}\mathbf{1} + \frac{a\mu-b}{ac-b^2}\,\mathbf{V}^{-1}\mathbf{m}\,. \tag{2.2}$$

The associated minimum value of σ^2 is

$$\begin{aligned}\sigma^2 &= \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) = (\alpha\,\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{V}^{-1}\mathbf{m})^{\mathrm{T}}\mathbf{V}(\alpha\,\mathbf{V}^{-1}\mathbf{1} + \beta\,\mathbf{V}^{-1}\mathbf{m})\\ &= \begin{pmatrix}\alpha & \beta\end{pmatrix}\begin{pmatrix} a & b\\ b & c\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{1}{ac-b^2}\begin{pmatrix}1 & \mu\end{pmatrix}\begin{pmatrix} c & -b\\ -b & a\end{pmatrix}\begin{pmatrix}1\\ \mu\end{pmatrix}\\ &= \frac{1}{a} + \frac{a}{ac-b^2}\left(\mu - \frac{b}{a}\right)^2.\end{aligned}$$

Constrained Minimization Problem

Remark. When calculating formulas such as (2.2) on a computer, we never compute the inverse of a matrix! Rather, we solve the linear algebraic systems

$$\mathbf{V}\mathbf{y} = \mathbf{1}\,, \qquad \mathbf{V}\mathbf{z} = \mathbf{m}\,.$$

We then generate a, b, and c by the formulas

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{y}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{z}\,, \qquad c = \mathbf{m}^{\mathrm{T}}\mathbf{z}\,.$$

Then formula (2.2) becomes

$$\mathbf{f}(\mu) = \frac{c - b\mu}{ac - b^2}\,\mathbf{y} + \frac{a\mu - b}{ac - b^2}\,\mathbf{z}\,.$$

Therefore when you see $\mathbf{V}^{-1}\mathbf{1}$ and $\mathbf{V}^{-1}\mathbf{m}$ in what follows, think of them as symbols for the vectors $\mathbf{y}$ and $\mathbf{z}$ which indicate that $\mathbf{y}$ and $\mathbf{z}$ are the solutions of certain linear algebraic systems!

Constrained Minimization Problem

Remark. When $\mathbf{1}$ and $\mathbf{m}$ are co-linear we have $\mathbf{m} = \mu\mathbf{1}$ for some μ. If we minimize $\frac{1}{2}\mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$ subject to the constraint $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ then, by Lagrange multipliers, we find that

$$\mathbf{V}\mathbf{f} = \alpha\mathbf{1}\,, \quad \text{for some } \alpha \in \mathbb{R}\,.$$

Because $\mathbf{V}$ is invertible, for every μ the unique minimizer is given by

$$\mathbf{f}(\mu) = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}}\,\mathbf{V}^{-1}\mathbf{1}\,.$$

The associated minimum value of σ^2 is

$$\sigma^2 = \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) = \frac{1}{\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1}} = \frac{1}{a}\,.$$

Remark. The mathematics used above is fairly elementary. Markowitz had cleverly indentified a class of portfolios which is analytically tractable by elemetary methods, and which provides a framework for useful models.

Frontier Portfolios

We have seen that for every μ there exists a unique Markowitz portfolio with mean μ that minimizes σ^2. This minimum value is

$$\sigma^2 = \frac{1}{a} + \frac{a}{ac - b^2}\left(\mu - \frac{b}{a}\right)^2 .$$

This is the equation of a hyperbola in the $\sigma\mu$-plane. Because volatility is nonnegative, we only consider the right half-plane $\sigma \geq 0$.

The volatility σ and mean μ of any Markowitz portfolio will be a point (σ, μ) in this half-plane that lies either on or to the right of this hyperbola.

Every point (σ, μ) on the hyperbola branch in this half-plane represents a unique Markowitz portfolio. These portfolios are called *frontier portfolios.*

Frontier Portfolios

We now replace a, b, and c with the more meaningful *frontier parameters*

$$\sigma_{\rm mv} = \frac{1}{\sqrt{a}}, \qquad \mu_{\rm mv} = \frac{b}{a}, \qquad \nu_{\rm as} = \sqrt{\frac{ac - b^2}{a}}.$$

The volatility σ for the frontier portfolio with mean μ is then given by

$$\sigma = \sigma_{\rm f}(\mu) \equiv \sqrt{\sigma_{\rm mv}^2 + \left(\frac{\mu - \mu_{\rm mv}}{\nu_{\rm as}}\right)^2}.$$

We see that this hyperbola branch has center $(0, \mu_{\rm mv})$, vertex $(\sigma_{\rm mv}, \mu_{\rm mv})$, and asymptotes

$$\mu = \mu_{\rm mv} \pm \nu_{\rm as}\sigma.$$

Frontier Portfolios

It is clear that no portfolio has a volatility σ that is less than $\sigma_{\rm mv}$. In other words, $\sigma_{\rm mv}$ is the minimum volatility attainable by diversification.

Markowitz interpreted $\sigma_{\rm mv}^2$ to be the contribution to the volatility due to the *systemic risk* of the market, and interpreted $(\mu - \mu_{\rm mv})^2/\nu_{\rm as}^2$ to be the contribution to the volatility due to the *specific risk* of the portfolio.

Remark. Markowitz attributed $\sigma_{\rm mv}$ to systemic risk of the market because he was considering the case when N was large enough that $\sigma_{\rm mv}$ would not be significantly reduced by introducing additional assets into the portfolio. *More generally, one should attribute $\sigma_{\rm mv}$ to those risks that are common to all of the N assets being considered for the portfolio.*

Frontier Portfolios

The frontier portfolio corresponding to $(\sigma_{\rm mv}, \mu_{\rm mv})$ is called the *minimum volatility portfolio.* Its associated allocation $\mathbf{f}_{\rm mv}$ is given by

$$\mathbf{f}_{\rm mv} = \mathbf{f}(\mu_{\rm mv}) = \mathbf{f}\left(\frac{b}{a}\right) = \frac{1}{a}\mathbf{V}^{-1}\mathbf{1} = \sigma_{\rm mv}^2 \mathbf{V}^{-1}\mathbf{1} \,.$$

This allocation depends only upon $\mathbf{V}$, and is therefore known with greater confidence than any allocation that also depends upon $\mathbf{m}$.

Frontier Portfolios

The allocation of the frontier portfolio with mean μ can be expressed as

$$\mathbf{f}_{\mathrm{f}}(\mu) \equiv \mathbf{f}_{\mathrm{mv}} + \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) \,.$$

Because $\mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) = b - \mu_{\mathrm{mv}} a = 0$, we see that

$$\mathbf{f}_{\mathrm{mv}}^{\mathrm{T}}\mathbf{V}\left(\mathbf{f}_{\mathrm{f}}(\mu) - \mathbf{f}_{\mathrm{mv}}\right) = \sigma_{\mathrm{mv}}^2 \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}^2} \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) = 0 \,.$$

The vectors $\mathbf{f}_{\mathrm{mv}}$ and $\mathbf{f}_{\mathrm{f}}(\mu) - \mathbf{f}_{\mathrm{mv}}$ are thereby orthogonal with respect to the $\mathbf{V}$-scalar product, which is given by $(\mathbf{f}_1 \,|\, \mathbf{f}_2)_{\mathbf{V}} = \mathbf{f}_1^{\mathrm{T}}\mathbf{V}\mathbf{f}_2$.

Frontier Portfolios

It follows that

$$\mathbf{f}_{\mathrm{f}}(\mu) \equiv \mathbf{f}_{\mathrm{mv}} + \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1})$$

is *an orthogonal decomposition of $\mathbf{f}_{\mathrm{f}}(\mu)$ with respect to the $\mathbf{V}$-scalar product given by $(\mathbf{f}_1 \,|\, \mathbf{f}_2)_{\mathbf{V}} = \mathbf{f}_1^{\mathrm{T}}\mathbf{V}\mathbf{f}_2$.*

In the associated $\mathbf{V}$-norm, which is given by $\|\mathbf{f}\|_{\mathbf{V}}^2 = (\mathbf{f} \,|\, \mathbf{f})_{\mathbf{V}}$, we find that

$$\|\mathbf{f}_{\mathrm{mv}}\|_{\mathbf{V}}^2 = \sigma_{\mathrm{mv}}^2 \,, \qquad \|\mathbf{f}_{\mathrm{f}}(\mu) - \mathbf{f}_{\mathrm{mv}}\|_{\mathbf{V}}^2 = \left(\frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}}\right)^2 .$$

Hence, $\mathbf{f}_{\mathrm{f}}(\mu) = \mathbf{f}_{\mathrm{mv}} + (\mathbf{f}_{\mathrm{f}}(\mu) - \mathbf{f}_{\mathrm{mv}})$ is the orthogonal decomposition of $\mathbf{f}_{\mathrm{f}}(\mu)$ into components that account for the contributions to the volatility due to systemic risk and specific risk respectively.

Frontier Portfolios

The fact that $\mathbf{f}_{\mathrm{f}}(\mu)$ is a linear function of μ leads to the following general property of frontier portfolios that contain two or more risky assets.

Fact. Let $\mathbf{f}_{\mathrm{f}}(\mu)$ be the frontier portfolios associated with $N \geq 2$ risky assets. Let μ_1 and μ_2 be any two return means with $\mu_1 < \mu_2$. Let $\mathbf{f}_1$ and $\mathbf{f}_2$ be the allocations of the frontier portfolios associated with μ_1 and μ_2. *Then $\mathbf{f}_{\mathrm{f}}(\mu)$ is given by*

$$\mathbf{f}_{\mathrm{f}}(\mu) = \frac{\mu_2 - \mu}{\mu_2 - \mu_1}\,\mathbf{f}_1 + \frac{\mu - \mu_1}{\mu_2 - \mu_1}\,\mathbf{f}_2\,,$$

Proof. Because $\mathbf{f}_{\mathrm{f}}(\mu)$ is a linear function of μ, it is uniquely determined by $\mathbf{f}_1 = \mathbf{f}_{\mathrm{f}}(\mu_1)$ and $\mathbf{f}_2 = \mathbf{f}_{\mathrm{f}}(\mu_2)$. □

Frontier Portfolios

Remark. *The foregoing property states that every frontier portfolio can be realized by holding positions in just two funds that have the portfolio allocations* $\mathbf{f}_1$ *and* $\mathbf{f}_2$*.* When $\mu \in (\mu_1, \mu_2)$ both funds are held long. When $\mu > \mu_2$ the first fund is held short while the second is held long. When $\mu < \mu_1$ the second fund is held short while the first is held long.

Remark. This property is often called the *Two Mutual Fund Theorem*, which is a label that elevates it to a higher status than it deserves. We will call it simply the *Two Fund Property.*

Portfolios with Two Risky Assets

Consider a portfolio of two risky assets with return mean vector **m** and return covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix}.$$

The constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ give the linear system

$$f_1 + f_2 = 1, \qquad m_1 f_1 + m_2 f_2 = \mu.$$

The vectors **m** and **1** are not co-linear if and only if $m_1 \neq m_2$. In that case **f** is uniquely determine by this linear system to be

$$\mathbf{f} = \mathbf{f}(\mu) = \begin{pmatrix} f_1(\mu) \\ f_2(\mu) \end{pmatrix} = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix}.$$

Portfolios with Two Risky Assets

Because there is just one portfolio for each μ, every portfolio is a frontier portfolio. In other words, $\mathbf{f}_{\mathrm{f}}(\mu) = \mathbf{f}(\mu)$.

These frontier portfolios trace out the hyperbola

$$\begin{aligned}\sigma^2 &= \mathbf{f}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}(\mu) \\ &= \frac{v_{11}(m_2-\mu)^2 + 2v_{12}(m_2-\mu)(\mu-m_1) + v_{22}(\mu-m_1)^2}{(m_2-m_1)^2}\,.\end{aligned}$$

Remark. Each (σ_i, m_i) lies on the frontier of every two-asset portfolio. Typically each (σ_i, m_i) lies strictly to the right of the frontier of a portfolio that contains more than two risky assets.

Portfolios with Two Risky Assets

The frontier parameters $\mu_{\rm mv}$, $\sigma_{\rm mv}$, and $\nu_{\rm as}$ are given by

$$\mu_{\rm mv} = \frac{(v_{22} - v_{12})m_1 + (v_{11} - v_{12})m_2}{v_{11} + v_{22} - 2v_{12}},$$

$$\sigma_{\rm mv}^2 = \frac{v_{11}v_{22} - v_{12}^2}{v_{11} + v_{22} - 2v_{12}}, \qquad \nu_{\rm as}^2 = \frac{(m_2 - m_1)^2}{v_{11} + v_{22} - 2v_{12}}.$$

Remark. The fact $\mathbf{V}$ is positive definite implies

$$v_{11}v_{22} - v_{12}^2 > 0 \qquad \text{and} \qquad v_{11} + v_{22} - 2v_{12} > 0.$$

Portfolios with Two Risky Assets

The minimum volatility portfolio is

$$\mathbf{f}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^2 \mathbf{V}^{-1}\mathbf{1} = \frac{\sigma_{\mathrm{mv}}^2}{v_{11}v_{22} - v_{12}^2} \begin{pmatrix} v_{22} & -v_{12} \\ -v_{12} & v_{11} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$= \frac{1}{v_{11} + v_{22} - 2v_{12}} \begin{pmatrix} v_{22} - v_{12} \\ v_{11} - v_{12} \end{pmatrix} .$$

Remark. Notice that $\mathbf{f}_{\mathrm{mv}}$ is a long portfolio if and only if

$$v_{11} - v_{12} \geq 0 \qquad \text{and} \qquad v_{22} - v_{12} \geq 0\,.$$

This holds when the two assets are anticorrelated — i.e. when $v_{12} \leq 0$.

Portfolios with Three Risky Assets

Consider a portfolio of three risky assets with return mean vector **m** and return covarience matrix **V** given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

The constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ give the linear system

$$f_1 + f_2 + f_3 = 1\,, \qquad m_1 f_1 + m_2 f_2 + m_3 f_3 = \mu\,.$$

The vectors **m** and **1** are not co-linear if and only if $m_i \neq m_j$ for some i and j. If we assume that $m_1 \neq m_3$ then a general solution is

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix} + \frac{\phi}{m_3 - m_1} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix},$$

where ϕ is an arbitrary real number.

Portfolios with Three Risky Assets

Therefore every allocation $\mathbf{f}$ that satisfies the constraints $\mathbf{1}^{\mathrm{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathrm{T}}\mathbf{f} = \mu$ can be expressed as the one-parameter family

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi \mathbf{n} \qquad \text{for some } \phi \in \mathbb{R}\,,$$

where $\mathbf{f}_{13}(\mu)$ and $\mathbf{n}$ are the linearly independent unitless vectors given by

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1}\begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1}\begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

It is easily checked that these vectors satisfy

$$\mathbf{1}^{\mathrm{T}}\mathbf{f}_{13}(\mu) = 1\,, \qquad \mathbf{1}^{\mathrm{T}}\mathbf{n} = 0\,,$$
$$\mathbf{m}^{\mathrm{T}}\mathbf{f}_{13}(\mu) = \mu\,, \qquad \mathbf{m}^{\mathrm{T}}\mathbf{n} = 0\,.$$

In particular, $\mathbf{f}_{13}(\mu)$ is the Markowitz portfolio with return mean μ that is generated by assets 1 and 3.

Portfolios with Three Risky Assets

We can use the family $\mathbf{f} = \mathbf{f}_{13}(\mu) + \phi\mathbf{n}$ to find an alternative expression for the frontier. Fix $\mu \in \mathbb{R}$. For Markowitz portfolios we obtain

$$\sigma^2 = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f} = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}\,.$$

Because $\mathbf{V}$ is positive definite we know that $\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n} > 0$, whereby the above quadratic function of ϕ has the unique minimizer at

$$\phi = -\frac{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,,$$

and minimum of

$$\sigma^2 = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) - \frac{\left(\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu)\right)^2}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}\,.$$

The first term on the right-hand side is the σ^2 of the unique portfolio with return mean μ that contains only assets 1 and 3. Hence, the minimum of σ^2 over all portfolios with return mean μ is less when $\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{13}(\mu) \neq 0$.

Portfolios with Three Risky Assets

Remark. Whenever $m_1 \neq m_2 \neq m_3 \neq m_1$ the Markowitz portfolios with return mean μ generated by assets 1 and 2 and assets 2 and 3 are

$$\mathbf{f}_{21}(\mu) = \frac{1}{m_2 - m_1}\begin{pmatrix} m_2 - \mu \\ \mu - m_1 \\ 0 \end{pmatrix}, \qquad \mathbf{f}_{32}(\mu) = \frac{1}{m_3 - m_2}\begin{pmatrix} 0 \\ m_3 - \mu \\ \mu - m_2 \end{pmatrix}.$$

Then we can show that the minimun of σ^2 over all portfolios with return mean μ that contain all three assets can be expressed as

$$\sigma^2 = \mathbf{f}_{13}(\mu)^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}_{13}(\mu) = \mathbf{f}_{21}(\mu)^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}_{21}(\mu) = \mathbf{f}_{32}(\mu)^{\mathrm{T}}\widetilde{\mathbf{V}}\mathbf{f}_{32}(\mu),$$

where

$$\widetilde{\mathbf{V}} = \mathbf{V} - \frac{\mathbf{V}\mathbf{n}\mathbf{n}^{\mathrm{T}}\mathbf{V}}{\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{n}}.$$

Portfolios with Three Risky Assets

Remark. The formula for σ^2 on the previous slide shows that for each $i = 1, 2, 3$ the point (σ_i, m_i) will lie to the right of the frontier hyperbola in the $\sigma\mu$-plane if and only if

$$\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{e}_i \neq 0\,.$$

This condition is almost always satisfied.

Remark. The same formula also shows that the minimum of σ^2 over all portfolios with return mean μ is strictly less than that of the unique portfolio with return mean μ containing only assets i and j when

$$\mathbf{n}^{\mathrm{T}}\mathbf{V}\mathbf{f}_{ij}(\mu) \neq 0\,.$$

Because this condition is linear in μ, typically it is satisfied at all but one value of μ.

Efficiency

Definition. When one portfolio has *greater mean and no greater volatility* than another portfolio then it is said to be *more efficient* because it promises *greater reward for no greater risk.*

Definition. The segment of the frontier with mean $\mu > \mu_{\rm mv}$ is called the *efficient frontier* and every frontier portfolio that it represents is called *efficient* because every such portfolio has no portfolios that are more efficient than it. Every other portfolio is called *inefficient* because there is an efficient portfolio that is more efficient than it.

Definition. The segment of the frontier with mean $\mu < \mu_{\rm mv}$ is called the *inefficient frontier* because every frontier portfolio that it represents is less efficient than every other portfolio with the same or less volatility.

Efficiency

The efficient frontier is the upper branch of the frontier hyperbola in the right-half $\sigma\mu$-plane. It is given as a function of σ by

$$\mu = \mu_{\mathrm{mv}} + \nu_{\mathrm{as}}\sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2}\,, \qquad \text{for } \sigma > \sigma_{\mathrm{mv}}\,.$$

This curve is increasing and concave and emerges vertically upward from the point $(\sigma_{\mathrm{mv}}, \mu_{\mathrm{mv}})$. As $\sigma \to \infty$ it becomes asymptotic to the line

$$\mu = \mu_{\mathrm{mv}} + \nu_{\mathrm{as}}\sigma\,.$$

Remark. *The efficient frontier quantifies the relationship between risk and reward that we mentioned in the first presentation.*

Efficiency

The inefficient frontier is the lower branch of the frontier hyperbola in the right-half $\sigma\mu$-plane. It is given as a function of σ by

$$\mu = \mu_{\mathrm{mv}} - \nu_{\mathrm{as}} \sqrt{\sigma^2 - \sigma_{\mathrm{mv}}^2}\,, \qquad \text{for } \sigma > \sigma_{\mathrm{mv}}\,.$$

This curve is decreasing and convex and emerges vertically downward from the point $(\sigma_{\mathrm{mv}}, \mu_{\mathrm{mv}})$. As $\sigma \to \infty$ it becomes asymptotic to the line

$$\mu = \mu_{\mathrm{mv}} - \nu_{\mathrm{as}}\sigma\,.$$

Remark. Markowitz introduced the notion of efficiency in his 1952 paper. However, its graphical representation in the $\sigma\mu$-plane came later.

Efficient Market Hypothesis

The *efficient-market hypothesis* (EMH) was framed by Eugene Fama in the early 1960's in his University of Chicago doctoral dissertation, which was published in 1965. It has several versions, the most basic is the following.

Given the information available when an investment is made, no investor will consistently beat market returns on a risk-adjusted basis over long periods except by chance.

This version of the EMH is called the *weak* EMH. The *semi-strong* and the *strong* versions of the EMH make bolder claims that markets reflect information instantly, even information that is not publicly available in the case of the strong EMH. While it is true that some investors react quickly, most investors do not act instantly to every piece of news. Consequently, *there is little evidence supporting these stronger versions of the EMH.*

Efficient Market Hypothesis

The EMH is an assertion about markets, not about investors. If the weak EMH is true then the only way for an actively-managed fund to beat the market is by chance. Of course, there is some debate regarding the truth of the weak EMH. It can be recast in the language of MPT as follows.

Markets for large classes of assets will lie on or near the efficient frontier.

Therefore you can test the weak EMH with MPT! If we understand "market" to mean a capitalization weighted collection of assets (i.e. an index fund) then the EMH can be tested by checking whether index funds lie on or near the efficient frontier. *You will see that this is often the case, but not always.*

Remark. *It is a common misconception that MPT assumes the weak EMH. It does not, which is why the EMH can be checked with MPT!*

Efficient Market Hypothesis

Given an index fund with volatility σ_I and return mean μ_I a nondimensional measure of its *efficiency* might be

$$\omega_I^\mu = \frac{\mu_I - \mu_{\text{if}}(\sigma_I)}{\mu_{\text{ef}}(\sigma_I) - \mu_{\text{if}}(\sigma_I)} .$$

Notice that $\omega_I^\mu \in [0,1]$, that if $\omega_I^\mu = 1$ then the index fund lies on the efficient frontier, and if $\omega_I^\mu = 0$ then it lies on the inefficient frontier.

A nondimensional measure of its *proximity* to the frontier might be

$$\omega_I^\sigma = \frac{\sigma_{\text{f}}(\mu_I)}{\sigma_I} .$$

Notice that $\omega_I^\sigma \in (0,1]$, that if $\omega_I^\sigma = 1$ then the index fund lies on the frontier, while if ω_I^σ is small then it lies far from the frontier.

Efficient Market Hypothesis

Remark. If the index funds did always lie on the efficient frontier then by the Two Fund Property we would be able to take any position on the efficient frontier simply by investing in index funds. This idea underpins the investment stategy advanced in *"A Random Walk Down Wall Street" by Burton G. Malkiel.*

Remark. It is often asserted that the EMH holds in *rational markets.* Such a market is one for which information regarding its assets is freely available to all investors. *This does not mean that investors will act rationally based on this information! Nor does it mean that markets price assets correctly.* Rational markets are subject to the greed and fear of its investors. That is why we have bubbles and crashes. Rational markets can behave irrationally because information is not knowledge!

Efficient Market Hypothesis

Remark. It is sometimes asserted that the EMH holds in *free markets.* Such markets have many agents, are rational, and subject to regulatory and legal oversight. *These are all elements in Adam Smith's notion of free market, which refers to the freedom of its agents to act, not to the freedom from any government role.* Indeed, his radical idea was that government should nurture free markets by playing the role of empowering individual agents. He had to write his book because free markets do not arise spontaneously, even though his *"invisible hand"* of agents pursuing their self interest insures that markets do.

Simple Portfolio with Three Risky Assets

Consider a portfolio of three risky assets with return mean vector $\mathbf{m}$ and return covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m - d \\ m \\ m + d \end{pmatrix}, \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix}.$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$. The last condition is equivalent to the condition that $\mathbf{V}$ is positive definite given $s > 0$. This portfolio has the unrealistic properties that (1) every asset has the same volatility s, (2) every pair of distinct assets has the same correlation r, and (3) the return means are uniformly spaced with difference $d = m_3 - m_2 = m_2 - m_1$. These simplifications will make it easier to follow the ensuing calculations than for the general three-asset portfolio. We will return to this simple portfolio in subsequent examples.

Simple Portfolio with Three Risky Assets

It is helpful to express $\mathbf{m}$ and $\mathbf{V}$ in terms of $\mathbf{1}$ and $\mathbf{n} = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ as

$$\mathbf{m} = m\,\mathbf{1} + d\,\mathbf{n}\,, \qquad \mathbf{V} = s^2(1-r)\left(\mathbf{I} + \frac{r}{1-r}\,\mathbf{1}\,\mathbf{1}^{\mathrm{T}}\right).$$

Notice that $\mathbf{1}^{\mathrm{T}}\mathbf{1} = 3$, $\mathbf{1}^{\mathrm{T}}\mathbf{n} = 0$, and $\mathbf{n}^{\mathrm{T}}\mathbf{n} = 2$. It can be checked that

$$\mathbf{V}^{-1} = \frac{1}{s^2(1-r)}\left(\mathbf{I} - \frac{r}{1+2r}\,\mathbf{1}\,\mathbf{1}^{\mathrm{T}}\right), \qquad \mathbf{V}^{-1}\mathbf{1} = \frac{1}{s^2(1+2r)}\,\mathbf{1}\,,$$

$$\mathbf{V}^{-1}\mathbf{n} = \frac{1}{s^2(1-r)}\,\mathbf{n}\,, \qquad \mathbf{V}^{-1}\mathbf{m} = \frac{m}{s^2(1+2r)}\,\mathbf{1} + \frac{d}{s^2(1-r)}\,\mathbf{n}\,.$$

Simple Portfolio with Three Risky Assets

The parameters a, b, and c are therefore given by

$$a = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{1} = \frac{3}{s^2(1+2r)}\,, \qquad b = \mathbf{1}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} = \frac{3m}{s^2(1+2r)}\,,$$

$$c = \mathbf{m}^{\mathrm{T}}\mathbf{V}^{-1}\mathbf{m} = \frac{3m^2}{s^2(1+2r)} + \frac{2d^2}{s^2(1-r)}\,.$$

The frontier parameters are then

$$\sigma_{\mathrm{mv}} = \sqrt{\frac{1}{a}} = s\sqrt{\frac{1+2r}{3}}\,, \qquad \mu_{\mathrm{mv}} = \frac{b}{a} = m\,,$$

$$\nu_{\mathrm{as}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s}\sqrt{\frac{2}{1-r}}\,.$$

The minimum volatility portfolio has allocation

$$\mathbf{f}_{\mathrm{mv}} = \sigma_{\mathrm{mv}}^2\mathbf{V}^{-1}\mathbf{1} = \tfrac{1}{3}\,\mathbf{1}\,.$$

Simple Portfolio with Three Risky Assets

The frontier is given by

$$\sigma = \sigma_{\mathrm{f}}(\mu) = s\sqrt{\frac{1+2r}{3} + \frac{1-r}{2}\left(\frac{\mu - m}{d}\right)^2} \qquad \text{for } \mu \in (-\infty, \infty)\,.$$

Notice that each (σ_i, m_i) lies strictly to the right of the frontier because

$$\sigma_{\mathrm{mv}} = \sigma_{\mathrm{f}}(m) = s\sqrt{\frac{1+2r}{3}} < \sigma_{\mathrm{f}}(m \pm d) = s\sqrt{\frac{5+r}{6}} < s\,.$$

Notice that as r decreases the frontier moves to the left for $|\mu - m| < \frac{2}{3}\sqrt{3}d$ and to the right for $|\mu - m| > \frac{2}{3}\sqrt{3}d$.

Simple Portfolio with Three Risky Assets

The allocation of the frontier portfolio with return mean μ is

$$\mathbf{f}_{\mathrm{f}}(\mu) = \mathbf{f}_{\mathrm{mv}} + \frac{\mu - \mu_{\mathrm{mv}}}{\nu_{\mathrm{as}}^2} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathrm{mv}}\mathbf{1}) = \begin{pmatrix} \frac{1}{3} - \frac{\mu - m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu - m}{2d} \end{pmatrix} .$$

Notice that the frontier portfolio will hold long postitions in all three assets when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. It will hold a short position in the first asset when $\mu > m + \frac{2}{3}d$, and a short position in the third asset when $\mu < m - \frac{2}{3}d$. In particular, in order to create a portfolio with return mean μ greater than that of any asset contained within it you must short sell the asset with the lowest return mean and invest the proceeds into the asset with the highest return mean. The fact that $\mathbf{f}_{\mathrm{f}}(\mu)$ is independent of $\mathbf{V}$ is a consequence of the simple forms of both $\mathbf{V}$ and $\mathbf{m}$. This is also why the fraction of the investment in the second asset is a constant.

Simple Portfolio with Three Risky Assets

Remark. The frontier portfolios for this example are independent of all the parameters in $\mathbf{V}$. While this is not generally true, it is generally true that they are independent of the overall market volatility. *Said another way, the frontier portfolios depend only upon the correlations c_{ij}, the volatility ratios σ_i/σ_j, and the means m_i. Moreover, the minimum volatility portfolio $\mathbf{f}_{\mathrm{mv}}$ depends only upon the correlations and the volatility ratios.* Because markets can exhibit periods of markedly different volatility, it is natural to ask when correlations and volatility ratios might be relatively stable across such periods.