Portfolios that Contain Risky Assets 3: Markowitz Portfolios

C. David Levermore

University of Maryland, College Park, MD

Math 420: *Mathematical Modeling*
March 21, 2018 version

Portfolios that Contain Risky Assets
Part I: Portfolio Models

1. Risk and Reward
2. Covariance Matrices
3. Markowitz Portfolios
4. Solvent Portfolios
5. Limited Portfolios
6. Markowitz Frontiers for Unlimited Portfolios
7. Markowitz Frontiers for Long Portfolios
8. Markowitz Frontiers for Limited Portfolios
9. Unlimited Portfolios with Risk-Free Assets
10. Limited Portfolios with Risk-Free Assets

Markowitz Portfolios

Portfolios

We will consider portfolios in which an investor can hold one of three positions with respect to any risky asset. The investor can:

- hold a *long position* by owning shares of the asset;
- hold a *short position* by selling borrowed shares of the asset;
- hold a *neutral position* by doing neither of the above.

In order to keep things simple, we will not consider derivative assets.

Remark. We hold a short position by borrowing shares of an asset from a lender (usually our broker) and selling them immediately. If the share price subsequently goes down then we can buy the same number of shares and give them to the lender, thereby paying off our loan and profiting by the price difference minus transaction costs. Of course, if the price goes up then our lender can force us either to increase our collateral or to pay off the loan by buying shares at this higher price, thereby taking a loss that might be larger than the original value of the shares.

Portfolios

We will consider portfolios that hold $n_i(d)$ shares of asset i throughout trading day d.

- If we hold a long position in asset i on day d then $n_i(d) > 0$.
- If we hold a short position in asset i on day d then $n_i(d) < 0$.
- If we hold a neutral position in asset i on day d then $n_i(d) = 0$.

The value of such a portfolio at the end of trading day d is

$$\pi(d) = \sum_{i=1}^{N} n_i(d)\, s_i(d)\,. \tag{1.1}$$

The return of such a portfolio for trading day d is

$$r(d) = \frac{\pi(d)}{\pi(d-1)} - 1 = \frac{\pi(d) - \pi(d-1)}{\pi(d-1)}\,, \tag{1.2}$$

where we assume that $\pi(d-1) > 0$ for every d.

Portfolios

Such a portfolio is called *self-financing* if $\{n_i(d)\}_{i=1}^N$ satisfies

$$\pi(d-1) = \sum_{i=1}^{N} n_i(d)\, s_i(d-1) \quad \text{for every } d\,. \tag{1.3}$$

This is an idealization that neglects trading costs and assumes that the opening price for a share of asset i on day d is $s_i(d-1)$.

If a portfolio is self-financing then by using (1.1) and (1.3) in (1.2) we see that its return for trading day d is

$$r(d) = \sum_{i=1}^{N} \frac{n_i(d)}{\pi(d-1)} \left(s_i(d) - s_i(d-1)\right). \tag{1.4}$$

Portfolios

A 1952 paper by Harry Markowitz had enormous influence on the theory and practice of portfolio management and financial engineering ever since.

- It presented his doctoral dissertation work at the Unversity of Chicago, for which he was awarded the Nobel Prize in Economics in 1990.
- It was the first work to quantify how diversifying a portfolio can reduce its risk without changing its potential reward. It did this because it was the first work to use the covariance between different assets in an essential way.

The key to carrying out this work was modeling. The first modeling step was to develop a class of idealized portfolios that is simple enough to analyze, yet is rich enough to yield useful results.

Portfolios

Markowitz carried out his analysis on a class of idealized portfolios that are each characterized by a set of real numbers $\{f_i\}_{i=1}^N$ that satisfy

$$\sum_{i=1}^{N} f_i = 1 . \tag{1.5}$$

The portfolio picks $n_i(d)$ at the beginning at each trading day d so that

$$\frac{n_i(d) s_i(d-1)}{\pi(d-1)} = f_i , \tag{1.6}$$

where $n_i(d)$ need not be an integer. We call these *Markowitz portfolios.*

The number f_i is called the *allocation of asset i.* It uniquely determines $n_i(d)$ at the start of each trading day.

Portfolios

Specifically, at the start of trading day d we determine $\{n_i(d)\}_{i=1}^{N}$ from (1.6) as

$$n_i(d) = \frac{f_i\,\pi(d-1)}{s_i(d-1)}\,. \tag{1.7}$$

We see that for so long as $\pi(d-1) > 0$ and $s_i(d-1) > 0$ the portfolio

- holds a long position in asset i if $f_i > 0$,
- holds a short position in asset i if $f_i < 0$,
- holds a neutral position in asset i if $f_i = 0$.

Remark. If every f_i is nonnegative then f_i is the fraction of the portfolio's value held in asset i at the start of each day.

Portfolio Returns

Because by (1.5) the allocations $\{f_i\}_{i=1}^N$ sum to 1, it follows from (1.6) that every Markowitz portfolio satifies (1.3) and thereby is self-financing.

Because Markowitz portfolios are self-financing, it follows from (1.4) and relationship (1.6) between $n_i(d)$ and f_i that the return $r(d)$ of a Markowitz portfolio for trading day d is

$$r(d) = \sum_{i=1}^{N} \frac{n_i(d)s_i(d-1)}{\pi(d-1)} \, \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \sum_{i=1}^{N} f_i \, r_i(d) \, . \tag{2.8}$$

The return $r(d)$ for the Markowitz portfolio characterized by $\{f_i\}_{i=1}^N$ is thereby simply the linear combination of the $r_i(d)$ with the coefficients f_i. *This relationship makes the class of Markowitz portfolios easy to analyze. It is the reason we will use Markowitz portfolios to model real portfolios.*

Portfolio Returns

Relationship (2.8) can be expressed in the compact form

$$r(d) = \mathbf{f}^{\mathrm{T}} \mathbf{r}(d) \,, \tag{2.9}$$

where $\mathbf{f}$ and $\mathbf{r}(d)$ are the N-vectors defined by

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix} , \qquad \mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix} .$$

We call $\mathbf{f}$ the *allocation vector* or simply the *allocation.*

The constraint (1.5) can be expressed in the compact form

$$\mathbf{1}^{\mathrm{T}} \mathbf{f} = 1 \,, \tag{2.10}$$

where $\mathbf{1}$ denotes the N-vector with each entry equal to 1.

Portfolio Statistics

Recall that if we assign weights $\{w(d)\}_{d=1}^{D}$ to the trading days of a daily return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ then the N-vector of return means $\mathbf{m}$ and the $N \times N$-matrix of return covariances $\mathbf{V}$ can be expressed in terms of $\mathbf{r}(d)$ as

$$\mathbf{m} = \begin{pmatrix} m_1 \\ \vdots \\ m_N \end{pmatrix} = \sum_{d=1}^{D} w(d)\, \mathbf{r}(d)\,,$$

$$\mathbf{V} = \begin{pmatrix} v_{11} & \cdots & v_{1N} \\ \vdots & \ddots & \vdots \\ v_{N1} & \cdots & v_{NN} \end{pmatrix} = \sum_{d=1}^{D} w(d)\, \big(\mathbf{r}(d) - \mathbf{m}\big)\, \big(\mathbf{r}(d) - \mathbf{m}\big)^{\mathrm{T}}\,.$$

The choices of the daily return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ and weights $\{w(d)\}_{d=1}^{D}$ specify the *calibration* of our models. *Ideally* $\mathbf{m}$ *and* $\mathbf{V}$ *should not be overly sensitive to these choices.*

Portfolio Statistics

Because $r(d) = \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$, the portfolio return mean μ for the Markowitz portfolio with allocation $\mathbf{f}$ is then given by

$$\mu = \sum_{d=1}^{D} w(d)\, r(d) = \sum_{d=1}^{D} w(d)\, \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$$
$$= \mathbf{f}^{\mathrm{T}}\left(\sum_{d=1}^{D} w(d)\, \mathbf{r}(d)\right) = \mathbf{f}^{\mathrm{T}}\mathbf{m}\,.$$

Hence, $\mu = \mathbf{f}^{\mathrm{T}}\mathbf{m}$.

Portfolio Statistics

Because $r(d) = \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$, the portfolio return variance v for the Markowitz portfolio with allocation $\mathbf{f}$ are then given by

$$
\begin{aligned}
v &= \sum_{d=1}^{D} w(d)\,\big(r(d) - \mu\big)^2 = \sum_{d=1}^{D} w(d)\,\big(\mathbf{f}^{\mathrm{T}}\mathbf{r}(d) - \mathbf{f}^{\mathrm{T}}\mathbf{m}\big)^2 \\
&= \sum_{d=1}^{D} w(d)\,\big(\mathbf{f}^{\mathrm{T}}\mathbf{r}(d) - \mathbf{f}^{\mathrm{T}}\mathbf{m}\big)\big(\mathbf{r}(d)^{\mathrm{T}}\mathbf{f} - \mathbf{m}^{\mathrm{T}}\mathbf{f}\big) \\
&= \mathbf{f}^{\mathrm{T}}\left(\sum_{d=1}^{D} w(d)\,\big(\mathbf{r}(d) - \mathbf{m}\big)\big(\mathbf{r}(d) - \mathbf{m}\big)^{\mathrm{T}}\right)\mathbf{f} = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}\,.
\end{aligned}
$$

Hence, $v = \mathbf{f}^{\mathrm{T}}\mathbf{V}\mathbf{f}$. Because $\mathbf{V}$ is positive definite, $v > 0$.

Remark. These simple formulas for μ and v are the reason that returns are preferred over growth rates when compiling statistics of markets.

Portfolio Statistics

Remark. This simplicity arises because $\mathbf{f}$ is independent of d and because the return $r(d)$ for the Markowitz portfolio with allocation $\mathbf{f}$ depends linearly upon the vector $\mathbf{r}(d)$ of returns as $r(d) = \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)$. In contrast, the growth rates $x(d)$ of a Markowitz portfolio are given by

$$
\begin{aligned}
x(d) &= \log\left(\frac{\pi(d)}{\pi(d-1)}\right) = \log\big(1 + r(d)\big) \\
&= \log\left(1 + \mathbf{f}^{\mathrm{T}}\mathbf{r}(d)\right) = \log\left(1 + \sum_{i=1}^{N} f_i \, r_i(d)\right) \\
&= \log\left(1 + \sum_{i=1}^{N} f_i \, \left(e^{x_i(d)} - 1\right)\right) = \log\left(\sum_{i=1}^{N} f_i \, e^{x_i(d)}\right) .
\end{aligned}
$$

Because the $x(d)$ are not linear functions of the $x_i(d)$, averages of $x(d)$ over d are not simply expressed in terms of averages of $x_i(d)$ over d.

Long Portfolios

Because the value of any portfolio with short positions can become negative, many investors will not hold a short position in any risky asset. Portfolios that hold no short positions are called *long portfolios*.

A Markowitz portfolio with allocation $\mathbf{f}$ *is long if and only if* $f_i \geq 0$ *for every* i. This can be expressed compactly as

$$\mathbf{f} \geq \mathbf{0}\,, \tag{4.11}$$

where $\mathbf{0}$ denotes the N-vector with each entry equal to 0 and the inequality is understood entrywise. Therefore the set of all long Markowitz portfolio allocations Λ is given by

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^{\mathrm{T}}\mathbf{f} = 1\,,\, \mathbf{f} \geq \mathbf{0} \right\}. \tag{4.12}$$

Long Portfolios

Let $\mathbf{e}_i$ denote the vector whose i^{th} entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where $f_i \geq 0$ for every $i = 1, \cdots, N$ and

$$\sum_{i=1}^{N} f_i = \mathbf{1}^{\mathrm{T}} \mathbf{f} = 1 \,.$$

This shows that Λ is simply all convex combinations of the vectors $\{\mathbf{e}_i\}_{i=1}^{N}$. We can visualize Λ when N is small.
When $N = 2$ it is the line segment that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} .$$

Long Portfolios

When $N = 3$ it is the triangle that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general N it is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$.

Critique

Aspects of Markowitz portfolios are unrealistic. These include:

- the fact portfolios can contain fractional shares of any asset;
- the fact portfolios are rebalanced every trading day;
- the fact transaction costs and taxes are neglected;
- the fact dividends are neglected.

By making these simplifications the subsequent analysis becomes easier. The idea is to find the Markowitz portfolio that is best for a given investor. The expectation is that any real portfolio with an allocation close to that for the optimal Markowitz portfolio will perform nearly as well. Consequently, most investors rebalance at most a few times per year, and not every asset is involved each time. Transaction costs and taxes are thereby limited. Similarly, borrowing costs are kept to a minimum by not borrowing often. The model can be modified to account for dividends.

Critique

Remark. Portfolios of risky assets can be designed for *trading or investing.*

Traders often take positions that require constant attention. They might buy and sell assets on short time scales in an attempt to profit from market fluctuations. They might also take highly leveraged positions that expose them to enormous gains or loses depending how the market moves. They must be ready to handle margin calls. Trading is often a full time job.

Investors operate on longer time scales. They buy or sell an asset based on their assessment of its fundamental value over time. Investing does not have to be a full time job. Indeed, most people who hold risky assets are investors who are saving for retirement. Lured by the promise of high returns, sometimes investors will buy shares in funds designed for traders. At that point they have become gamblers, whether they realize it or not.

The ideas presented in this course are designed to balance investment portfolios, not trading portfolios.