Portfolios that Contain Risky Assets 17

Fortune's Formulas

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Portfolios that Contain Risky Assets
Part II: Stochastic Models

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Portfolios 17. Fortune's Formulas

Introduction. We now consider some settings in which the optimization problem can be solved analytically. Specifically, we will derive explicit formulas for the solutions to the maximization problems for the family of parabolic objectives

$$\Gamma_{\mathsf{p}}^{\chi}(\mathbf{f}) = \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,, \tag{1a}$$

the family of quadratic objectives

$$\Gamma_{\mathsf{q}}^{\chi}(\mathbf{f}) = \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,, \tag{1b}$$

and the family of reasonable objectives

$$\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f}) = \log\left(1 + \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,, \tag{1c}$$

considered over their natural domains of allocations $\mathbf{f}$ for unlimited leverage portfolios with the One Risk-Free Rate model.

In the previous lecture we saw that the maximizer $\mathbf{f}_*$ for such a problem will correspond to a point (σ_*, μ_*) on the efficient frontier. Moreover, we saw that (σ_*, μ_*) is the point in the $\sigma\mu$-plane where the level curves of the objective are tangent to the efficient frontier. While this geometric picture gave insight into how optimal portfolio allocations arise, we do not yet have an algorthim by which to compute them.

The explicit formulas derived in this lecture for the maximizer $\mathbf{f}_*$ will confirm the general picture developed in the previous lecture. They will also give insight into the relative merits of the different families of objectives in (1). In particular, the maximizers when $\chi = 0$ give different realizations of the Kelly Criterion — so-called *fortune's formulas*. The maximizers when $\chi > 0$ will be corresponding fractional Kelly strategies. We will derive and analyze these formulas after reviewing the efficient frontier for unlimited leverage portfolios with the One Risk-Free Rate model.

Efficient Frontier. Recall that the frontier for unlimited leverage portfolios without risk-free assets is the hyperbola in the right-half of the $\sigma\mu$-plane given by

$$\sigma = \sqrt{\sigma_{\mathsf{mv}}^2 + \left(\frac{\mu - \mu_{\mathsf{mv}}}{\nu_{\mathsf{as}}}\right)^2}\,, \tag{2a}$$

where the so-called frontier parameters σ_{mv}, μ_{mv}, and ν_{as} are given by

$$\frac{1}{\sigma_{\mathsf{mv}}^2} = \mathbf{1}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{1}\,, \qquad \mu_{\mathsf{mv}} = \frac{\mathbf{1}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{m}}{\mathbf{1}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{1}}\,,$$
$$\nu_{\mathsf{as}}^2 = \mathbf{m}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{m} - \frac{(\mathbf{1}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{m})^2}{\mathbf{1}^{\mathsf{T}}\mathbf{V}^{-1}\mathbf{1}}\,. \tag{2b}$$

This so-called frontier hyperbola has vertex $(\sigma_{\mathsf{mv}}, \mu_{\mathsf{mv}})$ and asymptotes

$$\mu = \mu_{\mathsf{mv}} \pm \nu_{\mathsf{as}}\,\sigma \qquad \text{for } \sigma \geq 0\,.$$

The positive definiteness of $\mathbf{V}$ insures that $\sigma_{\mathsf{mv}} > 0$ and $\nu_{\mathsf{as}} > 0$.

If we introduce risk-free assets and use the One Risk-Free Rate model with risk-free return $\mu_{\text{rf}} < \mu_{\text{mv}}$ then the *efficient frontier* becomes the tangent half-line given by

$$\mu = \mu_{\text{rf}} + \nu_{\text{tg}}\,\sigma \qquad \text{for } \sigma \geq 0\,, \tag{3a}$$

where the slope is

$$\nu_{\text{tg}} = \nu_{\text{as}}\sqrt{1 + \left(\frac{\mu_{\text{mv}} - \mu_{\text{rf}}}{\nu_{\text{as}}\,\sigma_{\text{mv}}}\right)^2}\,. \tag{3b}$$

The *Sharpe ratio* of any portfolio with return mean μ and volatility σ is defined to be

$$\frac{\mu - \mu_{\text{rf}}}{\sigma}\,.$$

Clearly ν_{tg} is the Sharpe ratio of all portfolios on the efficient frontier (3a). Moreover, ν_{tg} is the largest possible Sharpe ratio for any portfolio.

The efficient frontier (3a) is tangent to the frontier hyperbola (2a) at the point $(\sigma_{\mathsf{tg}}, \mu_{\mathsf{tg}})$ where

$$\sigma_{\mathsf{tg}} = \sigma_{\mathsf{mv}} \sqrt{1 + \left(\frac{\nu_{\mathsf{as}}\, \sigma_{\mathsf{mv}}}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}} \right)^2} \,, \qquad \mu_{\mathsf{tg}} = \mu_{\mathsf{mv}} + \frac{\nu_{\mathsf{as}}^2\, \sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}} \,.$$

The unique *tangency portfolio* associated with this point has allocation

$$\mathbf{f}_{\mathsf{tg}} = \frac{\sigma_{\mathsf{mv}}^2}{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}} \mathbf{V}^{-1}(\mathbf{m} - \mu_{\mathsf{rf}} \mathbf{1}) \,. \tag{4}$$

Every portfolio on the efficient frontier (3a) can be viewed as holding a position in this tangency portfolio and a position in a risk-free asset.

We can select a particular portfolio on this efficient frontier by identifying an objective function to be maximized. In subsequent sections we derive and analyze explicit formulas for the maximizers for each family member of the parabolic, quadratic, and reasonable objectives given in (1).

Parabolic Objectives. First we consider the maximization problem

$$\mathbf{f}_* = \operatorname{arg\,max}\Big\{\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) \;:\; \mathbf{f} \in \mathbb{R}^N\Big\}\,, \tag{5a}$$

where $\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f})$ is the family of parabolic objectives parametrized by $\chi \geq 0$ and given by

$$\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) = \mu_{\mathrm{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,. \tag{5b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\Gamma_{\mathrm{p}}^{\chi}(\mathbf{f}) = \tilde{\mathbf{m}} - \mathbf{V}\mathbf{f} - \frac{\chi}{\sigma}\,\mathbf{V}\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} > 0$. By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = \tilde{\mathbf{m}} - \frac{\sigma_* + \chi}{\sigma_*}\,\mathbf{V}\mathbf{f}_*\,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^{\mathsf{T}}\mathbf{V}\mathbf{f}_*} > 0$.

By solving this equation for $\mathbf{f}_*$ we obtain

$$\mathbf{f}_* = \frac{\sigma_*}{\sigma_* + \chi}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,. \tag{6}$$

Because $\sigma_* = \sqrt{\mathbf{f}_*^\top \mathbf{V} \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^\top \mathbf{V} \mathbf{f}_* = \frac{\sigma_*^2}{(\sigma_* + \chi)^2}\,\tilde{\mathbf{m}}^\top \mathbf{V}^{-1}\tilde{\mathbf{m}} = \frac{\sigma_*^2}{(\sigma_* + \chi)^2}\,\nu_{\mathrm{tg}}^2\,,$$

we conclude that σ_* satisfies

$$(\sigma_* + \chi)^2 = \nu_{\mathrm{tg}}^2\,.$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < \nu_{\mathrm{tg}}\,, \tag{7}$$

and that σ_* is determined by

$$\sigma_* + \chi = \nu_{\mathrm{tg}}\,.$$

Then the maximizer $\mathbf{f}_*$ given by (6) becomes

$$\mathbf{f}_* = \left(1 - \frac{\chi}{\nu_{\mathrm{tg}}}\right) \mathbf{V}^{-1}\tilde{\mathbf{m}}\,. \tag{8}$$

Remark. Pure Kelly investors take $\chi = 0$, in which case (8) reduces to

$$\mathbf{f}_* = \mathbf{V}^{-1}\tilde{\mathbf{m}}\,. \tag{9}$$

Formula (9) is often called *fortune's formula* in the belief that it is a good approximation to the Kelly strategy. In this view formula (8) gives an explicit fractional Kelly strategy for every $\chi \in (0, \nu_{\mathrm{tg}})$. However, we will see that formula (9) gives an allocation that can be far from the Kelly strategy, and generally leads to overbetting.

The foregoing analysis did not yield a maximzier when $\chi \geq \nu_{\mathsf{tg}}$. We now show that in this case $\mathbf{f}_* = \mathbf{0}$. The key to doing this is the *Cauchy inequality* in the form

$$\left|\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\right| \leq \sqrt{\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,. \tag{10}$$

When $\chi \geq \nu_{\mathsf{tg}}$ we use the positive definiteness of $\mathbf{V}$, the fact $\chi \geq \nu_{\mathsf{tg}}$, and the above Cauchy inequality to show

$$\begin{aligned}
\Gamma_{\mathsf{p}}^{\chi}(\mathbf{f}) &= \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \nu_{\mathsf{tg}}\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&= \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \sqrt{\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathsf{rf}} = \Gamma_{\mathsf{p}}^{\chi}(\mathbf{0})\,.
\end{aligned}$$

This implies that $\mathbf{f}_* = \mathbf{0}$.

Therefore the solution $\mathbf{f}_*$ of the maximization problem (5) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \mathbf{V}^{-1}\tilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathsf{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathsf{tg}}\,. \end{cases} \tag{11}$$

This solution lies on the efficient frontier (3a). It allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathsf{tg}}$ given by (4) and $1 - f_{\mathsf{tg}}^{\chi}$ times the portfolio value in a risk-free asset, where

$$f_{\mathsf{tg}}^{\chi} = \left(1 - \frac{\chi}{\nu_{\mathsf{tg}}}\right) \frac{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}{\sigma_{\mathsf{mv}}^2}\,. \tag{12}$$

Quadratic Objectives. Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\Big\{\Gamma_\mathrm{q}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N\Big\}\,, \tag{13a}$$

where $\Gamma_\mathrm{q}^{\chi}(\mathbf{f})$ is the family of quadratic objectives parametrized by $\chi \geq 0$ and given by

$$\Gamma_\mathrm{q}^{\chi}(\mathbf{f}) = \mu_\mathrm{rf} + \tilde{\mathbf{m}}^\mathsf{T}\mathbf{f} - \tfrac{1}{2}\left(\mu_\mathrm{rf} + \tilde{\mathbf{m}}^\mathsf{T}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^\mathsf{T}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^\mathsf{T}\mathbf{V}\mathbf{f}}\,. \tag{13b}$$

If $\mathbf{f} \neq 0$ then the gradient of $\Gamma_\mathrm{q}^{\chi}(\mathbf{f})$ is

$$\nabla_\mathbf{f}\Gamma_\mathrm{q}^{\chi}(\mathbf{f}) = (1 - \mu_\mathrm{rf})\tilde{\mathbf{m}} - \tilde{\mathbf{m}}\,\tilde{\mathbf{m}}^\mathsf{T}\mathbf{f} - \mathbf{V}\mathbf{f} - \frac{\chi}{\sigma}\mathbf{V}\mathbf{f}\,,$$

where $\sigma = \sqrt{\mathbf{f}^\mathsf{T}\mathbf{V}\mathbf{f}} > 0$. By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{0} = (1 - \mu_\mathrm{rf})\tilde{\mathbf{m}} - \tilde{\mathbf{m}}\,\tilde{\mathbf{m}}^\mathsf{T}\mathbf{f}_* - \frac{\sigma_* + \chi}{\sigma_*}\mathbf{V}\mathbf{f}_*\,,$$

where $\sigma_* = \sqrt{\mathbf{f}_*^\mathsf{T}\mathbf{V}\mathbf{f}_*} > 0$.

After multiplying this relation by $\mathbf{V}^{-1}$ and bringing the terms involving $\mathbf{f}_*$ to the left-hand side, we obtain

$$\frac{\sigma_* + \chi}{\sigma_*}\,\mathbf{f}_* + \mathbf{V}^{-1}\tilde{\mathbf{m}}\,\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}_* = (1 - \mu_{\mathsf{rf}})\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,. \tag{14}$$

Now multiply this by $\sigma_*\,\tilde{\mathbf{m}}^{\mathsf{T}}$ and use the fact that $\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}} = \nu_{\mathsf{tg}}^2$ to obtain

$$\left(\sigma_* + \chi + \nu_{\mathsf{tg}}^2\,\sigma_*\right)\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}_* = (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}^2\,\sigma_*\,,$$

which implies that

$$\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}_* = (1 - \mu_{\mathsf{rf}})\,\frac{\nu_{\mathsf{tg}}^2\,\sigma_*}{\sigma_* + \chi + \nu_{\mathsf{tg}}^2\,\sigma_*}\,.$$

When this expression is placed into (14) we can solve for $\mathbf{f}_*$ to find

$$\mathbf{f}_* = (1 - \mu_{\mathsf{rf}})\,\frac{\sigma_*}{\sigma_* + \chi + \nu_{\mathsf{tg}}^2\,\sigma_*}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,. \tag{15}$$

Because $\sigma_* = \sqrt{\mathbf{f}_*^\mathsf{T} \mathbf{V} \mathbf{f}_*}$ we have

$$\sigma_*^2 = \mathbf{f}_*^\mathsf{T} \mathbf{V} \mathbf{f}_* = \frac{(1-\mu_{\mathsf{rf}})^2 \, \sigma_*^2}{\Big((1+\nu_{\mathsf{tg}}^2)\,\sigma_* + \chi\Big)^2} \, \tilde{\mathbf{m}}_*^\mathsf{T} \mathbf{V}^{-1} \tilde{\mathbf{m}}$$
$$= \frac{(1-\mu_{\mathsf{rf}})^2 \, \sigma_*^2}{\Big((1+\nu_{\mathsf{tg}}^2)\,\sigma_* + \chi\Big)^2} \, \nu_{\mathsf{tg}}^2 \,,$$

we conclude that σ_* satisfies

$$\Big((1+\nu_{\mathsf{tg}}^2)\,\sigma_* + \chi\Big)^2 = (1-\mu_{\mathsf{rf}})^2 \, \nu_{\mathsf{tg}}^2 \,.$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < (1-\mu_{\mathsf{rf}}) \, \nu_{\mathsf{tg}} \,, \tag{16}$$

and that σ_* is determined by

$$(1+\nu_{\mathsf{tg}}^2)\,\sigma_* + \chi = (1-\mu_{\mathsf{rf}}) \, \nu_{\mathsf{tg}} \,.$$

Then the maximizer $\mathbf{f}_*$ given by (15) becomes

$$\mathbf{f}_* = \left(1 - \mu_{\mathsf{rf}} - \frac{\chi}{\nu_{\mathsf{tg}}}\right) \frac{1}{1 + \nu_{\mathsf{tg}}^2} \mathbf{V}^{-1}\tilde{\mathbf{m}} \,. \tag{17}$$

Remark. Pure Kelly investors take $\chi = 0$, in which case (17) reduces to

$$\mathbf{f}_* = \frac{1 - \mu_{\mathsf{rf}}}{1 + \nu_{\mathsf{tg}}^2} \mathbf{V}^{-1}\tilde{\mathbf{m}} \,. \tag{18}$$

Formula (18) differs significantly from formula (9) whenever the Sharpe ratio ν_{tg} is not small. Sharpe ratios are often near 1 and sometimes can be as large as 3. So which of these should be called *fortune's formula*? Certainly not formula (9)! To see why, set $\mathbf{f} = \mathbf{V}^{-1}\tilde{\mathbf{m}}$ into the quadratic objective (13b) with $\chi = 0$ to obtain

$$\Gamma_{\mathsf{q}}^0\left(\mathbf{V}^{-1}\tilde{\mathbf{m}}\right) = \mu_{\mathsf{rf}} + \tfrac{1}{2}\,\nu_{\mathsf{tg}}^2 - \tfrac{1}{2}\left(\mu_{\mathsf{rf}} + \nu_{\mathsf{tg}}^2\right)^2 ,$$

which can be negative when ν_{tg} is near 1. So formula (9) can overbet!

The foregoing analysis did not yield a maximzier when $\chi \geq (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}$. We now show that in this case $\mathbf{f}_* = \mathbf{0}$. The key to doing this is again the *Cauchy inequality* (10). When $\chi \geq (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}$ we use the positive definiteness of $\mathbf{V}$, the fact $\chi \geq (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}$, and the Cauchy inequality to show

$$
\begin{aligned}
\Gamma_{\mathsf{q}}^{\chi}(\mathbf{f}) &= \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\right)^2 - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \tfrac{1}{2}\left(\mu_{\mathsf{rf}}^2 + 2\mu_{\mathsf{rf}}\,\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\right) - \chi\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&= \mu_{\mathsf{rf}} - \tfrac{1}{2}\,\mu_{\mathsf{rf}}^2 + (1 - \mu_{\mathsf{rf}})\,\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \mu_{\mathsf{rf}} - \tfrac{1}{2}\,\mu_{\mathsf{rf}}^2 + (1 - \mu_{\mathsf{rf}})\,\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&= \mu_{\mathsf{rf}} - \tfrac{1}{2}\,\mu_{\mathsf{rf}}^2 + (1 - \mu_{\mathsf{rf}})\left(\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \sqrt{\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}}}\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\right) \\
&\leq \mu_{\mathsf{rf}} - \tfrac{1}{2}\,\mu_{\mathsf{rf}}^2 = \Gamma_{\mathsf{q}}^{\chi}(\mathbf{0})\,.
\end{aligned}
$$

This implies that $\mathbf{f}_* = \mathbf{0}$.

Therefore the solution $\mathbf{f}_*$ of the maximization problem (13) is

$$\mathbf{f}_* = \begin{cases} \left(1 - \mu_{\mathsf{rf}} - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \dfrac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{1 + \nu_{\mathsf{tg}}^2} & \text{if } \chi < (1 - \mu_{\mathsf{rf}})\, \nu_{\mathsf{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq (1 - \mu_{\mathsf{rf}})\, \nu_{\mathsf{tg}}\,. \end{cases} \tag{19}$$

This solution lies on the efficient frontier (3a). It allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\mathsf{tg}}$ given by (4) and $1 - f_{\mathsf{tg}}^{\chi}$ times the portfolio value in a risk-free asset, where

$$f_{\mathsf{tg}}^{\chi} = \left(1 - \mu_{\mathsf{rf}} - \frac{\chi}{\nu_{\mathsf{tg}}}\right) \frac{1}{1 + \nu_{\mathsf{tg}}^2} \, \frac{\mu_{\mathsf{mv}} - \mu_{\mathsf{rf}}}{\sigma_{\mathsf{mv}}^2}\,. \tag{20}$$

Reasonable Objectives. Next we consider the maximization problem

$$\mathbf{f}_* = \arg\max\Big\{\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f}) \,:\, \mathbf{f} \in \mathbb{R}^N\,,\, 1 + \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} > 0\Big\}\,, \tag{21a}$$

where $\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f})$ is the family of reasonable objectives parametrized by $\chi \geq 0$ and given by

$$\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f}) = \log\Big(1 + \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\Big) - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,. \tag{21b}$$

Because $\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f}) \to -\infty$ as $\mathbf{f}$ approaches the boundary of the domain being considered in (21a), the maximizer $\mathbf{f}_*$ must lie in the interior of the domain. If $\mathbf{f} \neq 0$ then the gradient of $\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f})$ is

$$\nabla_{\mathbf{f}}\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f}) = \frac{1}{1+\mu}\,\tilde{\mathbf{m}} - \mathbf{V}\mathbf{f} - \frac{\chi}{\sigma}\,\mathbf{V}\mathbf{f}\,,$$

where $\mu = \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}$ and $\sigma = \sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} > 0$.

By setting this gradient equal to zero we see that if the maximizer $\mathbf{f}_*$ is nonzero then it satisfies

$$\mathbf{f}_* = \frac{1}{1+\mu_*} \frac{\sigma_*}{\sigma_* + \chi} \mathbf{V}^{-1}\tilde{\mathbf{m}} \,, \tag{22}$$

where $\mu_* = \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}_*$ and $\sigma_* = \sqrt{\mathbf{f}_*^{\mathsf{T}}\mathbf{V}\mathbf{f}_*} > 0$.

Because $\sigma_* = \sqrt{\mathbf{f}_*^{\mathsf{T}}\mathbf{V}\mathbf{f}_*}$ we have

$$\begin{aligned}\sigma_*^2 = \mathbf{f}_*^{\mathsf{T}}\mathbf{V}\mathbf{f}_* &= \frac{1}{(1+\mu_*)^2} \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}} \\ &= \frac{1}{(1+\mu_*)^2} \frac{\sigma_*^2}{(\sigma_* + \chi)^2} \nu_{\mathsf{tg}}^2 \,.\end{aligned}$$

From this we conclude that μ_* and σ_* satisfy

$$(\sigma_* + \chi)^2 = \frac{\nu_{\mathsf{tg}}^2}{(1+\mu_*)^2} \,.$$

Because $\sigma_* > 0$ and $\chi \geq 0$ we see that

$$0 \leq \chi < \frac{\nu_{\mathsf{tg}}}{1+\mu_*}\,, \tag{23}$$

and that we can determine σ_* in terms of μ_* from

$$\sigma_* + \chi = \frac{\nu_{\mathsf{tg}}}{1+\mu_*}\,.$$

Then the maximizer $\mathbf{f}_*$ given by (22) becomes

$$\mathbf{f}_* = \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathsf{tg}}}\right) \mathbf{V}^{-1}\tilde{\mathbf{m}}\,, \tag{24}$$

Because $\mu_* = \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}_*$ we have

$$\begin{aligned}\mu_* = \mu_{\mathsf{rf}} + \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}_* &= \mu_{\mathsf{rf}} + \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathsf{tg}}}\right) \tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}} \\ &= \mu_{\mathsf{rf}} + \left(\frac{1}{1+\mu_*} - \frac{\chi}{\nu_{\mathsf{tg}}}\right) \nu_{\mathsf{tg}}^2\,.\end{aligned}$$

This can be reduced to the quadratic equation

$$\left(\frac{\nu_{\mathrm{tg}}}{1+\mu_*}\right)^2 + \left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right)\frac{\nu_{\mathrm{tg}}}{1+\mu_*} = 1\,,$$

which has the unique positive root

$$\frac{\nu_{\mathrm{tg}}}{1+\mu_*} = -\frac{1}{2}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right)^2}\,. \tag{25}$$

Then condition (23) is satisfied if and only if

$$\begin{aligned} 0 &< \frac{\nu_{\mathrm{tg}}}{1+\mu_*} - \chi \\ &= -\frac{1}{2}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right)^2}\,. \end{aligned}$$

This inequality holds if and only if

$$0 < 1 + \frac{1}{4}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right)^2 - \frac{1}{4}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} + \chi\right)^2 = 1 - \frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}}\,\chi\,.$$

This holds if and only if χ satisfies the bounds

$$0 \leq \chi < \frac{\nu_{\mathrm{tg}}}{1+\mu_{\mathrm{rf}}}\,. \tag{26}$$

By using (25) to eliminate μ_* from the maximizer $\mathbf{f}_*$ given by (24) we find

$$\mathbf{f}_* = \left[-\frac{1}{2}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} + \chi\right) + \sqrt{1 + \frac{1}{4}\left(\frac{1+\mu_{\mathrm{rf}}}{\nu_{\mathrm{tg}}} - \chi\right)^2}\,\right]\frac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{\nu_{\mathrm{tg}}}\,.$$

This becomes

$$\mathbf{f}_* = \left(\frac{1}{1+\mu_{\mathsf{rf}}} - \frac{\chi}{\nu_{\mathsf{tg}}} \right) \frac{1}{D\!\left(\chi, \dfrac{\nu_{\mathsf{tg}}}{1+\mu_{\mathsf{rf}}}\right)} \mathbf{V}^{-1}\tilde{\mathbf{m}} \,, \tag{27a}$$

where

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2} \,. \tag{27b}$$

Remark. Pure Kelly investors take $\chi = 0$, in which case (27) reduces to

$$\mathbf{f}_* = \frac{1}{\frac{1}{2}(1+\mu_{\mathsf{rf}}) + \frac{1}{2}\sqrt{(1+\mu_{\mathsf{rf}})^2 + 4\nu_{\mathsf{tg}}^2}} \mathbf{V}^{-1}\tilde{\mathbf{m}} \,. \tag{28}$$

This candidate for *fortune's formula* will be compared with the others later.

The foregoing analysis did not yield a maximzier when $(1+\mu_{\mathsf{rf}})\,\chi \geq \nu_{\mathsf{tg}}$. We now show that in this case $\mathbf{f}_* = \mathbf{0}$. The key to doing this is again the *Cauchy inequality* (10). When $(1+\mu_{\mathsf{rf}})\,\chi \geq \nu_{\mathsf{tg}}$ we use the positive definitness of $\mathbf{V}$, the concavity of $\log(x)$, the fact $(1+\mu_{\mathsf{rf}})\,\chi \geq \nu_{\mathsf{tg}}$, and the Cauchy inequality to show

$$
\begin{aligned}
\Gamma_{\mathsf{r}}^{\chi}(\mathbf{f}) &= \log\left(1+\mu_{\mathsf{rf}}+\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}\right) - \tfrac{1}{2}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \log(1+\mu_{\mathsf{rf}}) + \frac{\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}}{1+\mu_{\mathsf{rf}}} - \chi\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&\leq \log(1+\mu_{\mathsf{rf}}) + \frac{\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f}}{1+\mu_{\mathsf{rf}}} - \frac{\nu_{\mathsf{tg}}}{1+\mu_{\mathsf{rf}}}\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \\
&= \log(1+\mu_{\mathsf{rf}}) + \frac{1}{1+\mu_{\mathsf{rf}}}\left(\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{f} - \sqrt{\tilde{\mathbf{m}}^{\mathsf{T}}\mathbf{V}^{-1}\tilde{\mathbf{m}}}\,\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\right) \\
&\leq \log(1+\mu_{\mathsf{rf}}) = \Gamma_{\mathsf{r}}^{\chi}(\mathbf{0})\,.
\end{aligned}
$$

This implies that $\mathbf{f}_* = \mathbf{0}$.

Therefore the solution $\mathbf{f}_*$ of the maximization problem (21) is

$$\mathbf{f}_* = \begin{cases} \left(\dfrac{1}{1+\mu_{\text{rf}}} - \dfrac{\chi}{\nu_{\text{tg}}}\right) \dfrac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{D\left(\chi, \dfrac{\nu_{\text{tg}}}{1+\mu_{\text{rf}}}\right)} & \text{if } \chi < \dfrac{\nu_{\text{tg}}}{1+\mu_{\text{rf}}}\,, \\ \mathbf{0} & \text{if } \chi \geq \dfrac{\nu_{\text{tg}}}{1+\mu_{\text{rf}}}\,, \end{cases} \tag{29}$$

where $D(\chi, y)$ was defined by (27b).

This solution lies on the efficient frontier (3a). It allocates f_{tg}^{χ} times the portfolio value in the tangent portfolio $\mathbf{f}_{\text{tg}}$ given by (4) and $1 - f_{\text{tg}}^{\chi}$ times the portfolio value in a risk-free asset, where

$$f_{\text{tg}}^{\chi} = \left(\frac{1}{1+\mu_{\text{rf}}} - \frac{\chi}{\nu_{\text{tg}}}\right) \frac{1}{D\left(\chi, \dfrac{\nu_{\text{tg}}}{1+\mu_{\text{rf}}}\right)} \frac{\mu_{\text{mv}} - \mu_{\text{rf}}}{\sigma_{\text{mv}}^2}\,. \tag{30}$$

Comparisons. The maximizers for the parabolic, quadratic, and reasonable objectives are given by (11), (19), and (29) respectively. They are

$$\mathbf{f}_*^{\mathsf{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \mathbf{V}^{-1}\tilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathsf{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathsf{tg}}\,, \end{cases} \tag{31a}$$

$$\mathbf{f}_*^{\mathsf{q}} = \begin{cases} \left(1 - \mu_{\mathsf{rf}} - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \dfrac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{1 + \nu_{\mathsf{tg}}^2} & \text{if } \chi < (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}\,, \\ \mathbf{0} & \text{if } \chi \geq (1 - \mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}}\,, \end{cases} \tag{31b}$$

$$\mathbf{f}_*^{\mathsf{r}} = \begin{cases} \left(\dfrac{1}{1 + \mu_{\mathsf{rf}}} - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \dfrac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{D\left(\chi, \dfrac{\nu_{\mathsf{tg}}}{1 + \mu_{\mathsf{rf}}}\right)} & \text{if } \chi < \dfrac{\nu_{\mathsf{tg}}}{1 + \mu_{\mathsf{rf}}}\,, \\ \mathbf{0} & \text{if } \chi \geq \dfrac{\nu_{\mathsf{tg}}}{1 + \mu_{\mathsf{rf}}}\,, \end{cases} \tag{31c}$$

where $D(\chi, y)$ was defined by (27b).

Fact 1. If $\mu_{\mathsf{rf}} \in [0,1)$ then $\mathbf{f}_*^{\mathsf{q}}$ is the most conservative of these allocations and $\mathbf{f}_*^{\mathsf{p}}$ is the most agressive.

Proof. First observe that because $\mu_{\mathsf{rf}} \in [0,1)$ we have

$$1 - \mu_{\mathsf{rf}} \leq \frac{1}{1+\mu_{\mathsf{rf}}} \leq 1\,.$$

These inequalities imply that

$$(1-\mu_{\mathsf{rf}})\,\nu_{\mathsf{tg}} \leq \frac{\nu_{\mathsf{tg}}}{1+\mu_{\mathsf{rf}}} \leq \nu_{\mathsf{tg}}\,, \tag{32}$$

and that

$$1 - \mu_{\mathsf{rf}} - \frac{\chi}{\nu_{\mathsf{tg}}} \leq \frac{1}{1+\mu_{\mathsf{rf}}} - \frac{\chi}{\nu_{\mathsf{tg}}} \leq 1 - \frac{\chi}{\nu_{\mathsf{tg}}}\,. \tag{33}$$

Each of these inequalities is strict when $\mu_{\mathsf{rf}} \in (0,1)$.

Recall from (27b) that

$$D(\chi, y) = \tfrac{1}{2}(1 + \chi\, y) + \tfrac{1}{2}\sqrt{(1 - \chi\, y)^2 + 4y^2}\,. \tag{34}$$

For every $y > 0$ we have

$$\partial_\chi D(\chi, y) = \tfrac{1}{2} y \left(1 - \frac{1 - \chi\, y}{\sqrt{(1 - \chi\, y)^2 + 4y^2}} \right) > 0\,,$$

whereby $D(\chi, y)$ is an strictly increasing function of χ. Hence, for every $\chi \in [0, y)$ we have

$$1 < D(0, y) \leq D(\chi, y) < D(y, y) = 1 + y^2\,. \tag{35}$$

Therefore

$$1 < D\!\left(\chi, \frac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}} \right) < 1 + \frac{\nu_{\mathrm{tg}}^2}{(1 + \mu_{\mathrm{rf}})^2} \leq 1 + \nu_{\mathrm{tg}}^2 \quad \text{if } \chi < \frac{\nu_{\mathrm{tg}}}{1 + \mu_{\mathrm{rf}}}\,. \tag{36}$$

The inequalities (32) imply that $\mathbf{f}_*^{\mathsf{q}}$ given by (31b) has the smallest critical value of χ at which it becomes $\mathbf{0}$, and that $\mathbf{f}_*^{\mathsf{p}}$ given by (31a) has the largest critical value of χ at which it becomes $\mathbf{0}$.

The inequalities (33) and (36) imply that the factor multiplying $\mathbf{V}^{-1}\tilde{\mathbf{m}}$ in the expression for $\mathbf{f}_*^{\mathsf{q}}$ given by (31b) is smaller than the factor multiplying $\mathbf{V}^{-1}\tilde{\mathbf{m}}$ in the expression for $\mathbf{f}_*^{\mathsf{r}}$ given by (31c), which is smaller than the factor multiplying $\mathbf{V}^{-1}\tilde{\mathbf{m}}$ in the expression for $\mathbf{f}_*^{\mathsf{p}}$ given by (31a). Hence, $\mathbf{f}_*^{\mathsf{q}}$ is more conservative than $\mathbf{f}_*^{\mathsf{r}}$, which is more conservative than $\mathbf{f}_*^{\mathsf{p}}$. $\square$

Remark. The risk-free return μ_{rf} is usually much smaller than the Sharpe ratio ν_{tg}. This means that the main differences between the maximizers given by formulas (31) arise due to their dependence upon ν_{tg}.

In practice μ_{rf} is often small enough that it can be neglected. By setting $\mu_{\mathsf{rf}} = 0$ in (31) we get

$$\mathbf{f}_*^{\mathsf{p}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \mathbf{V}^{-1}\tilde{\mathbf{m}} & \text{if } \chi < \nu_{\mathsf{tg}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathsf{tg}} \,, \end{cases} \tag{37a}$$

$$\mathbf{f}_*^{\mathsf{q}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \dfrac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{1 + \nu_{\mathsf{tg}}^2} & \text{if } \chi < \nu_{\mathsf{tg}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathsf{tg}} \,, \end{cases} \tag{37b}$$

$$\mathbf{f}_*^{\mathsf{r}} = \begin{cases} \left(1 - \dfrac{\chi}{\nu_{\mathsf{tg}}}\right) \dfrac{\mathbf{V}^{-1}\tilde{\mathbf{m}}}{D(\chi, \nu_{\mathsf{tg}})} & \text{if } \chi < \nu_{\mathsf{tg}} \,, \\ \mathbf{0} & \text{if } \chi \geq \nu_{\mathsf{tg}} \,, \end{cases} \tag{37c}$$

where $D(\chi, y)$ is given by (34). These all are nonzero for $\chi < \nu_{\mathsf{tg}}$ and all vanish for $\chi \geq \nu_{\mathsf{tg}}$. We see from (35) that $1 < D(\chi, \nu_{\mathsf{tg}}) < 1 + \nu_{\mathsf{tg}}^2$.

We now use formulas (37b) and (37c) to isolate the dependence of the maximizers $\mathbf{f}_*^{\mathsf{q}}$ and $\mathbf{f}_*^{\mathsf{r}}$ upon ν_{tg}.

Fact 2. For every $\chi \in [0, \nu_{\mathrm{tg}})$ we have

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}}{1 + \nu_{\mathrm{tg}}^2} \leq \frac{D(\chi, \nu_{\mathrm{tg}})}{1 + \nu_{\mathrm{tg}}^2} < 1\,, \tag{38}$$

where the left-hand side is a strictly decreasing function of ν_{tg}.

Proof. By (35) we have

$$1 + \nu_{\mathrm{tg}}^2 > D(\chi, \nu_{\mathrm{tg}}) \geq D(0, \nu_{\mathrm{tg}}) = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{1 + 4\nu_{\mathrm{tg}}^2}\,.$$

The inequalities (38) follow. The task of proving the left-hand side of (38) is a strictly decreasing function of ν_{tg} is left as an exercise. □

We now use **Fact 2** to show that $\mathbf{f}_*^{\mathsf{q}}$ and $\mathbf{f}_*^{\mathsf{r}}$ are nearly equal when ν_{tg} is not too large.

Fact 3. If $\nu_{\mathsf{tg}} \leq \frac{2}{3}$ then for every $\chi \in [0, \nu_{\mathsf{tg}})$ we have

$$\tfrac{12}{13} \leq \frac{D(\chi, \nu_{\mathsf{tg}})}{1 + \nu_{\mathsf{tg}}^2} < 1\,. \tag{39}$$

Proof. By the monotonicity asserted in **Fact 2** if $\nu_{\mathsf{tg}} \leq \frac{2}{3}$ then

$$\frac{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4\nu_{\mathsf{tg}}^2}}{1 + \nu_{\mathsf{tg}}^2} \geq \frac{\frac{1}{2} + \frac{1}{2} \cdot \frac{5}{3}}{1 + \frac{4}{9}} = \frac{\frac{4}{3}}{\frac{13}{9}} = \frac{12}{13}\,.$$

Then (39) follows from inequality (38) of **Fact 2**. □

Remark. This fact implies that there is little difference between $\mathbf{f}_*^{\mathsf{q}}$ and $\mathbf{f}_*^{\mathsf{r}}$ when $\nu_{\mathsf{tg}} \leq \frac{2}{3}$.

Remark. A pure Kelly investor would set $\chi = 0$, in which case (31) gives

$$\mathbf{f}_*^{\mathsf{p}} = \mathbf{V}^{-1}\tilde{\mathbf{m}}\,, \tag{40a}$$

$$\mathbf{f}_*^{\mathsf{q}} = \frac{1-\mu_{\mathsf{rf}}}{1+\nu_{\mathsf{tg}}^2}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,, \tag{40b}$$

$$\mathbf{f}_*^{\mathsf{r}} = \frac{1}{\frac{1}{2}(1+\mu_{\mathsf{rf}}) + \frac{1}{2}\sqrt{(1+\mu_{\mathsf{rf}})^2 + 4\nu_{\mathsf{tg}}^2}}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,. \tag{40c}$$

This is the case for which the difference between $\mathbf{f}_*^{\mathsf{q}}$ and $\mathbf{f}_*^{\mathsf{r}}$ is greatest. To get a feel for this difference, when $\mu_{\mathsf{rf}} = 0$ and $\nu_{\mathsf{tg}} = \sqrt{2}$ these become

$$\mathbf{f}_*^{\mathsf{q}} = \tfrac{1}{3}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,, \qquad \mathbf{f}_*^{\mathsf{r}} = \tfrac{1}{2}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,,$$

while when $\mu_{\mathsf{rf}} = 0$ and $\nu_{\mathsf{tg}} = \sqrt{6}$ these become

$$\mathbf{f}_*^{\mathsf{q}} = \tfrac{1}{7}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,, \qquad \mathbf{f}_*^{\mathsf{r}} = \tfrac{1}{3}\,\mathbf{V}^{-1}\tilde{\mathbf{m}}\,.$$

This suggests that these differences might become significant for Sharpe ratios greater than 2.

What We Have Learned. Here are some insights that we have gained.

1. The Sharpe ratio ν_{tg} and the caution coefficient χ play a large role in determining the optimal allocation. In particular, when $\chi \geq \nu_{\mathrm{tg}}$ the optimal allocation is entirely in risk-free assets. The risk-free return μ_{rf} plays a much smaller role in determining the optimal allocation.

2. For any choice of χ the maximizer for the quadratic objective is more conservative than the maximizer for the reasonable objective, which is more conservative than the maximizer for the parabolic objective.

3. The maximizer for a parabolic objective is agressive and will overbet when the Sharpe ratio ν_{tg} is not small.

4. The maximizers for quadratic and reasonable objectives are close when the Sharpe ratio ν_{tg} is not large. As χ approaches ν_{tg}, the maximizers for the quadratic and reasonable objectives will become closer.

5. We will have greater confidence in the computed Sharpe ratio ν_{tg} when the tangency portfolio lies towards the "nose" of the efficient frontier. This translates into greater confidence in the maximizers for the quadratic and reasonable objectives.

6. Analyzing the maximizers for both the quadratic and reasonable objectives gave greater insights than analyzing each of them separately. Together they are *fortune's formulas*.