Portfolios that Contain Risky Assets 15

Central Limit Theorem Objectives

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Portfolios 14. Central Limit Theorem Objectives

Introduction. Because the value of the associated portfolio is

$$\Pi(d) = \Pi(0)\,\exp(Y(d)d)\,, \quad \text{where} \quad Y(d) = \frac{1}{d}\sum_{d'=1}^{d} X(d')\,,$$

we see that $Y(d)$ is the growth rate of the portfolio at day d.

The Law of Large Numbers implies that $Y(d)$ is likely to approach γ as $d \to \infty$. *The Kelly criterion stated that investors whose goal is to maximize the value of their portfolio over an extended period should maximize* γ*. More precisely, it suggests that such investors should select an allocation* $\mathbf{f}$ *that maximizes* $\widehat{\gamma}(\mathbf{f})$*.*

The suggestion to maximize $\widehat{\gamma}(\mathbf{f})$ rests upon the assumptions that $\widehat{\gamma}(\mathbf{f})$ is accurate and that the investor will hold the portfolio for an extended period. The first assumption is foolhardy. The second might be a suitable assumption for most young investors, but not for those older investors who depend upon their portfolios for their income.

The development of objective functions that are better suited for more cautious investors requires more information about $Y(d)$ than the Law of Large Numbers provides. However, this additional information can be estimated with the aid of the *Central Limit Theorem.*

Central Limit Theorem. Let $\{X(d)\}_{d=1}^{\infty}$ be any sequence of IID random variables drawn from a probability density $p(X)$ with mean γ and variance $\theta > 0$. Let $\{Y(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y(d) = \frac{1}{d} \sum_{d'=1}^{d} X(d') \qquad \text{for every } d = 1, \cdots, \infty\,.$$

Recall that

$$\mathrm{Ex}(Y(d)) = \gamma\,, \qquad \mathrm{Var}(Y(d)) = \frac{\theta}{d}\,.$$

Now let $\{Z(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Z(d) = \frac{Y(d) - \gamma}{\sqrt{\theta/d}} \qquad \text{for every } d = 1, \cdots, \infty\,.$$

These random variables have been normalized so that

$$\mathrm{Ex}(Z(d)) = 0\,, \qquad \mathrm{Var}(Z(d)) = 1\,.$$

The *Central Limit Theorem* states that as $d \to \infty$ the limiting distribution of $Z(d)$ will be the mean-zero, variance-one normal distribution. Specifically, for every $\zeta \in \mathbb{R}$ it implies that

$$\lim_{d\to\infty} \Pr\Big\{Z(d) \geq -\zeta\Big\} = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z\,.$$

This can be expressed in terms of $Y(d)$ as

$$\lim_{d\to\infty} \Pr\Big\{Y(d) \geq \gamma - \zeta\sqrt{\theta/d}\Big\} = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z\,. \qquad (1)$$

Remark. The power of the Central Limit Theorem is that it assumes so little about the underlying probability density $p(X)$. Specifically, it assumes that

$$\int_{-\infty}^{\infty} X^2 p(X)\, \mathrm{d}X < \infty\,,$$

and that

$$0 < \theta = \int_{-\infty}^{\infty} (X-\gamma)^2 p(X)\, \mathrm{d}X\,, \quad \text{where} \quad \gamma = \int_{-\infty}^{\infty} X\, p(X)\, \mathrm{d}X\,.$$

Remark. The Central Limit Theorem does not estimate how fast this limit is approached. Any such estimate would require additional assumptions about the underlying probability density $p(X)$. Roughly speaking, the rate of convergence will be slower when $p(X)$ is further from a normal density.

Remark. In an IID model of a portfolio $Y(d)$ is the growth rate of the portfolio when it is held for d days. The Central Limit Theorem shows that as $d \to \infty$ the values of $Y(d)$ become strongly peak around γ. This behavior seems to be consistent with the idea that a reasonable approach towards portfolio management is to select $\mathbf{f}$ to maximize the estimator $\widehat{\gamma}(\mathbf{f})$. However, by taking $\zeta = 0$ we see that the Central Limit Theorem implies

$$\lim_{d \to \infty} \mathrm{Pr}\Big\{Y(d) \geq \gamma\Big\} = \tfrac{1}{2}\,.$$

This shows that in the long run the growth rate of a portfolio will exceed γ with a probability of only $\frac{1}{2}$. A cautious investor might want the portfolio to exceed the optimized growth rate with a higher probability.

Growth Rate Exceeded with Probability. We consider the family of objectives $\Gamma(\lambda, T, \mathbf{f})$ that is the growth rate exceeded by a portfolio with probability λ after time T in trading days. *Here we will use the Central Limit Theorem to construct an estimator $\widehat{\Gamma}(\lambda, T, \mathbf{f})$ of this quantity.* We do this by assuming that T is large enough that we can use the approximation

$$\mathrm{Pr}\left\{Y(T) \geq \gamma - \zeta\sqrt{\theta/T}\right\} \approx \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z\,. \tag{2}$$

Given any probability $\lambda \in (0, 1)$, we set

$$\lambda = \int_{-\zeta}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z = \int_{-\infty}^{\zeta} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}Z^2}\, \mathrm{d}Z \equiv \mathrm{N}(\zeta)\,.$$

Our approximation can then be expressed as

$$\mathrm{Pr}\left\{Y(d) \geq \gamma - \frac{\zeta}{\sqrt{T}}\sqrt{\theta}\right\} \approx \lambda\,, \tag{3}$$

where $\zeta = \mathrm{N}^{-1}(\lambda)$.

For a Markowitz portfolio with allocation $\mathbf{f}$ we can replace γ and θ in the above approximation by the estimators $\widehat{\gamma}(\mathbf{f})$ and $\widehat{\theta}(\mathbf{f})$ given by

$$\begin{aligned}\widehat{\gamma}(\mathbf{f}) &= \sum_{d=1}^{D} w(d)x(d,\mathbf{f})\,,\\ \widehat{\theta}(\mathbf{f}) &= \frac{1}{1-\bar{w}}\sum_{d=1}^{D} w(d)\Big(x(d,\mathbf{f})-\widehat{\gamma}(\mathbf{f})\Big)^2\,,\end{aligned} \tag{4}$$

where $x(d,\mathbf{f}) = \log(1 + r(d,\mathbf{f}))$ with

$$r(d,\mathbf{f}) = \mu_{\mathsf{rf}}(\mathbf{f})\Big(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\Big) + \mathbf{r}(d)^{\mathsf{T}}\mathbf{f}\,.$$

We would then want select $\mathbf{f}$ so as to maximize the objective

$$\widehat{\Gamma}(\lambda,T,\mathbf{f}) = \widehat{\gamma}(\mathbf{f}) - \frac{\mathsf{N}^{-1}(\lambda)}{\sqrt{T}}\sqrt{\widehat{\theta}(\mathbf{f})}\,. \tag{5}$$

Remark. *The only new assumption we have made in order to construct this objective is that T is large enough for the Central Limit Theorem to yield a good approximation of the distribution of growth rates.* Investors often choose T to be the interval at which the portfolio will be rebalanced, regardless of whether T is large enough for the approximation to be valid. If an investor plans to rebalance once a year then $T = 252$, twice a year then $T = 126$, and four times a year then $T = 63$. The smaller T, the less likely it is that the Central Limit Theorem approximation is valid.

The idea now will be to select the admissible Markowitz allocation $\mathbf{f}$ that maximizes $\widehat{\Gamma}(\lambda, T, \mathbf{f})$ given a choice of λ and T by the investor. In other words, the objective will be to maximize the growth rate that will be exceeded by the portfolio with probability λ when it is held for T trading days. Because $1 - \lambda$ is the fraction of times the investor is willing to experience a downside tail event, the choice of λ characterizes how cautious the investor feels. More cautious investors will select a higher λ.

Remark. The caution of an investor generally increases with age. Retirees whose portfolio provides them with an income that covers much of their living expenses will generally be extremely cautious. Investors within ten years of retirement will be fairly cautious because they have less time for their portfolio to recover from any economic downturn. In constrast, young investors can be less cautious because they have more time to experience economic upturns and because they are typically far from their peak earning capacity.

Remark. The caution of an investor should also depend on a careful reading of economic factors or an analysis of the historical data. For example, if the historical data shows evidence of a bubble then an investor should be more cautious.

An investor can simply select $\zeta = \mathsf{N}^{-1}(\lambda)$ such that λ is a probability that reflects his or her caution. For example, based on the tabulations

$$\begin{aligned} &\mathsf{N}(0) = .5000\,, \quad \mathsf{N}\big(\tfrac{1}{4}\big) \approx .5987\,, \quad \mathsf{N}\big(\tfrac{1}{2}\big) \approx .6915\,, \quad \mathsf{N}\big(\tfrac{3}{4}\big) \approx .7734\,, \\ &\mathsf{N}(1) \approx .8413\,, \quad \mathsf{N}\big(\tfrac{5}{4}\big) \approx .8944\,, \quad \mathsf{N}\big(\tfrac{3}{2}\big) \approx .9332\,, \quad \mathsf{N}\big(\tfrac{7}{4}\big) \approx .9505\,, \end{aligned}$$

an investor who is willing to experience a downside tail event roughly

$$\begin{aligned} \text{once every two years might select } \zeta &= 0\,, \\ \text{twice every five years might select } \zeta &= \tfrac{1}{4}\,, \\ \text{thrice every ten years might select } \zeta &= \tfrac{1}{2}\,, \\ \text{twice every nine years might select } \zeta &= \tfrac{3}{4}\,, \\ \text{once every six years might select } \zeta &= 1\,, \\ \text{once every ten years might select } \zeta &= \tfrac{5}{4}\,, \\ \text{once every fifteen years might select } \zeta &= \tfrac{3}{2}\,, \\ \text{once every twenty years might select } \zeta &= \tfrac{7}{4}\,. \end{aligned}$$

Remark. *The Central Limit Theorem approximation generally degrades badly as ζ increases because $p(X)$ typically decays much more slowly than a normal density as $X \to -\infty$.* Therefore it is a bad idea to pick $\zeta > 2$ based on this approximation. Fortunately, $\zeta = \frac{7}{4}$ already corresponds to a fairly conservative investor.

Remark. *You should pick a larger value of ζ whenever your analysis of the historical data gives you less confidence either in the calibration of $\mathbf{m}$ and $\mathbf{V}$ or in the validity of an IID model.*

Remark. This approach is similar to something in financal management called *value at risk*. The finance problem is much harder because the time horizon T considered there is much shorter, typically on the order of days. In that setting the Central Limit Theorem approximation is certainly invalid.

Other Estimators of These Objectives. We now use the estimator $\widehat{\Gamma}(\lambda, T, \mathbf{f})$ to derive new estimators of $\Gamma(\lambda, T, \mathbf{f})$ in terms of sample estimators of the return mean and variance given by

$$\widehat{\mu}(\mathbf{f}) = \mu_{\mathsf{rf}}(\mathbf{f})\Big(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\Big) + \mathbf{m}^{\mathsf{T}}\mathbf{f}\,, \qquad \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,, \tag{6}$$

where $\mathbf{m}$ and $\mathbf{V}$ are given by

$$\begin{aligned}
\mathbf{m} &= \sum_{d=1}^{D} w(d)\mathbf{r}(d)\,, \\
\mathbf{V} &= \sum_{d=1}^{D} w(d)\Big(\mathbf{r}(d) - \mathbf{m}\Big)\Big(\mathbf{r}(d) - \mathbf{m}\Big)^{\mathsf{T}}\,.
\end{aligned} \tag{7}$$

These new return mean-variance estimators of $\Gamma(\lambda, T, \mathbf{f})$ will allow us to work within the framework of Markowitz portfolio theory.

Recall that $\widehat{\mu}(\mathbf{f})$ is the sample mean of of the history $\{r(d, \mathbf{f})\}_{d=1}^{D}$ and that

$$r(d, \mathbf{f}) - \widehat{\mu}(\mathbf{f}) = \tilde{\mathbf{r}}(d)^{\mathsf{T}}\mathbf{f}\,,$$

where $\tilde{\mathbf{r}}(d) = \mathbf{r}(d) - \mathbf{m}$. In words, $\tilde{\mathbf{r}}(d)$ is the deviation of $\mathbf{r}(d)$ from its sample mean $\mathbf{m}$. Then we can write

$$\begin{aligned} x(d, \mathbf{f}) &= \log\Bigl(1 + r(d, \mathbf{f})\Bigr) \\ &= \log\Bigl(1 + \widehat{\mu}(\mathbf{f})\Bigr) + \log\left(1 + \frac{\tilde{\mathbf{r}}(d)^{\mathsf{T}}\mathbf{f}}{1 + \widehat{\mu}(\mathbf{f})}\right)\,. \end{aligned} \tag{8}$$

We used this expression in the last lecture to derive several estimators of $\widehat{\gamma}(\mathbf{f})$. We now use it to derive several estimators of $\widehat{\theta}(\mathbf{f})$.

We use the first-order Taylor approximation $\log(1+r) \approx r$ in the second term of (8) to obtain

$$x(d, \mathbf{f}) \approx \log\Big(1 + \widehat{\mu}(\mathbf{f})\Big) + \frac{\tilde{\mathbf{r}}(d)^\top \mathbf{f}}{1 + \widehat{\mu}(\mathbf{f})}\,. \tag{9}$$

This yields the approximation

$$\begin{aligned}\widehat{\theta}(\mathbf{f}) &= \frac{1}{1-\bar{w}} \sum_{d=1}^{D} w(d)\Big(x(d,\mathbf{f}) - \widehat{\gamma}(\mathbf{f})\Big)^2 \\ &\approx \frac{1}{1-\bar{w}} \, \frac{\mathbf{f}^\top \mathbf{V} \mathbf{f}}{(1+\widehat{\mu}(\mathbf{f}))^2}\,,\end{aligned}$$

which leads to the estimator

$$\widehat{\theta}_{\mathsf{t}}(\mathbf{f}) = \frac{1}{1-\bar{w}} \frac{\mathbf{f}^\top \mathbf{V} \mathbf{f}}{(1+\widehat{\mu}(\mathbf{f}))^2}\,. \tag{10}$$

Like the second-order estimator $\widehat{\gamma}_{\mathsf{s}}(\mathbf{f})$, this estimator is not well-behaved. The simplest thing to do is drop the $\widehat{\mu}(\mathbf{f})$ term in the denominator.

We introduce the *caution coefficient* χ by

$$\chi = \frac{1}{\sqrt{1-\bar{\bar{w}}}}\,\frac{\zeta}{\sqrt{T}}\,. \tag{11}$$

When our approximation is combined with the reasonable estimator $\widehat{\gamma}_{\mathsf{r}}(\mathbf{f})$ we obtain

$$\widehat{\Gamma}_{\mathsf{r}}(\chi,\mathbf{f}) = \log\Big(1+\widehat{\mu}(\mathbf{f})\Big) - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \quad \text{over} \quad 1+\widehat{\mu}(\mathbf{f}) > 0\,. \tag{12}$$

When it is combined with the quadratic estimator $\widehat{\gamma}_{\mathsf{q}}(\mathbf{f})$ we obtain

$$\widehat{\Gamma}_{\mathsf{q}}(\chi,\mathbf{f}) = \widehat{\mu}(\mathbf{f}) - \tfrac{1}{2}\widehat{\mu}(\mathbf{f})^2 - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \quad \text{over} \quad \widehat{\mu}(\mathbf{f}) < 1\,. \tag{13}$$

When it is combined with the parabolic estimator $\widehat{\gamma}_{\mathsf{p}}(\mathbf{f})$ we obtain

$$\widehat{\Gamma}_{\mathsf{p}}(\chi,\mathbf{f}) = \widehat{\mu}(\mathbf{f}) - \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} - \chi\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}}\,. \tag{14}$$