Portfolios that Contain Risky Assets 12

Growth Rate Mean and Variance Estimators

C. David Levermore
University of Maryland, College Park

Math 420: *Mathematical Modeling*
April 11, 2017 version

Portfolios that Contain Risky Assets
Part II: Stochastic Models

Portfolios 12. Growth Rate Mean and Variance Estimators

1. Moment and Cumulant Generating Functions
2. Estimators from Moment Generating Functions
3. Estimators from Cumulent Generating Functions

Portfolios 12. Growth Rate Estimators

The idea now is to treat the Markowitz portfolio associated with $\mathbf{f}$ *as a single risky asset that can be modeled by the IID process associated with the growth rate probability density* $p_{\mathbf{f}}(X)$ *given by*

$$p_{\mathbf{f}}(X) = q_{\mathbf{f}}\Big(e^{X} - 1\Big)\; e^{X} \,.$$

The mean γ and variance θ of X are given by

$$\gamma = \int X \, p_{\mathbf{f}}(X) \, \mathrm{d}X \,, \qquad \theta = \int (X - \gamma)^2 p_{\mathbf{f}}(X) \, \mathrm{d}X \,.$$

We know from our study of one risky asset that γ is a good proxy for reward, while $\sqrt{\theta}$ is a good proxy for risk. Therefore we would like to estimate γ and θ in terms of the estimators $\widehat{\mu}$ and $\widehat{\xi}$ that we studied last time.

Moment and Cumulant Generating Functions. Estimators for γ and θ will be constructed from the positive function

$$M(\tau) = \mathsf{Ex}\Big(e^{\tau X}\Big) = \int e^{\tau X}\, p_{\mathbf{f}}(X)\, \mathrm{d}X\,.$$

We will assume $M(\tau)$ is defined for every τ in an open interval $(\tau_{\mathsf{mn}}, \tau_{\mathsf{mx}})$ that contains the interval $[0, 2]$. It can then be shown that $M(\tau)$ is infinitely differentiable over $(\tau_{\mathsf{mn}}, \tau_{\mathsf{mx}})$ with

$$M^{(m)}(\tau) = \mathsf{Ex}\Big(X^m\, e^{\tau X}\Big) = \int X^m\, e^{\tau X}\, p_{\mathbf{f}}(X)\, \mathrm{d}X\,.$$

We call $M(\tau)$ the *moment generating function* for X because, by setting $\tau = 0$ in the above expression, we see that the *moments* $\{\mathsf{Ex}(X^m)\}_{m=1}^{\infty}$ are generated from $M(\tau)$ by the formula

$$\mathsf{Ex}\Big(X^m\Big) = \int X^m p_{\mathbf{f}}(X)\, \mathrm{d}X = M^{(m)}(0)\,.$$

A related inifinitely differentiable function over $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ is

$$K(\tau) = \log(M(\tau)) = \log\Big(\mathrm{Ex}\Big(e^{\tau X}\Big)\Big) \; .$$

We call $K(\tau)$ the *cumulant generating function* because the *cumulants* $\{\kappa_m\}_{m=1}^{\infty}$ of X are generated by the formula $\kappa_m = K^{(m)}(0)$. Because

$$K'(\tau) = \frac{\mathrm{Ex}\Big(X\, e^{\tau X}\Big)}{\mathrm{Ex}\Big(e^{\tau X}\Big)} \,,$$

$$K''(\tau) = \frac{\mathrm{Ex}\Big((X - K'(\tau))^2 e^{\tau X}\Big)}{\mathrm{Ex}\Big(e^{\tau X}\Big)} \,,$$

$$K'''(\tau) = \frac{\mathrm{Ex}\Big((X - K'(\tau))^3 e^{\tau X}\Big)}{\mathrm{Ex}\Big(e^{\tau X}\Big)} \,,$$

$$K''''(\tau) = \frac{\mathrm{Ex}\Big((X - K'(\tau))^4 e^{\tau X}\Big)}{\mathrm{Ex}\Big(e^{\tau X}\Big)} - 3K''(\tau)^2 \,,$$

we see that the first four cumulants of X are

$$\begin{aligned}
\kappa_1 &= \quad K'(0) = \mathrm{Ex}(X) = \gamma\,, \\
\kappa_2 &= \quad K''(0) = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \theta\,, \\
\kappa_3 &= \quad K'''(0) = \mathrm{Ex}\Big((X-\gamma)^3\Big)\,, \\
\kappa_4 &= \quad K''''(0) = \mathrm{Ex}\Big((X-\gamma)^4\Big) - 3\theta^2\,.
\end{aligned}$$

We call these respectively the mean, variance, skewness, and kurtosis. The skewness measures asymmetry in the tails of the distribution. It is positive or negative depending on whether the fatter tail is to the right or left respectively. The kurtosis measures a relationship between the tails and the center of the distribution. It is greater for distributions with greater weight in the tails than in the center.

Remark. It should be evident from the formulas on the previous slide that $K'(\tau)$, $K''(\tau)$, $K'''(\tau)$, and $K''''(\tau)$ are the mean, variance, skewness, and kurtosis for the probability density $e^{\tau X} p_{\mathbf{f}}(X)/\mathrm{Ex}(e^{\tau X})$.

Remark. If X is normally distributed with mean γ and variance θ then

$$p_{\mathrm{f}}(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

A direct calculation then shows that

$$\begin{aligned}
\mathsf{Ex}\left(e^{\tau X}\right) &= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma)^2}{2\theta} + \tau X\right) \mathrm{d}X \\
&= \frac{1}{\sqrt{2\pi\theta}} \int \exp\left(-\frac{(X-\gamma-\tau\theta)^2}{2\theta} + \tau\gamma + \tfrac{1}{2}\tau^2\theta\right) \mathrm{d}X \\
&= \exp\left(\tau\gamma + \tfrac{1}{2}\tau^2\theta\right) ,
\end{aligned}$$

Therefore $K(\tau) = \log(\mathsf{Ex}(e^{\tau X})) = \tau\gamma + \frac{1}{2}\tau^2\theta$. *This shows that when X is normally distributed the skewness, kurtosis, and all other higher-order cumulants vanish. Conversely, if all these higher-order cumulants vanish then X is normally distributed.*

Remark. The cumulent generating function $K(\tau)$ is *strictly convex* over the interval $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ because $K''(\tau) > 0$.

Remark. We can also see that $K(\tau)$ is convex over $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$ as follows. Let τ_0, $\tau_1 \in (\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$. By applying the *Hölder inequality* with $p = \frac{1}{1-s}$ and $p^* = \frac{1}{s}$, we see that for every $s \in (0,1)$ we have

$$\begin{aligned} M\Big((1-s)\tau_0 + s\tau_1)\Big) &= \int e^{(1-s)\tau_0 X}\, e^{s\tau_1 X}\, p_{\mathbf{f}}(X)\,\mathrm{d}X \\ &\leq \left(\int e^{\tau_0 X} p_{\mathbf{f}}(X)\,\mathrm{d}X\right)^{1-s} \left(\int e^{\tau_1 X} p_{\mathbf{f}}(X)\,\mathrm{d}X\right)^{s} \\ &= M(\tau_0)^{1-s} M(\tau_1)^s\,. \end{aligned}$$

By taking the logarithm of this inequality we obtain

$$K((1-s)\tau_0 + s\tau_1) \leq (1-s)K(\tau_0) + sK(\tau_1) \quad \text{for every } s \in (0,1)\,.$$

Therefore $K(\tau)$ is a convex function over $(\tau_{\mathrm{mn}}, \tau_{\mathrm{mx}})$.

Estimators from Moment Generating Functions. We will now construct estimators for γ and θ by using the moment generating function

$$M(\tau) = \text{Ex}\Big(e^{\tau X}\Big) \,.$$

Because $R = e^X - 1$ and $\text{Ex}(e^X) = M(1)$, we have

$$\mu = \text{Ex}(R) = M(1) - 1 \,.$$

Because $R - \mu = e^X - M(1)$ and $\text{Ex}(e^{2X}) = M(2)$, we have

$$\xi = \text{Ex}\Big((R-\mu)^2\Big) = \Big(M(2) - M(1)^2\Big) \,.$$

These equations can be solved for $M(1)$ and $M(2)$ as

$$M(1) = 1 + \mu \,, \qquad M(2) = (1+\mu)^2 + \xi \,.$$

Therefore knowing μ and ξ is equivalent to knowing $M(1)$ and $M(2)$.

Because $\mathrm{Ex}(X) = M'(0)$ and $\mathrm{Ex}(X^2) = M''(0)$, we see that

$$\gamma = \mathrm{Ex}(X) = M'(0)\,,$$
$$\theta = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \mathrm{Ex}\Big(X^2\Big) - \gamma^2 = M''(0) - M'(0)^2\,.$$

Because $M(0) = 1$, we construct an estimator of $M(\tau)$ by interpolating the values $M(0)$, $M(1)$, and $M(2)$ with a quadratic polynomial as

$$\begin{aligned}\widehat{M}(\tau) &= 1 + \tau\Big(M(1) - 1\Big) + \tau(\tau-1)\tfrac{1}{2}\Big(M(2) - 2M(1) + 1\Big) \\ &= 1 + \tau\mu + \tfrac{1}{2}\tau(\tau-1)\left(\mu^2 + \xi\right)\,.\end{aligned}$$

By direct calculation we see that

$$\widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,, \qquad \widehat{M}''(0) = \mu^2 + \xi\,. \tag{1}$$

We then construct estimators $\widehat{\gamma}$ and $\widehat{\theta}$ as functions of μ and ξ by

$$\widehat{\gamma} = \widehat{M}'(0) = \mu - \tfrac{1}{2}(\mu^2 + \xi)\,,$$
$$\widehat{\theta} = \widehat{M}''(0) - \widehat{M}'(0)^2 = \mu^2 + \xi - \left(\mu - \tfrac{1}{2}(\mu^2 + \xi)\right)^2\,.$$

By replacing the μ and ξ that appear in the foregoing estimators with the estimators

$$\widehat{\mu} = \mu_{\mathsf{rf}}(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}) + \mathbf{m}^{\mathsf{T}}\mathbf{f}\,, \qquad \widehat{\xi} = \frac{1}{1-\bar{w}}\,\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,. \tag{2a}$$

we obtain the estimators

$$\begin{aligned} \widehat{\gamma} &= \widehat{\mu} - \tfrac{1}{2}\left(\widehat{\mu}^2 + \widehat{\xi}\right)\,, \\ \widehat{\theta} &= \widehat{\mu}^2 + \widehat{\xi} - \left(\widehat{\mu} - \tfrac{1}{2}\left(\widehat{\mu}^2 + \widehat{\xi}\right)\right)^2\,, \end{aligned} \tag{2b}$$

The variance θ is generally positive, but the estimator $\widehat{\theta}$ given above is not intrinsically positive.

Expanding the above expression for $\widehat{\theta}$ in powers of $\widehat{\mu}$ and $\widehat{\xi}$ yields

$$\widehat{\theta} = \widehat{\xi} + \widehat{\mu}\left(\widehat{\mu}^2 + \widehat{\xi}\right) - \tfrac{1}{4}\left(\widehat{\mu}^2 + \widehat{\xi}\right)^2 .$$

The only term in this expansion that is intrinsically positive is the first one. *Therefore we make the smallness assumptions*

$$|\widehat{\mu}| \ll 1 , \qquad \widehat{\xi} \ll 1 ,$$

and keep only through quadratic statistics — i.e. through quadratic in $\widehat{\mu}$ and linear in $\widehat{\xi}$. We thereby arrive at the *quadratic estimators*

$$\widehat{\gamma} = \widehat{\mu} - \tfrac{1}{2}\left(\widehat{\mu}^2 + \widehat{\xi}\right) , \qquad \widehat{\theta} = \widehat{\xi} , \tag{3}$$

where $\widehat{\mu}$ and $\widehat{\xi}$ are given by (2a).

Remark. These smallness assumptions are very easy to check.

Remark. The estimators $\widehat{\gamma}$ and $\widehat{\theta}$ given above have at least three potential sources of error:

- the estimators $\widehat{M}'(0)$ and $\widehat{M}''(0)$ as functions of μ and ξ given by (1),
- the estimators $\widehat{\mu}$ and $\widehat{\xi}$ used in (2) to approximate μ and ξ,
- the smallness assumptions that lead to (3).

The derivation of the first estimators assumes that the returns for each Markowitz portfolio are described by a density $q_{\mathbf{f}}(\mathbf{R})$ that is narrow enough for some moment beyond the second to exist. All of these approximations should be examined carefully, especially when markets are highly volatile.

Estimators from Cumulent Generating Functions. We will now give an alternative derivation of estimators that uses the cumulent generating function $K(\tau) = \log(M(\tau))$ and is based on the fact that $\gamma = K'(0)$ and $\theta = K''(0)$. It begins by observing that

$$
\begin{aligned}
K(1) &= \log\Big(M(1)\Big) = \log(1+\mu)\,, \\
K(2) &= \log\Big(M(2)\Big) = \log\Big((1+\mu)^2 + \xi\Big)\,.
\end{aligned}
$$

Therefore knowing μ and ξ is equivalent to knowing $K(1)$ and $K(2)$.

Because $K(0) = 0$, we construct an estimator of $K(\tau)$ by interpolating the values $K(0)$, $K(1)$, and $K(2)$ with a quadratic polynomial as

$$
\begin{aligned}
\widehat{K}(\tau) &= \tau K(1) + \tau(\tau-1)\tfrac{1}{2}\Big(K(2) - 2K(1)\Big) \\
&= \tau \log(1+\mu) + \tau(\tau-1)\tfrac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right)\,.
\end{aligned}
$$

This yields the estimators

$$
\begin{aligned}
\widehat{\gamma} &= \widehat{K}'(0) = \log(1+\mu) - \tfrac{1}{2}\log\left(1 + \frac{\xi}{(1+\mu)^2}\right) , \\
\widehat{\theta} &= \widehat{K}''(0) = \log\left(1 + \frac{\xi}{(1+\mu)^2}\right) .
\end{aligned}
\tag{4}
$$

By replacing the μ and ξ that appear above with the estimators $\widehat{\mu}$ and $\widehat{\xi}$ given by (2a), we obtain the new estimators

$$
\begin{aligned}
\widehat{\gamma} &= \log(1+\widehat{\mu}) - \tfrac{1}{2}\log\left(1 + \frac{\widehat{\xi}}{(1+\widehat{\mu})^2}\right) , \\
\widehat{\theta} &= \log\left(1 + \frac{\widehat{\xi}}{(1+\widehat{\mu})^2}\right) .
\end{aligned}
\tag{5}
$$

So long as $1 + \widehat{\mu} > 0$ these estimators will be well defined and $\widehat{\theta}$ will be positive.

If $1+\widehat{\mu} > 0$ and we make the smallness assumption

$$\frac{\widehat{\xi}}{(1+\widehat{\mu})^2} \ll 1\,,$$

then we obtain the estimators

$$\widehat{\gamma} = \log(1+\widehat{\mu}) - \tfrac{1}{2}\frac{\widehat{\xi}}{(1+\widehat{\mu})^2}\,, \qquad \widehat{\theta} = \frac{\widehat{\xi}}{(1+\widehat{\mu})^2}\,. \tag{6}$$

Finally, if we make the additional smallness assumption

$$|\widehat{\mu}| \ll 1\,,$$

use the fact

$$\log(1+\widehat{\mu}) = \widehat{\mu} - \tfrac{1}{2}\widehat{\mu}^2 + \tfrac{1}{3}\widehat{\mu}^3 + \cdots\,,$$

and keep only through quadratic statistics then we obtain the *quadratic estimators* derived earlier in (3)

Remark. *The fact that both derivations lead to the same estimators gives us greater confidence in the validity the quadratic estimators.*

Remark. If the Markowitz portfolio specified by $\mathbf{f}$ has growth rates X that are normally distributed with mean γ and variance θ then we have seen that $K(\tau) = \tau\gamma + \frac{1}{2}\tau^2\theta$. In this case we have $\widehat{K}(\tau) = K(\tau)$, so the estimators $\widehat{\gamma} = \widehat{K}'(0)$ and $\widehat{\theta} = \widehat{K}''(0)$ are exact.

Remark. The biggest uncertainty associated with these estimators for $\widehat{\gamma}$ and $\widehat{\theta}$ is usually the uncertainty inherited from the estimators for $\widehat{\mu}$ and $\widehat{\xi}$.

Exercise. When the quadratic estimators $\widehat{\gamma}$ and $\widehat{\theta}$ are applied to a single risky asset, they reduce to

$$\widehat{\gamma} = \widehat{\mu} - \tfrac{1}{2}\Big(\widehat{\mu}^2 + \widehat{\xi}\Big)\,, \qquad \widehat{\theta} = \widehat{\xi}\,.$$

Use these to estimate γ and θ for each of the following assets given the share price history $\{s(d)\}_{d=0}^{D}$. How do these $\widehat{\gamma}$ and $\widehat{\theta}$ compare with the unbiased estimators for γ and θ that you obtained in the previous problem?

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2009;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2007;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2009;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2007.

Exercise. Compute $\widehat{\gamma}$ and $\widehat{\theta}$ based on daily data for the Markowitz portfolio with value equally distributed among the assets in each of the groups given in the previous exercise.