Portfolios that Contain Risky Assets 11.

Independent, Identically-Distributed Models

C. David Levermore
University of Maryland, College Park

Math 420: *Mathematical Modeling*
April 11, 2017 version

Portfolios that Contain Risky Assets
Part II: Stochastic Models

Portfolios 11. Independent, Identically-Distributed Models

1. IID Models for an Asset
2. Return Probability Densities
3. Growth Rate Probability Densities
4. Normal Growth Rate Model
5. IID Models for Markets
6. IID Models for Markowitz Portfolios

Portfolios 11. Independent, Identically-Distributed Models

Investors have long followed the old adage “don’t put all your eggs in one basket” by holding diversified portfolios. However, before MPT the value of diversification had not been quantified. Key aspects of MPT are:

1. it uses the return mean as a proxy for reward;
2. it uses volatility as a proxy for risk;
3. it analyzes Markowitz portfolios;
4. it shows diversification reduces volatility through covariances;
5. it identifies the efficient frontier as the place to be.

The orignial form of MPT did not give guidance to investors about where to be on the efficient frontier. We will now begin to build stochasitc models that can be used in conjunction with the original MPT to address this question. *By doing so, we will see that maximizing the return mean is not the best strategy for maximizing your reward.*

IID Models for an Asset. We begin by building models of one risky asset with a share price history $\{s(d)\}_{d=0}^{D}$. Let $\{r(d)\}_{d=1}^{D}$ be the associated return history. Because each $s(d)$ is positive, each $r(d)$ lies in the interval $(-1, \infty)$. An *independent, identically-distributed (IID)* model for this history simply independently draws D random numbers $\{R(d)\}_{d=1}^{D}$ from $(-1, \infty)$ in accord with a fixed probability density $q(R)$ over $(-1, \infty)$. This means that $q(R)$ is a nonnegative integrable function such that

$$\int_{-1}^{\infty} q(R)\, \mathrm{d}R = 1\,,$$

and that the probability that each $R(d)$ takes a value inside any interval $[R_1, R_2] \subset (-1, \infty)$ is given by

$$\mathrm{Pr}\Big\{R(d) \in [R_1, R_2]\Big\} = \int_{R_1}^{R_2} q(R)\, \mathrm{d}R\,.$$

Here capital letters $R(d)$ denote random numbers drawn from $(-1, \infty)$ in accord with the probability density $q(R)$ rather than real return data.

Because the random numbers $\{R(d)\}_{d=1}^{D}$ are drawn from $(-1, \infty)$ in accord with the probability density $q(R)$ *independent* of each other, there is no correlation of $R(d)$ with $R(d')$ when $d \neq d'$. In particular, if we plot the points $\{(R(d), R(d+c))\}_{d=1}^{D-c}$ in the rr'-plane for any $c > 0$ they will be distributed in accord with the probability density $q(r)q(r')$. *Therefore if the return history $\{r(d)\}_{d=1}^{D}$ is mimiced by such a model then the points $\{(r(d), r(d+c))\}_{d=1}^{D-c}$ plotted in the rr'-plane should appear to be distributed in a way consistant with the probability density $q(r)q(r')$.* Such plots are called *scatter plots.*

In general a scatter plot will not show independence when c is small. This is because the behavior of an asset on any given trading day generally correlates with its behavior on the previous trading day. However, if a scatter plot shows independence for some c that is small compared to D the an IID model might still be good. Such a time c is called the *correlation time.*

Because the random numbers $\{R(d)\}_{d=1}^{D}$ are each drawn from $(-1, \infty)$ in accord with the *same* probability density $q(R)$, if we plot the points $\{(d, R(d))\}_{d=1}^{D}$ in the dr-plane they will usually be distributed in a way that looks uniform in d. *Therefore if the return history $\{r(d)\}_{d=1}^{D}$ is mimiced by such a model then the points $\{(d, r(d))\}_{d=1}^{D}$ plotted in the dr-plane should appear to be distributed in a way that is unifrom in d.*

Exercise. Plot $\{(r(d), r(d+1))\}_{d=1}^{D-1}$ and $\{(d, r(d))\}_{d=1}^{D}$ for each of the following assets and explain which might be good candidates to be mimiced by an IID model.

(a) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2013;

(b) Google, Microsoft, Exxon-Mobil, UPS, GE, and Ford stock in 2008;

(c) S&P 500 and Russell 1000 and 2000 index funds in 2013;

(d) S&P 500 and Russell 1000 and 2000 index funds in 2008.

Remark. We have adopted IID models because they are simple. It is not hard to develop more complicated stochastic models. For example, we could use a different probability density for each day of the week rather than treating all trading days the same way. Because there are usually five trading days per week, Monday through Friday, such a model would require calibrating five times as many means and covariances with one fifth as much data. There would then be greater uncertainty associated with the calibration. Moreover, we then have to figure out how to treat weeks that have less than five trading days due to holidays. Perhaps just the first and last trading days of each week should get their own probability density, no matter on which day of the week they fall. *Before increasing the complexity of a model, you should investigate whether the costs of doing so outweigh the benefits.* Specifically, you should investigate whether or not there is benefit in treating any one trading day of the week differently than the others before building a more complicated models.

Remark. *IID models are the simplest models that are consistent with the way any portfolio theory is used.* Specifically, to use any portfolio theory we must first calibrate a model from historical data. This model is then used to suggest how a set of ideal portfolios might behave in the future. Based on these suggestions we select the ideal portfolio that optimizes some objective. *This strategy assumes that in the future the market will behave statistically as it did in the past.*

This assumption requires the market statistics to be stable relative to its dynamics. But this requires future states to decorrelate from past states. Markov models are characterized by the assumption that possible future states are independent of past states, which maximizes this decorrelation. IID models are the simplest Markov models. All the models discussed in the previous remark are also Markov models. We will use only IID models.

Return Probability Densities. Once you have decided to use an IID model for a particular asset, you might think the next goal is to pick an appropriate probability density $q(R)$. However, that is neither practical nor necessary. *Rather, the goal is to identify appropriate statistical information about $q(R)$ that sheds light on the market. Ideally this information should be insensitive to details of $q(R)$ within a large class of probability densities.* Statisticians call such an approach *nonparametric.*

The expected value of any function $\psi(R)$ is given by

$$\mathrm{Ex}\Big(\psi(R)\Big) = \int_{-1}^{\infty} \psi(R)\, q(R)\, \mathrm{d}R\,,$$

provided $|\psi(R)|\, q(R)$ is integrable. Because we have been computing return means and variances, we will assume that the probability densities satisfy

$$\int_{-1}^{\infty} R^2 q(R)\, \mathrm{d}R < \infty\,.$$

The mean μ and variance ξ of R are then

$$\mu = \mathsf{Ex}(R) = \int_{-1}^{\infty} R\, q(R)\, \mathrm{d}R\,,$$
$$\xi = \mathsf{Var}(R) = \mathsf{Ex}\Big((R-\mu)^2\Big) = \int_{-1}^{\infty} (R-\mu)^2\, q(R)\, \mathrm{d}R\,.$$

However we do not know these. Rather, we must infer them from the data, at least approximately. Given D samples $\{R(d)\}_{d=1}^{D}$ that are drawn from the density $q(R)$, we can construct an *estimator* $\widehat{\mu}$ of μ by

$$\widehat{\mu} = \sum_{d=1}^{D} w(d)\, R(d)\,.$$

This is so-called *sample mean* is an *unbiased estimator* of μ because

$$\mathsf{Ex}(\widehat{\mu}) = \sum_{d=1}^{D} w(d)\, \mathsf{Ex}\Big(R(d)\Big) = \sum_{d=1}^{D} w(d)\, \mu = \mu\,.$$

We can estimate how close $\widehat{\mu}$ is to μ by computing its variance as

$$\begin{aligned}
\mathsf{Var}(\widehat{\mu}) &= \mathsf{Ex}\Big((\widehat{\mu} - \mu)^2\Big) \\
&= \mathsf{Ex}\left(\sum_{d=1}^{D} \sum_{d'=1}^{D} w(d)\, w(d')\, (R(d) - \mu)\, (R(d') - \mu)\right) \\
&= \sum_{d=1}^{D} \sum_{d'=1}^{D} w(d)\, w(d')\, \mathsf{Ex}\Big((R(d) - \mu)\, (R(d') - \mu)\Big) \\
&= \sum_{d=1}^{D} w(d)^2\, \mathsf{Ex}\Big((R(d) - \mu)^2\Big) \\
&= \sum_{d=1}^{D} w(d)^2\, \mathsf{Var}(R) = \sum_{d=1}^{D} w(d)^2\, \xi\,.
\end{aligned}$$

Here the off-diagonal terms in the double sum vanish because

$$\mathsf{Ex}\Big((R(d) - \mu)\, (R(d') - \mu)\Big) = 0 \quad \text{when } d \neq d'\,.$$

We now define $\bar{w}$ by

$$\bar{w} = \sum_{d=1}^{D} w(d)^2 \,.$$

Then the result of the calculation on the last slide can be expressed as

$$\text{Var}(\widehat{\mu}) = \bar{w}\,\xi \,.$$

This implies that $\widehat{\mu}$ converges to μ like $\sqrt{\bar{w}}$ as $D \to \infty$.

The Cauchy inequality implies that

$$1 = \left(\sum_{d=1}^{D} w(d) \right)^2 \leq \left(\sum_{d=1}^{D} 1^2 \right) \left(\sum_{d=1}^{D} w(d)^2 \right) = D\,\bar{w} \,.$$

Therefore $1/D \leq \bar{w}$ for any choice of weights. Because $\bar{w} = 1/D$ for uniform weights, we see that the rate of convergence of $\widehat{\mu}$ to μ is fastest for uniform weights, when it is $1/\sqrt{D}$ as $D \to \infty$.

We can construct an *unbiased estimator* of ξ that is proportional to the so-called *sample variance* as

$$\widehat{\xi} = \frac{1}{1-\bar{w}} \sum_{d=1}^{D} w(d)\Big(R(d) - \widehat{\mu}\Big)^2 .$$

Indeed, by using the fact that $\mathrm{Var}(\widehat{\mu}) = \bar{w}\,\xi$ we confirm that

$$\begin{aligned}
\mathrm{Ex}\big(\widehat{\xi}\big) &= \frac{1}{1-\bar{w}}\,\mathrm{Ex}\left(\sum_{d=1}^{D} w(d)\Big(R(d)-\mu\Big)^2 - (\widehat{\mu}-\mu)^2\right) \\
&= \sum_{d=1}^{D} \frac{w(d)}{1-\bar{w}}\,\mathrm{Ex}\Big(\Big(R(d)-\mu\Big)^2\Big) - \frac{\mathrm{Ex}\big((\widehat{\mu}-\mu)^2\big)}{1-\bar{w}} \\
&= \sum_{d=1}^{D} \frac{w(d)}{1-\bar{w}}\,\mathrm{Var}(R) - \frac{\mathrm{Var}(\widehat{\mu})}{1-\bar{w}} \\
&= \frac{\xi}{1-\bar{w}} - \frac{\bar{w}\,\xi}{1-\bar{w}} = \xi\,.
\end{aligned}$$

Growth Rate Probability Densities. Given D samples $\{R(d)\}_{d=1}^{D}$ that are drawn from the return probability density $q(R)$, the associated simulated share prices satisfy

$$S(d) = (1 + R(d))\, S(d-1)\,, \quad \text{for } d = 1, \cdots, D\,.$$

If we set $S(0) = s(0)$ then you can easily see that

$$S(d) = \prod_{d'=1}^{d} \Big(1 + R(d')\Big)\, s(0)\,.$$

The *growth rate* $X(d)$ is related to the return $R(d)$ by

$$e^{X(d)} = 1 + R(d)\,.$$

In other words, $X(d)$ is the growth rate that yeilds a return $R(d)$ on trading day d. The formula for $S(d)$ then takes the form

$$S(d) = \exp\left(\sum_{d'=1}^{d} X(d')\right) s(0)\,.$$

If the samples $\{R(d)\}_{d=1}^{D}$ are drawn from a density $q(R)$ over $(-1, \infty)$ then the $\{X(d)\}_{d=1}^{D}$ are drawn from a density $p(X)$ over $(-\infty, \infty)$ where $p(X)\,\mathrm{d}X = q(R)\,\mathrm{d}R$ with X and R related by

$$X = \log(1+R)\,, \qquad R = e^{X} - 1\,.$$

More explicitly, the densities $p(X)$ and $q(R)$ are related by

$$p(X) = q\Big(e^{X} - 1\Big)\, e^{X}\,, \qquad q(R) = \frac{p(\log(1+R))}{1+R}\,.$$

Because our models will involve means and variances, we will require that

$$\int_{-\infty}^{\infty} X^2 p(X)\,\mathrm{d}X = \int_{-1}^{\infty} \log(1+R)^2\, q(R)\,\mathrm{d}R < \infty\,,$$
$$\int_{-\infty}^{\infty} \Big(e^{X} - 1\Big)^2 p(X)\,\mathrm{d}X = \int_{-1}^{\infty} R^2 q(R)\,\mathrm{d}R < \infty\,.$$

The big advantage of working with $p(X)$ rather than $q(R)$ is the fact that

$$\log\left(\frac{S(d)}{s(0)}\right) = \sum_{d'=1}^{d} X(d')\,.$$

In other words, $\log(S(d)/s(0))$ *is a sum of an IID process.* It is easy to compute the mean and variance of this quantity in terms of those of X.

The mean γ and variance θ of X are

$$\gamma = \mathrm{Ex}(X) = \int_{-\infty}^{\infty} X\,p(X)\,\mathrm{d}X\,,$$

$$\theta = \mathrm{Var}(X) = \mathrm{Ex}\Big((X-\gamma)^2\Big) = \int_{-\infty}^{\infty} (X-\gamma)^2\,p(X)\,\mathrm{d}X\,.$$

For the mean of $\log(S(d)/s(0))$ we find that

$$\mathrm{Ex}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \sum_{d'=1}^{d} \mathrm{Ex}\Big(X(d')\Big) = d\,\gamma\,,$$

For the variance of $\log(S(d)/s(0))$ we find that

$$\begin{aligned}
\mathsf{Var}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) &= \mathsf{Ex}\left(\left(\sum_{d'=1}^{d} X(d') - d\,\gamma\right)^2\right) \\
&= \mathsf{Ex}\left(\left(\sum_{d'=1}^{d} \big(X(d') - \gamma\big)\right)^2\right) \\
&= \mathsf{Ex}\left(\sum_{d'=1}^{d} \sum_{d''=1}^{d} \big(X(d') - \gamma\big)\big(X(d'') - \gamma\big)\right) \\
&= \sum_{d'=1}^{d} \mathsf{Ex}\Big(\big(X(d') - \gamma\big)^2\Big) = d\,\theta\,.
\end{aligned}$$

Here the off-diagonal terms in the double sum vanish because

$$\mathsf{Ex}\Big(\big(X(d') - \gamma\big)\,\big(X(d'') - \gamma\big)\Big) = 0 \qquad \text{when } d'' \neq d'\,.$$

Therefore the expected growth and variance of the IID model asset at day d is

$$\mathrm{Ex}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \gamma\, d\,, \qquad \mathrm{Var}\left(\log\left(\frac{S(d)}{s(0)}\right)\right) = \theta\, d\,.$$

Remark. *The IID model suggests that the growth rate mean γ is a good proxy for the reward of an asset and that $\sqrt{\theta}$ is a good proxy for its risk. However, these are not the proxies chosen by MPT when it is applied to a portfolio consisting of one risky asset.* These proxies can be approximated by $\widehat{\gamma}$ and $\sqrt{\widehat{\theta}}$ where $\widehat{\gamma}$ and $\widehat{\theta}$ are the unbiased estimators of γ and θ given by

$$\widehat{\gamma} = \sum_{d=1}^{D} w(d)\, X(d)\,, \qquad \widehat{\theta} = \sum_{d=1}^{D} \frac{w(d)}{1-\bar{w}} \Big(X(d) - \widehat{\gamma}\Big)^2 .$$

Normal Growth Rate Model. We can illustrate what is going on with the simple IID model where $p(X)$ is the *normal* or *Gaussian* density with mean γ and variance θ, which is given by

$$p(X) = \frac{1}{\sqrt{2\pi\theta}} \exp\left(-\frac{(X-\gamma)^2}{2\theta}\right) .$$

Let $\{X(d)\}_{d=1}^{\infty}$ be a sequence of IID random variables drawn from $p(X)$. Let $\{Y(d)\}_{d=1}^{\infty}$ be the sequence of random variables defined by

$$Y(d) = \frac{1}{d} \sum_{d'=1}^{d} X(d') \qquad \text{for every } d = 1, \cdots, \infty .$$

You can easily check that

$$\mathsf{Ex}(Y(d)) = \gamma , \qquad \mathsf{Var}(Y(d)) = \frac{\theta}{d} .$$

You can also check that $\mathsf{Ex}(Y(d)|Y(d-1)) = \frac{d-1}{d}Y(d-1) + \frac{1}{d}\gamma$. So the variables $Y(d)$ are neither independent nor identically distributed.

It can be shown (the details are not given here) that $Y(d)$ is drawn from the normal density with mean γ and variance θ/d, which is given by

$$p_d(Y) = \sqrt{\frac{d}{2\pi\theta}}\, \text{exp}\left(-\frac{(Y-\gamma)^2 d}{2\theta}\right) .$$

Because $S(d)/s(0) = e^{dY(d)}$, the mean return at day d is

$$\begin{aligned}
\text{Ex}\left(e^{dY(d)}\right) &= \sqrt{\frac{d}{2\pi\theta}} \int \text{exp}\left(-\frac{(Y-\gamma)^2 d}{2\theta} + dY\right) \text{d}Y \\
&= \sqrt{\frac{d}{2\pi\theta}} \int \text{exp}\left(-\frac{(Y-\gamma-\theta)^2 d}{2\theta} + d(\gamma + \tfrac{1}{2}\theta)\right) \text{d}Y \\
&= \text{exp}\left(d(\gamma + \tfrac{1}{2}\theta)\right) .
\end{aligned}$$

Because $p_d(Y)$ becomes sharply peaked around $Y = \gamma$ as d increases, most investors will see the lower growth rate γ rather than $\gamma + \frac{1}{2}\theta$.

By setting $d = 1$ in the above formula, we see that the return mean is

$$\mu = \text{Ex}(R) = \text{Ex}\left(e^X - 1\right) = \exp\left(\gamma + \tfrac{1}{2}\theta\right) - 1\,.$$

Hence, $\mu > \gamma + \frac{1}{2}\theta$, with $\mu \approx \gamma + \frac{1}{2}\theta$ when $(\gamma + \frac{1}{2}\theta) << 1$. *Therefore most investors will see a return that is below the return mean μ — far below in volatile markets.* This is because e^X amplifies the tail of the normal density. For a more realistic IID model with a density $p(X)$ that decays more slowly than a normal density as $X \to \infty$, this difference can be more striking. Said another way, most investors will not see the same return as Warren Buffett, but his return will boost the mean.

The normal growth rate model confirms that γ is a better proxy for how well a risky asset might perform than μ because $p_d(Y)$ becomes more peaked around $Y = \gamma$ as d increases. We will extend this result to a general class of IID models that are more realistic.

IID Models for Markets. We now consider a market with N risky assets. Let $\{s_i(d)\}_{d=0}^{D}$ be the share price history of asset i. The associated return and growth rate histories are $\{r_i(d)\}_{d=1}^{D}$ and $\{x_i(d)\}_{d=1}^{D}$ where

$$r_i(d) = \frac{s_i(d)}{s_i(d-1)} - 1\,, \qquad x_i(d) = \log\left(\frac{s_i(d)}{s_i(d-1)}\right)\,.$$

Because each $s_i(d)$ is positive, each $r_i(d)$ is in $(-1, \infty)$, and each $x_i(d)$ is in $(-\infty, \infty)$. Let $\mathbf{r}(d)$ and $\mathbf{x}(d)$ be the N-vectors

$$\mathbf{r}(d) = \begin{pmatrix} r_1(d) \\ \vdots \\ r_N(d) \end{pmatrix}\,, \qquad \mathbf{x}(d) = \begin{pmatrix} x_1(d) \\ \vdots \\ x_N(d) \end{pmatrix}\,.$$

The market return and growth rate histories can then be expressed simply as $\{\mathbf{r}(d)\}_{d=1}^{D}$ and $\{\mathbf{x}(d)\}_{d=1}^{D}$ respectively.

An IID model for this market draws D random vectors $\{\mathbf{R}(d)\}_{d=1}^{D}$ from a fixed probablity density $q(\mathbf{R})$ over $(-1, \infty)^N$. Such a model is reasonable when the points $\{(d, \mathbf{r}(d))\}_{d=1}^{D}$ are distributed uniformly in d. This is hard to visualize when N is not small. You might think a necessary condition for the entire market to have an IID model is that each asset has an IID model. *This can be visualized for each asset by plotting the points $\{(d, r_i(d))\}_{d=1}^{D}$ in the dr-plane and seeing if they appear to be distributed uniformly in d.* Similar visual tests based on pairs of assets can be carried out by plotting the points $\{(d, r_i(d), r_j(d))\}_{d=1}^{D}$ in $\mathbb{R}^3$ with an interactive 3D graphics package.

Visual tests like those described above often show that funds behave more like IID models than individual stocks or bonds. This means that portfolio balancing strategies based on IID models might work better for portfolios composed largely of funds. This is one reason why some investors prefer investing in funds over investing in individual stocks and bonds.

A better lesson to be drawn from the observation in the last paragraph is that every sufficiently diverse portfolio of assets in a market will behave more like an IID model than many of the individual assets in that market. In other words, IID models for a market can be used to develop portfolio balancing strategies when the portfolios considered are sufficiently diverse, even when the behavior of individual assets in that market may not be well described by the model. This is another reason to prefer holding diverse, broad-based portfolios. *More importantly, this suggests that it is better to apply visual tests like those described above to representative portfolios rather than to individual assets in the market.*

Remark. Such visual tests can only warn you when IID models might not be appropriate for describing the data. There are also statistical tests that can play this role. *There is no visual or statistical test that can insure the validity of using an IID model for a market. However, due to their simplicity, IID models are often used unless there is a good reason not to use them.*

After you have decided to use an IID model for the market, you must gather statistical information about the return probability density $q(\mathbf{R})$. The mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Xi}$ of $\mathbf{R}$ are given by

$$\boldsymbol{\mu} = \int \mathbf{R}\, q(\mathbf{R})\, \mathrm{d}\mathbf{R}\,, \qquad \boldsymbol{\Xi} = \int (\mathbf{R} - \boldsymbol{\mu})(\mathbf{R} - \boldsymbol{\mu})^{\mathsf{T}} q(\mathbf{R})\, \mathrm{d}\mathbf{R}\,.$$

Given any sample $\{\mathbf{R}(d)\}_{d=1}^{D}$ drawn from $q(\mathbf{R})$, these have the unbiased estimators

$$\hat{\boldsymbol{\mu}} = \sum_{d=1}^{D} w(d)\, \mathbf{R}(d)\,, \qquad \hat{\boldsymbol{\Xi}} = \sum_{d=1}^{D} \frac{w(d)}{1 - \bar{w}} \left(\mathbf{R}(d) - \hat{\boldsymbol{\mu}}\right) \left(\mathbf{R}(d) - \hat{\boldsymbol{\mu}}\right)^{\mathsf{T}} .$$

If we assume that such a sample is given by the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ then these estimators are given in terms of the vector $\mathbf{m}$ and matrix $\mathbf{V}$ by

$$\hat{\boldsymbol{\mu}} = \mathbf{m}\,, \qquad \hat{\boldsymbol{\Xi}} = \frac{1}{1 - \bar{w}} \mathbf{V}\,.$$

IID Models for Markowitz Portfolios. Recall that the value of a portfolio that holds a risk-free balance $b_{\mathsf{rf}}(d)$ with return μ_{rf} and $n_i(d)$ shares of asset i during trading day d is

$$\pi(d) = b_{\mathsf{rf}}(d)\,(1 + \mu_{\mathsf{rf}}) + \sum_{i=1}^{N} n_i(d) s_i(d)\,.$$

We will assume that $\pi(d) > 0$ for every d. Then the return $r(d)$ and growth rate $x(d)$ for this portfolio on trading day d are given by

$$r(d) = \frac{\pi(d)}{\pi(d-1)} - 1\,, \qquad x(d) = \log\left(\frac{\pi(d)}{\pi(d-1)}\right)\,.$$

Recall that the return $r(d)$ for the Markowitz portfolio with allocation $\mathbf{f}$ can be expressed in terms of the vector $\mathbf{r}(d)$ as

$$r(d) = (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}}\mathbf{r}(d)\,.$$

This implies that if the underlying market has an IID model with return probability density $q(\mathbf{R})$ then the Markowitz portfolio with allocation $\mathbf{f}$ has the IID model with return probability density $q_{\mathbf{f}}(R)$ given by

$$q_{\mathbf{f}}(R) = \int \delta\Big(R - (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} - \mathbf{R}^{\mathsf{T}}\mathbf{f}\Big)\, q(\mathbf{R})\, \mathsf{d}\mathbf{R}\,.$$

Here $\delta(\,\cdot\,)$ denotes the *Dirac delta distribution*, which can be defined by the property that for every sufficiently nice function $\psi(R)$

$$\int \psi(R)\, \delta\Big(R - (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} - \mathbf{R}^{\mathsf{T}}\mathbf{f}\Big)\, \mathsf{d}R = \psi\Big((1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{R}^{\mathsf{T}}\mathbf{f}\Big)\,.$$

Hence, for every sufficiently nice function $\psi(R)$ we have the formula

$$\begin{aligned}
\mathsf{Ex}\Big(\psi(R)\Big) &= \int \psi(R)\, q_{\mathbf{f}}(R)\, \mathsf{d}R \\
&= \iint \psi(R)\, \delta\Big(R - (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} - \mathbf{R}^{\mathsf{T}}\mathbf{f}\Big)\, q(\mathbf{R})\, \mathsf{d}\mathbf{R}\, \mathsf{d}R \\
&= \int \psi\Big((1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{R}^{\mathsf{T}}\mathbf{f}\Big)\, q(\mathbf{R})\, \mathsf{d}\mathbf{R}\,.
\end{aligned}$$

We can thereby compute the mean μ and variance ξ of $q_{\mathbf{f}}(R)$ as

$$\begin{aligned}
\mu = \mathsf{Ex}(R) &= \int \Big((1-\mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{R}^{\mathsf{T}}\mathbf{f}\Big)\, q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= (1-\mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} \int q(\mathbf{R})\,\mathrm{d}\mathbf{R} + \Big(\int \mathbf{R}\, q(\mathbf{R})\,\mathrm{d}\mathbf{R}\Big)^{\mathsf{T}}\, \mathbf{f} \\
&= (1-\mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \boldsymbol{\mu}^{\mathsf{T}}\mathbf{f}\,, \\
\xi = \mathsf{Ex}\Big((R-\mu)^2\Big) &= \int \Big((1-\mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{R}^{\mathsf{T}}\mathbf{f} - \mu\Big)^2 q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= \int \Big(\mathbf{R}^{\mathsf{T}}\mathbf{f} - \boldsymbol{\mu}^{\mathsf{T}}\mathbf{f}\Big)^2 q(\mathbf{R})\,\mathrm{d}\mathbf{R} = \int \mathbf{f}^{\mathsf{T}}(\mathbf{R}-\boldsymbol{\mu})(\mathbf{R}-\boldsymbol{\mu})^{\mathsf{T}}\mathbf{f}\, q(\mathbf{R})\,\mathrm{d}\mathbf{R} \\
&= \mathbf{f}^{\mathsf{T}}\Big(\int (\mathbf{R}-\boldsymbol{\mu})(\mathbf{R}-\boldsymbol{\mu})^{\mathsf{T}} q(\mathbf{R})\,\mathrm{d}\mathbf{R}\Big)\,\mathbf{f} = \mathbf{f}^{\mathsf{T}}\boldsymbol{\Xi}\,\mathbf{f}\,,
\end{aligned}$$

where we have used the facts that

$$\int q(\mathbf{R})\,\mathrm{d}\mathbf{R} = 1\,, \qquad \int \mathbf{R}\, q(\mathbf{R})\,\mathrm{d}\mathbf{R} = \boldsymbol{\mu}\,,$$

$$\int (\mathbf{R}-\boldsymbol{\mu})(\mathbf{R}-\boldsymbol{\mu})^{\mathsf{T}} q(\mathbf{R})\,\mathrm{d}\mathbf{R} = \boldsymbol{\Xi}\,.$$

If we assume that the return history $\{\mathbf{r}(d)\}_{d=1}^{D}$ is an IID sample drawn from a probability density $q(\mathbf{R})$ then unbiased estimators of the associated mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\Xi}$ are given in terms of $\mathbf{m}$ and $\mathbf{V}$ by

$$\widehat{\boldsymbol{\mu}} = \mathbf{m}\,, \qquad \widehat{\boldsymbol{\Xi}} = \frac{1}{1-\bar{w}}\mathbf{V}\,.$$

Moreover, the Markowitz portfolio with allocation $\mathbf{f}$ has the return history $\{r(d)\}_{d=1}^{D}$ where

$$r(d) = (1 - \mathbf{1}^{\mathsf{T}}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{f}^{\mathsf{T}}\mathbf{r}(d)\,.$$

This return history is an IID sample drawn from the probability density $q_{\mathbf{f}}(R)$ and the formulas on the previous page show that the mean μ and variance ξ of $q_{\mathbf{f}}(R)$ have the unbiased estimators

$$\widehat{\mu} = \mu_{\mathsf{rf}}(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}) + \mathbf{m}^{\mathsf{T}}\mathbf{f}\,, \qquad \widehat{\xi} = \frac{1}{1-\bar{w}}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}\,.$$