Portfolios that Contain Risky Assets

Portfolio Models 10.

Survey of Markowitz Portfolio Models

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Survey of Markowitz Portfolio Models

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Survey of Markowitz Portfolio Models

1. Introduction. So far we have considered Markowitz portfolios that are either long, leveraged, unlimited, or solvent. Here we survey the sets of allocation vectors $\mathbf{f}$ for the risky assets of these portfolio models.

These portfolios may or may not contain a risk-free asset. If there is no risk-free asset then $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$. If there are risk-free assets then $1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}$ is the allocation in the risk-free asset.

These portfolio models are built upon *value ratios* for portfolios, which are built from *price ratios* for individual assets. These notions were used to construct solvent portfolios. We will review both of these notions and the construction of these portfolio models.

2. Long Portfolios. For long Markowitz portfolios with no risk-free asset the set of allocation vectors for the risky assets is

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \,,\, \mathbf{f} \geq \mathbf{0} \right\} . \tag{1}$$

For long Markowitz portfolios with a risk-free asset the set of allocation vectors for the risky assets is

$$\Lambda^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} \leq 1 \,,\, \mathbf{f} \geq \mathbf{0} \right\} . \tag{2}$$

It is clear that $\Lambda \subset \Lambda^+$.

3. Leveraged Portfolios. For Markowitz portfolios with no risk-free asset and with a leverage limit $\ell \in [0, \infty)$ the set of allocation vectors for the risky assets is

$$\Pi_\ell = \left\{ \mathbf{f} \in \mathbb{R}^N \, : \, \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \, , \, |\mathbf{f}| \leq 1 + 2\ell \right\} . \tag{3}$$

For Markowitz portfolios with a risk-free asset and with a leverage limit $\ell \in [0, \infty)$ the set of allocation vectors for the risky assets is

$$\Pi_\ell^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \, : \, |1 - \mathbf{1}^\mathsf{T}\mathbf{f}| + |\mathbf{f}| \leq 1 + 2\ell \right\} . \tag{4}$$

It is clear that $\Pi_\ell \subset \Pi_\ell^+$ for every $\ell \in [0, \infty)$. It is also clear that if $\ell, \ell' \in [0, \infty)$ then $\ell \leq \ell'$ implies that

$$\Pi_\ell \subset \Pi_{\ell'} \qquad \text{and} \qquad \Pi_\ell^+ \subset \Pi_{\ell'}^+ .$$

Finally, we saw earlier that

$$\Lambda = \Pi_0 \qquad \text{and} \qquad \Lambda^+ = \Pi_0^+ .$$

4. Unlimited Portfolios. For Markowitz portfolios with no risk-free asset and no leverage limit the set of allocation vectors for the risky assets is

$$\Pi_\infty = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \right\}. \tag{5}$$

For Markowitz portfolios with a risk-free asset and with no leverage limit the set of allocation vectors for the risky assets is

$$\Pi_\infty^+ = \mathbb{R}^N. \tag{6}$$

It is clear that $\Pi_\infty \subset \Pi_\infty^+$. As the notation suggests, if $\ell \in [0, \infty)$ then

$$\Pi_\ell \subset \Pi_\infty \qquad \text{and} \qquad \Pi_\ell^+ \subset \Pi_\infty^+.$$

Moreover, Π_∞ is the union of all the Π_ℓ over $\ell > 0$ and Π_∞^+ is the union of all the Π_ℓ^+ over $\ell > 0$. These models are easy to analyze because they have no inequality constraints.

5. Price Ratios of Assets. Given a share price history $\{s_i(d)\}_{d=0}^D$ for N risky assets indexed by $i = 1, \cdots, N$, we define the price ratio history $\{\rho_i(d)\}_{d=0}^D$ by

$$\rho_i(d) = \frac{s_i(d)}{s_i(d-1)} \quad \text{for every } i = 1, \cdots, N \text{ and } d = 1, \cdots, D\,.$$

Because return rates $r_i(d)$ were defined by

$$r_i(d) = \frac{s_i(d) - s_i(d-1)}{s_i(d-1)} = \frac{s_i(d)}{s_i(d-1)} - 1\,,$$

we see that price ratios are related to the return rates by

$$\rho_i(d) = 1 + r_i(d) \quad \text{for every } i = 1, \cdots, N \text{ and } d = 1, \cdots, D\,.$$

Because share prices typically do not change much on any trading day, most price ratios will be close to 1. Because each share price is positive, every price ratio is positive.

6. Value Ratios of Markowitz Portfolios. The Markowitz portfolios with no risk-free asset are specified by allocation vectors $\mathbf{f}$ that satisfy $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$. Earlier we saw that if this portfolio has value history $\{\pi(d)\}_{d=1}^{D}$ then its value ratio on trading day d is

$$\frac{\pi(d)}{\pi(d-1)} = \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}\,,$$

where $\boldsymbol{\rho}(d)$ is the N-vector of price ratios on day d, which is

$$\boldsymbol{\rho}(d) = \begin{pmatrix} \rho_1(d) \\ \vdots \\ \rho_N(d) \end{pmatrix} .$$

The Markowitz portfolios with a risk-free asset are specified by allocation vectors $\mathbf{f}$. Its value ratio on trading day d is

$$\frac{\pi(d)}{\pi(d-1)} = (1+\mu_{\mathsf{rf}})\left(1 - \mathbf{1}^{\mathsf{T}}\mathbf{f}\right) + \boldsymbol{\rho}(d)^{\mathsf{T}}\mathbf{f}\,.$$

7. Solvent Portfolios. For solvent Markowitz portfolios with no risk-free asset the set of allocation vectors for the risky assets is

$$\Omega = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\, 0 < \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d \right\}. \tag{7}$$

For solvent Markowitz portfolios with a risk-free asset the set of allocation vectors for the risky assets is

$$\Omega^+ = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, 0 < (1 + \mu_{\mathsf{rf}})\left(1 - \mathbf{1}^\mathsf{T}\mathbf{f}\right) + \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d \right\}. \tag{8}$$

It is clear that $\Omega \subset \Omega^+$. Earlier we saw that

$$\Lambda \subset \Omega \qquad \text{and} \qquad \Lambda^+ \subset \Omega^+\,.$$

The relationships between Π_ℓ and Ω and between Π_ℓ^+ and Ω^+ are less clear when $\ell > 0$. We will identify these relationships with the help of a more refined set of portfolio models that are also built upon value ratios.

8. Bounded Value-Ratio Portfolios. For Markowitz portfolios with no risk-free asset and with value ratios bounded within $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega_{[\underline{\rho},\overline{\rho}]} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \,,\; \underline{\rho} \leq \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \leq \overline{\rho} \;\; \forall d \right\}. \tag{9}$$

For Markowitz portfolios with a risk-free asset and with value ratios bounded within $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega^+_{[\underline{\rho},\overline{\rho}]} = \left\{ \mathbf{f} \in \mathbb{R}^N \,:\, \underline{\rho} \leq (1 + \mu_{\mathsf{rf}})\left(1 - \mathbf{1}^\mathsf{T}\mathbf{f}\right) + \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \leq \overline{\rho} \;\; \forall d \right\}. \tag{10}$$

It is clear that $\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega^+_{[\underline{\rho},\overline{\rho}]}$ for every $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$. It is also clear that if $[\underline{\rho}, \overline{\rho}]$ and $[\underline{\rho}', \overline{\rho}']$ are subsets of $(0, \infty)$ then $[\underline{\rho}, \overline{\rho}] \subset [\underline{\rho}', \overline{\rho}']$ implies that

$$\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega_{[\underline{\rho}',\overline{\rho}']} \qquad \text{and} \qquad \Omega^+_{[\underline{\rho},\overline{\rho}]} \subset \Omega^+_{[\underline{\rho}',\overline{\rho}']} \,.$$

Finally, it is clear that each of these portfolios are solvent. Specifically, we have $\Omega_{[\underline{\rho},\overline{\rho}]} \subset \Omega$ and $\Omega^+_{[\underline{\rho},\overline{\rho}]} \subset \Omega^+$ for every $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$.

9. Bounded Below Value-Ratio Portfolios. For Markowitz portfolios with no risk-free asset and with value ratios bounded below by $\rho \in (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega_\rho = \Big\{ \mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\; \rho \leq \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d \Big\}\,. \tag{11}$$

For Markowitz portfolios with a risk-free asset and with value ratios bounded below by $\rho \in (0, \infty)$ the set of allocation vectors for the risky assets is

$$\Omega_\rho^+ = \Big\{ \mathbf{f} \in \mathbb{R}^N \,:\, \rho \leq (1 + \mu_{\mathsf{rf}})\Big(1 - \mathbf{1}^\mathsf{T}\mathbf{f}\Big) + \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d \Big\}\,. \tag{12}$$

It is clear that $\Omega_\rho \subset \Omega_\rho^+$ for every $\rho \in (0, \infty)$. It is also clear that if $\rho, \rho' \in (0, \infty)$ then $\rho \leq \rho'$ implies that

$$\Omega_\rho \subset \Omega_{\rho'} \qquad \text{and} \qquad \Omega_\rho^+ \subset \Omega_{\rho'}^+\,.$$

Finally, it is clear that each of these portfolios are solvent. Specifically, we have $\Omega_\rho \subset \Omega$ and $\Omega_\rho^+ \subset \Omega^+$ for every $\rho \in (0, \infty)$.

10. Frontiers. Let Π be the set of allocation vectors for the risky assets in any Markowitz portfolio model. For example, Π can be Λ, Λ^+, Π_ℓ, Π_ℓ^+, Π_∞, Π_∞^+, Ω, Ω^+, $\Omega_{[\underline{\rho},\overline{\rho}]}$, $\Omega^+_{[\underline{\rho},\overline{\rho}]}$, Ω_ρ, or Ω_ρ^+.

For every $\mathbf{f} \in \mathbb{R}^N$ define $\mu(\mathbf{f})$ by

$$\mu(\mathbf{f}) = (1 - \mathbf{1}^\mathsf{T}\mathbf{f})\mu_{\mathsf{rf}} + \mathbf{f}^\mathsf{T}\mathbf{m}\,,$$

where μ_{rf} is either the one-rate or the two-rate model for risk-free rates. Let $\mu_{\mathsf{mn}}(\Pi)$ and $\mu_{\mathsf{mx}}(\Pi)$ be given by

$$\mu_{\mathsf{mn}}(\Pi) = \inf\left\{\mu(\mathbf{f})\ :\ \mathbf{f} \in \Pi\right\},$$
$$\mu_{\mathsf{mx}}(\Pi) = \sup\left\{\mu(\mathbf{f})\ :\ \mathbf{f} \in \Pi\right\},$$

The interval $(\mu_{\mathsf{mn}}(\Pi), \mu_{\mathsf{mx}}(\Pi))$ will be bounded for most choices of Π. Let $I_\mu(\Pi)$ denote its closure.

For every $\mu \in I_\mu(\Pi)$ define $\sigma_{\mathsf{f}}(\Pi)$ by

$$\sigma_{\mathsf{f}}(\mu;\Pi) = \min\left\{\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \,:\, \mu(\mathbf{f}) = \mu\,,\, \mathbf{f} \in \Pi\right\}.$$

Then the *frontier* for Π in the $\sigma\mu$-plane is the set

$$\left\{\left(\sigma_{\mathsf{f}}(\mu;\Pi)\,,\,\mu\right) \,:\, \mu \in I_\mu(\Pi)\right\}.$$

This set cannot be computed analytically for most choices of Π. However, it can be approximated numerically by solving the quadratic programming problem

$$\arg\min\left\{\tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \,:\, \mu(\mathbf{f}) = \mu\,,\, \mathbf{f} \in \Pi\right\}. \tag{13}$$

To use quadprog we must express the constraints $\mu(\mathbf{f}) = \mu$ and $\mathbf{f} \in \Pi$ as a combination of equality and inequality constraints.

When $\Pi = \Lambda$ the quadratic programming problem (13) becomes

$$\arg\min\left\{\frac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \,:\, \mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu\,,\ \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1\,,\ \mathbf{f} \geq \mathbf{0}\right\}\,.$$

This is expressed with two equality and N inequality constraints.

When $\Pi = \Lambda^{+}$ then (because the safe investment is the only risk-free asset that can be held) the quadratic programming problem (13) becomes

$$\arg\min\left\{\frac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \,:\, (\mathbf{m} - \mu_{\mathsf{si}}\mathbf{1})^{\mathsf{T}}\mathbf{f} = \mu - \mu_{\mathsf{si}}\,,\ \mathbf{1}^{\mathsf{T}}\mathbf{f} \leq 1\,,\ \mathbf{f} \geq \mathbf{0}\right\}\,.$$

This is expressed with one equality and $N + 1$ inequality constraints.

When $\Pi = \Omega_\rho$ for some $\rho \in (0, \infty)$ the quadratic programming problem (13) becomes

$$\text{arg}\min\Big\{\tfrac{1}{2}\mathbf{f}^\mathsf{T}\mathbf{V}\mathbf{f} \,:\, \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\,,\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\, \rho \leq \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \;\; \forall d\Big\}\,.$$

This is expressed with two equality and D inequality constraints.

When $\Pi = \Omega_{[\underline{\rho},\overline{\rho}]}$ for some $[\underline{\rho}, \overline{\rho}] \subset (0, \infty)$ the quadratic programming problem (13) becomes

$$\text{arg}\min\Big\{\tfrac{1}{2}\mathbf{f}^\mathsf{T}\mathbf{V}\mathbf{f} \,:\, \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\,,\, \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\, \underline{\rho} \leq \boldsymbol{\rho}(d)^\mathsf{T}\mathbf{f} \leq \overline{\rho} \;\; \forall d\Big\}\,.$$

This is expressed with two equality and $2D$ inequality constraints. When $D = 252$ this is 504 inequality constraints.

When $\Pi = \Pi_\ell$ for some $\ell \in (0, \infty)$ the quadratic programming problem (13) becomes

$$\arg\min\left\{\tfrac{1}{2}\mathbf{f}^\mathsf{T}\mathbf{V}\mathbf{f} \,:\, \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\,,\ \mathbf{1}^\mathsf{T}\mathbf{f} = 1\,,\ |\mathbf{f}| \leq 1 + 2\ell\right\}\,.$$

The constraint $|\mathbf{f}| \leq 1 + 2\ell$ describes a polyhedron in $\mathbb{R}^N$ with 2^N faces. (The polyhedron is a diamond when $N = 2$ and is a octahedron when $N = 3$.) The constraint can be expressed as the 2^N inequality constraints (one for each face of the polyhedron)

$$\pm f_1 \pm f_2 \pm \cdots \pm f_N \leq 1 + 2\ell\,,$$

where the N signs $\pm$ are chosen independently. Thereby the quadratic programming problem is expressed with two equality and 2^N inequality constraints. When $N = 9$ this is 512 inequality constraints.