Portfolios that Contain Risky Assets

Portfolio Models 8.

Long Portfolios without Risk-Free Assets

C. David Levermore
University of Maryland, College Park

Math 420: *Mathematical Modeling*
February 25, 2017 version

Portfolios that Contain Risky Assets
Part I: Portfolio Models

1. Risk and Reward
2. Covariance Matrices
3. Markowitz Portfolios
4. Solvent Portfolios
5. Leveraged Portfolios
6. Basic Markowitz Portfolio Theory
7. Unlimited Portfolios with Risk-Free Assets
8. Long Portfolios without Risk-Free Assets
9. Long Portfolios with a Safe Investment
10. Limited Leverage Portfolios

Portfolio Models 8. Long Portfolios without Risk-Free Assets

1. Long Portfolio Constraints
2. Long Frontiers
3. General Portfolio with Two Risky Assets
4. General Portfolio with Three Risky Assets
5. Simple Portfolio with Three Risky Assets

Portfolio Models 8. Long Portfolios without Risk-Free Assets

Long Portfolio Constraints. *Because the value of any portfolio with short positions has the potential to go negative, many investors will not hold a short position in any risky asset.* For these investors we consider only portfolios that hold either a long or neutral position in each risky asset. These so-called *long portfolios* satisfy the inequality constraints $\mathbf{f} \geq \mathbf{0}$.

Let Λ be the set of all long portfolio allocations and $\Lambda(\mu)$ be the set of all long portfolio allocations with return mean μ. These sets are given by

$$\Lambda = \left\{ \mathbf{f} \in \mathbb{R}^N \ : \ \mathbf{f} \geq \mathbf{0}\,, \ \mathbf{1}^\mathsf{T}\mathbf{f} = 1 \right\},$$
$$\Lambda(\mu) = \left\{ \mathbf{f} \in \Lambda \ : \ \mathbf{m}^\mathsf{T}\mathbf{f} = \mu \right\}.$$

Clearly $\Lambda(\mu) \subset \Lambda$ for every $\mu \in \mathbb{R}$.

We first consider the set Λ. Let $\mathbf{e}_i$ denote the vector whose i^{th} entry is 1 while every other entry is 0. For every $\mathbf{f} \in \Lambda$ we have

$$\mathbf{f} = \sum_{i=1}^{N} f_i \mathbf{e}_i \,,$$

where $f_i \geq 0$ for every $i = 1, \cdots, N$ and

$$\sum_{i=1}^{N} f_i = \mathbf{1}^{\mathsf{T}} \mathbf{f} = 1 \,.$$

This shows that Λ is simply all convex combinations of the vectors $\{\mathbf{e}_i\}_{i=1}^{N}$. Moreover, Λ is closed. Indeed, for any $\mathbf{f}$ in the closure of Λ there exists a sequence $\{\mathbf{f}_n\}_{n \in \mathbb{N}} \subset \Lambda$ such that $\mathbf{f}_n \to \mathbf{f}$. Because $\mathbf{f}_n \geq \mathbf{0}$ and $\mathbf{1}^{\mathsf{T}} \mathbf{f}_n = 1$ for every $n \in \mathbb{N}$, we see that $\mathbf{f} \in \Lambda$ because

$$\mathbf{f} = \lim_{n \to \infty} \mathbf{f}_n \geq \mathbf{0} \,, \qquad \mathbf{1}^{\mathsf{T}} \mathbf{f} = \lim_{n \to \infty} \mathbf{1}^{\mathsf{T}} \mathbf{f}_n = 1 \,.$$

Therefore Λ is a nonempty, closed, bounded, convex set.

We can visualize Λ when N is small. When $N = 2$ it is the line segment that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

When $N = 3$ it is the triangle that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

When $N = 4$ it is the tetrahedron that connects the unit vectors

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

For general N it is the simplex that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$.

Remark. When $N = 4$ it is easy to check that the tetrahedron $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ given by

$$\mathcal{T} = \left\{ \mathbf{z} \in \mathbb{R}^3 \;:\; \mathbf{w}_k^\top \mathbf{z} \le 1 \text{ for } k = 1, 2, 3, 4 \right\},$$

where

$$\mathbf{w}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{w}_2 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{w}_4 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix},$$

under the one-to-one affine mapping $\mathbf{\Phi} : \mathbb{R}^3 \to \mathbb{R}^4$ given by

$$\mathbf{\Phi}(\mathbf{z}) = \frac{1}{4} \begin{pmatrix} 1 - \mathbf{w}_1^\top \mathbf{z} \\ 1 - \mathbf{w}_2^\top \mathbf{z} \\ 1 - \mathbf{w}_3^\top \mathbf{z} \\ 1 - \mathbf{w}_4^\top \mathbf{z} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -z_1 \\ -z_2 \\ -z_3 \end{pmatrix}.$$

Therefore the set Λ in $\mathbb{R}^4$ can be visualized in $\mathbb{R}^3$ as the tetrahedron $\mathcal{T}$.

We saw that the simplex Λ is a nonempty, closed, bounded, convex set. For every $\mu \in \mathbb{R}$ the set $\Lambda(\mu)$ is the intersection of the simplex Λ with the hyperplane $\{\mathbf{f} \in \mathbb{R}^N \,:\, \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\}$. *This intersection might be empty.*

We now derive a condition that μ must satisfy for $\Lambda(\mu)$ to be nonempty. Let

$$\mu_{\mathsf{mn}} = \min\{m_i \,:\, i = 1, \cdots, N\}\,,$$
$$\mu_{\mathsf{mx}} = \max\{m_i \,:\, i = 1, \cdots, N\}\,.$$

Because $\mathbf{f} \geq \mathbf{0}$ and $\mathbf{1}^\mathsf{T}\mathbf{f} = 1$, for every $\mathbf{f} \in \Lambda(\mu)$ we have the inequalities

$$\mu_{\mathsf{mn}} = \mu_{\mathsf{mn}}\mathbf{1}^\mathsf{T}\mathbf{f} = \mu_{\mathsf{mn}} \sum_{i=1}^{N} f_i \leq \sum_{i=1}^{N} m_i f_i = \mathbf{m}^\mathsf{T}\mathbf{f} = \mu\,,$$
$$\mu = \mathbf{m}^\mathsf{T}\mathbf{f} = \sum_{i=1}^{N} m_i f_i \leq \mu_{\mathsf{mx}} \sum_{i=1}^{N} f_i = \mu_{\mathsf{mx}}\mathbf{1}^\mathsf{T}\mathbf{f} = \mu_{\mathsf{mx}}\,.$$

Therefore if $\Lambda(\mu)$ is nonempty then $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$.

Conversely, let $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ and set

$$\mathbf{f} = \frac{\mu_{\mathsf{mx}} - \mu}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mathbf{e}_{\mathsf{mn}} + \frac{\mu - \mu_{\mathsf{mn}}}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mathbf{e}_{\mathsf{mx}} \,.$$

where

$$\mathbf{e}_{\mathsf{mn}} = \mathbf{e}_i \quad \text{for any } i \text{ that satisfies } m_i = \mu_{\mathsf{mn}} \,,$$
$$\mathbf{e}_{\mathsf{mx}} = \mathbf{e}_j \quad \text{for any } j \text{ that satisfies } m_j = \mu_{\mathsf{mx}} \,.$$

Clearly $\mathbf{f} \geq \mathbf{0}$. Because $\mathbf{1}^{\mathsf{T}}\mathbf{e}_{\mathsf{mn}} = \mathbf{1}^{\mathsf{T}}\mathbf{e}_{\mathsf{mx}} = 1$, $\mathbf{m}^{\mathsf{T}}\mathbf{e}_{\mathsf{mn}} = \mu_{\mathsf{mn}}$, and $\mathbf{m}^{\mathsf{T}}\mathbf{e}_{\mathsf{mx}} = \mu_{\mathsf{mx}}$, we see that

$$\begin{aligned}
\mathbf{1}^{\mathsf{T}}\mathbf{f} &= \frac{\mu_{\mathsf{mx}} - \mu}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mathbf{1}^{\mathsf{T}}\mathbf{e}_{\mathsf{mn}} + \frac{\mu - \mu_{\mathsf{mn}}}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mathbf{1}^{\mathsf{T}}\mathbf{e}_{\mathsf{mx}} \\
&= \frac{\mu_{\mathsf{mx}} - \mu}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}} + \frac{\mu - \mu_{\mathsf{mn}}}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}} = 1 \,, \\
\mathbf{m}^{\mathsf{T}}\mathbf{f} &= \frac{\mu_{\mathsf{mx}} - \mu}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mathbf{m}^{\mathsf{T}}\mathbf{e}_{\mathsf{mn}} + \frac{\mu - \mu_{\mathsf{mn}}}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mathbf{m}^{\mathsf{T}}\mathbf{e}_{\mathsf{mx}} \\
&= \frac{\mu_{\mathsf{mx}} - \mu}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mu_{\mathsf{mn}} + \frac{\mu - \mu_{\mathsf{mn}}}{\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}}}\, \mu_{\mathsf{mx}} = \mu \,.
\end{aligned}$$

Hence, $\mathbf{f} \in \Lambda(\mu)$. *Therefore if $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ then $\Lambda(\mu)$ is nonempty.*

Because we have assumed that $\mathbf{m}$ is not proportional to $\mathbf{1}$, the return means $\{m_i\}_{i=1}^N$ are not identical. This implies that $\mu_{\mathsf{mn}} < \mu_{\mathsf{mx}}$, which implies that the interval $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ does not reduce to a point.

Because Λ is the simplex in $\mathbb{R}^N$ that connects the unit vectors $\{\mathbf{e}_i\}_{i=1}^N$, it is a closed, bounded, convex set of dimension $N-1$. The set $\Lambda(\mu)$ is nonempty for every $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Because it is the intersection of Λ and the hyperplane $\{\mathbf{f} \in \mathbb{R}^N : \mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu\}$, the set $\Lambda(\mu)$ will be closed, bounded, and convex. Because it is defined by linear constraints, *the set $\Lambda(\mu)$ will be a nonempty convex polytope of dimension at most $N-2$.*

We can visualize the polytope $\Lambda(\mu)$ when N is small. When $N = 2$ it is a point because it is the intersection of the line segment Λ with a transverse line. When $N = 3$ it is either a point or line segment because it is the intersection of the triangle Λ with a transverse plane. When $N = 4$ it is either a point, line segment, triangle, or convex quadralateral because it is the intersection of the tetrahedron Λ with a transverse hyperplane.

Remark. Recall from our last remark that when $N = 4$ the set $\Lambda \subset \mathbb{R}^4$ is the image of the tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ under the one-to-one affine mapping $\mathbf{\Phi} : \mathbb{R}^3 \to \mathbb{R}^4$ given there. The set $\Lambda(\mu) \subset \mathbb{R}^4$ is thereby the image under $\mathbf{\Phi}$ of the intersection of $\mathcal{T}$ with the hyperplane H_μ given by

$$H_\mu = \left\{ \mathbf{z} \in \mathbb{R}^3 \,;\; \mathbf{m}^{\mathsf{T}} \mathbf{\Phi}(\mathbf{z}) = \mu \right\} .$$

Hence, the set $\Lambda(\mu)$ in $\mathbb{R}^4$ can be visualized in $\mathbb{R}^3$ as the set $\mathcal{T}_\mu = \mathcal{T} \cap H_\mu$. Because $\mathbf{\Phi}$ is one-to-one and $\mathbf{m}$ is arbitrary, H_μ can be any hyperplane in $\mathbb{R}^3$. Therefore $\mathcal{T}_\mu$ can be the intersection of the tetrahedron $\mathcal{T}$ with any hyperplane in $\mathbb{R}^3$. When such an intersection is nonempty it can be either

1. a *point* that is a vertex of $\mathcal{T}$,
2. a *line segment* that is an edge of $\mathcal{T}$,
3. a *triangle* with vertices on edges of $\mathcal{T}$,
4. a *convex quadrilateral* with vertices on edges of $\mathcal{T}$.

These are each convex polytopes of dimension at most 2.

Long Frontiers. The set Λ in $\mathbb{R}^N$ of all long portfolios is associated with the set Σ in the $\sigma\mu$-plane of volatilities and return means given by

$$\Sigma = \left\{ \, (\sigma, \mu) \in \mathbb{R}^2 \; : \; \sigma = \sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \, , \; \mu = \mathbf{m}^{\mathsf{T}}\mathbf{f} \, , \; \mathbf{f} \in \Lambda \, \right\} .$$

The set Σ is the image in $\mathbb{R}^2$ of the simplex Λ in $\mathbb{R}^N$ under the mapping $\mathbf{f} \mapsto (\sigma, \mu)$. Because the set Λ is compact (closed and bounded) and the mapping $\mathbf{f} \mapsto (\sigma, \mu)$ is continuous, the set Σ is compact.

We have seen that the set $\Lambda(\mu)$ of all long portfolios with return mean μ is nonempty if and only if $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Hence, Σ can be expressed as

$$\Sigma = \left\{ \, \left(\sqrt{\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}} \, , \, \mu \right) \; : \; \mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}] \, , \; \mathbf{f} \in \Lambda(\mu) \, \right\} .$$

The points on the boundary of Σ that correspond to those long portfolios that have less volatility than every other long portfolio with the same return mean is called the *long frontier.*

The point of the long frontier associated with $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ is $(\sigma_{\text{lf}}(\mu), \mu)$ where $\sigma_{\text{lf}}(\mu)$ is obtained by solving the constrained minimization problem

$$\sigma_{\text{lf}}(\mu)^2 = \min\Big\{\, \sigma^2 \;:\; (\sigma, \mu) \in \Sigma \,\Big\} = \min\Big\{\, \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \in \Lambda(\mu) \,\Big\}\,.$$

This problem can not be solved by Lagrange multipliers because of the inequality constraints $\mathbf{f} \geq \mathbf{0}$ associated with the set of long portfolios $\Lambda(\mu)$.

Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ is continuous over the compact set $\Lambda(\mu)$, *a minimizer exists.* Because $\mathbf{V}$ is positive definite, the function $\mathbf{f} \mapsto \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Lambda(\mu)$. Because the function $\mathbf{f} \mapsto \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ is strictly convex over the convex set $\Lambda(\mu)$, *the minimizer is unique.* If this unique minimizer is denoted by $\mathbf{f}_{\text{lf}}(\mu)$ then the long frontier is given by the equation $\sigma = \sigma_{\text{lf}}(\mu)$ over $\mu \in [\mu_{\text{mn}}, \mu_{\text{mx}}]$ where $\sigma_{\text{lf}}(\mu)$ is given by

$$\sigma_{\text{lf}}(\mu) = \sqrt{\mathbf{f}_{\text{lf}}(\mu)^{\mathsf{T}}\mathbf{V}\mathbf{f}_{\text{lf}}(\mu)}\,.$$

For every $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ the portfolio $\mathbf{f}_{\mathsf{lf}}(\mu)$ can be expressed as

$$\mathbf{f}_{\mathsf{lf}}(\mu) = \arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \;:\; \mathbf{f} \geq \mathbf{0}\,,\; \mathbf{1}^{\mathsf{T}}\mathbf{f} = 1\,,\; \mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu \right\} .$$

Here $\arg\min$ is read *"the argument that minimizes".* It means that $\mathbf{f}_{\mathsf{lf}}(\mu)$ is the minimizer of the function $\mathbf{f} \mapsto \frac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ subject to the given constraints. Because the function being minimized is quadratic in $\mathbf{f}$ while the constraints are linear in $\mathbf{f}$, this is called a *quadratic programming problem*.

This problem can be solved for a particular $\mathbf{V}$, $\mathbf{m}$, and μ by using the Matlab command "quadprog". In general quadprog$(\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{d}, \mathbf{C}_{\mathsf{eq}}, \mathbf{d}_{\mathsf{eq}})$ returns the solution of the quadratic programming problem given by

$$\arg\min\left\{ \tfrac{1}{2}\mathbf{f}^{\mathsf{T}}\mathbf{A}\mathbf{f} + \mathbf{b}^{\mathsf{T}}\mathbf{f} \;:\; \mathbf{C}\mathbf{f} \leq \mathbf{d}\,,\; \mathbf{C}_{\mathsf{eq}}\mathbf{f} = \mathbf{d}_{\mathsf{eq}} \right\} ,$$

where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is Hermitian positive, $\mathbf{b} \in \mathbb{R}^{N}$, $\mathbf{C} \in \mathbb{R}^{M\times N}$, $\mathbf{d} \in \mathbb{R}^{M}$, $\mathbf{C}_{\mathsf{eq}} \in \mathbb{R}^{M_{\mathsf{eq}}\times N}$, and $\mathbf{d}_{\mathsf{eq}} \in \mathbb{R}^{M_{\mathsf{eq}}}$. Here M and M_{eq} are the number of inequality and equality constraints respectively.

By comparing this general quadratic programming problem with the one above it that yields $\mathbf{f}_{\mathsf{lf}}(\mu)$ for a given $\mathbf{V}$, $\mathbf{m}$, and μ, we see that

$$\mathbf{A} = \mathbf{V}\,, \quad \mathbf{b} = \mathbf{0}\,, \quad \mathbf{C} = -\mathbf{I}\,, \quad \mathbf{d} = \mathbf{0}\,, \quad \mathbf{C}_{\mathsf{eq}} = \begin{pmatrix} \mathbf{1}^{\mathsf{T}} \\ \mathbf{m}^{\mathsf{T}} \end{pmatrix}, \quad \mathbf{d}_{\mathsf{eq}} = \begin{pmatrix} 1 \\ \mu \end{pmatrix},$$

where $\mathbf{I}$ is the $N \times N$ identity. Here $M = N$ because $\mathbf{f} \geq \mathbf{0}$ gives N inequality constraints while $M_{\mathsf{eq}} = 2$ because $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu$ are two equality constraints. There are other ways to use quadprog to obtain $\mathbf{f}_{\mathsf{lf}}(\mu)$. Documentation for this command is easy to find on the web.

In practice $\mathbf{f}_{\mathsf{lf}}(\mu)$ can be obtained as the output f of a quadprog command that is formated as

$$\mathsf{f} = \mathsf{quadprog}(\mathsf{V}, \mathsf{z}, -\mathrm{I}, \mathsf{z}, \mathsf{C}, \mathsf{d})\,,$$

where the matrices V, I, and C, and vectors z and d are given by

$$\mathsf{V} = \mathbf{V}\,, \quad \mathsf{z} = \mathbf{0}\,, \quad \mathrm{I} = \mathbf{I}\,, \quad \mathsf{C} = \begin{pmatrix} \mathbf{1}^{\mathsf{T}} \\ \mathbf{m}^{\mathsf{T}} \end{pmatrix}, \quad \mathsf{d} = \begin{pmatrix} 1 \\ \mu \end{pmatrix}.$$

The long frontier can be computed numerically with the Matlab command quadprog. First, partition the interval $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ as

$$\mu_{\mathsf{mn}} = \mu_0 < \mu_1 < \cdots < \mu_{n-1} < \mu_n = \mu_{\mathsf{mx}} \,.$$

For example, set $\mu_k = \mu_{\mathsf{mn}} + k(\mu_{\mathsf{mx}} - \mu_{\mathsf{mn}})/n$ for a uniform partition. Second, compute σ_0 and σ_n from the minimization problems

$$\sigma_0 = \min\{\sqrt{v_{ii}} \,:\, m_i = \mu_0\} \,, \qquad \sigma_n = \min\{\sqrt{v_{ii}} \,:\, m_i = \mu_n\} \,.$$

Typically there is just one asset to consider in each of these problems. Third, for every $k = 1, \cdots, n-1$ use quadprog to compute $\mathbf{f}_{\mathsf{lf}}(\mu_k)$ and compute σ_k by

$$\sigma_k = \sigma_{\mathsf{lf}}(\mu_k) = \sqrt{\mathbf{f}_{\mathsf{lf}}(\mu_k)^{\mathsf{T}} \mathbf{V} \mathbf{f}_{\mathsf{lf}}(\mu_k)} \,.$$

Finally, "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^{n}$ to build an approximation to the long frontier in the $\sigma\mu$-plane. Here n should be large enough to resolve the features of the long frontier.

When computing a long frontier, it helps to know some general properties of the function $\sigma_{\mathsf{lf}}(\mu)$. These include:

- $\sigma_{\mathsf{lf}}(\mu)$ is *continuous* over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$;

- $\sigma_{\mathsf{lf}}(\mu)$ is *strictly convex* over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$;

- $\sigma_{\mathsf{lf}}(\mu)$ is *piecewise hyperbolic* over $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$.

This means that $\sigma_{\mathsf{lf}}(\mu)$ is built up from segments of hyperbolas that are connected at a finite number of *nodes* that correspond to points in the interval $(\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}})$ where $\sigma_{\mathsf{lf}}(\mu)$ has either *a jump discontinuity in its first derivative* or *a jump discontinuity in its second derivative.*

Remark. The way to "connect the dots" between the points $\{(\sigma_k, \mu_k)\}_{k=0}^n$ is motivated by the two-fund property. Specifically, for every $\mu \in (\mu_{k-1}, \mu_k)$ we set

$$\tilde{\mathbf{f}}_{\mathsf{lf}}(\mu) = \frac{\mu_k - \mu}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathsf{lf}}(\mu_{k-1}) + \frac{\mu - \mu_{k-1}}{\mu_k - \mu_{k-1}}\, \mathbf{f}_{\mathsf{lf}}(\mu_k)\,,$$

and then set

$$\tilde{\sigma}_{\mathsf{lf}}(\mu) = \sqrt{\tilde{\mathbf{f}}_{\mathsf{lf}}(\mu)^{\mathsf{T}} \mathbf{V} \tilde{\mathbf{f}}_{\mathsf{lf}}(\mu)}\,.$$

If the graph of $\sigma_{\mathsf{lf}}(\mu)$ over the interval (μ_{k-1}, μ_k) lies within one of the segments of hyperbolas that comprise the long frontier (i.e. it does not contain a node) then $\tilde{\sigma}_{\mathsf{lf}}(\mu)$ will recover the exact long frontier. Otherwise $\tilde{\sigma}_{\mathsf{lf}}(\mu)$ will give an approximation to the long frontier that will lie to the right of the long frontier in the $\sigma\mu$-plane. This approximation will not be too bad if n is large enough.

General Portfolio with Two Risky Assets. Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} , \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix} .$$

Without loss of generality we can assume that $m_1 < m_2$. Then $\mu_{\mathsf{mn}} = m_1$ and $\mu_{\mathsf{mx}} = m_2$. Recall that for every $\mu \in \mathbb{R}$ the unique portfolio that satisfies the constraints $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu$ is

$$\mathbf{f} = \mathbf{f}(\mu) = \frac{1}{m_2 - m_1} \begin{pmatrix} m_2 - \mu \\ \mu - m_1 \end{pmatrix} .$$

Clearly $\mathbf{f}(\mu) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_2] = [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$. Therefore the set Λ of long portfolios is given by

$$\Lambda = \Big\{ \mathbf{f}(\mu) \, : \, \mu \in [m_1, m_2] \Big\} .$$

In other words, the line segment Λ in $\mathbb{R}^2$ is the image of the interval $[m_1, m_2]$ under the affine mapping $\mu \mapsto \mathbf{f}(\mu)$. Then for every $\mu \in [m_1, m_2]$ the set $\Lambda(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$.

Because for every $\mu \in [m_1, m_2]$ the set $\Lambda(\mu)$ consists of the single portfolio $\mathbf{f}(\mu)$, the minimizer of $\mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f}$ over $\Lambda(\mu)$ is $\mathbf{f}(\mu)$. Therefore the long frontier portfolios are

$$\mathbf{f}_{\mathsf{lf}}(\mu) = \mathbf{f}(\mu) \qquad \text{for } \mu \in [m_1, m_2]\,,$$

and the long frontier is given by

$$\sigma = \sigma_{\mathsf{lf}}(\mu) = \sqrt{\mathbf{f}(\mu)^{\mathsf{T}}\mathbf{V}\,\mathbf{f}(\mu)} \qquad \text{for } \mu \in [m_1, m_2]\,.$$

Hence, the long frontier is simply a segment of the frontier hyperbola. It has no nodes.

General Portfolio with Three Risky Assets. Recall the portfolio of two risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix}, \qquad \mathbf{V} = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{12} & v_{22} & v_{23} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Without loss of generality we can assume that

$$m_1 \leq m_2 \leq m_3 , \qquad m_1 < m_3 .$$

Then $\mu_{\mathsf{mn}} = m_1$ and $\mu_{\mathsf{mx}} = m_3$. Recall that for every $\mu \in \mathbb{R}$ the portfolios that satisfies the constraints $\mathbf{1}^{\mathsf{T}}\mathbf{f} = 1$ and $\mathbf{m}^{\mathsf{T}}\mathbf{f} = \mu$ are

$$\mathbf{f} = \mathbf{f}(\mu, \phi) = \mathbf{f}_{13}(\mu) + \phi\mathbf{n} , \qquad \text{for some } \phi \in \mathbb{R} ,$$

where

$$\mathbf{f}_{13}(\mu) = \frac{1}{m_3 - m_1} \begin{pmatrix} m_3 - \mu \\ 0 \\ \mu - m_1 \end{pmatrix}, \qquad \mathbf{n} = \frac{1}{m_3 - m_1} \begin{pmatrix} m_2 - m_3 \\ m_3 - m_1 \\ m_1 - m_2 \end{pmatrix}.$$

Clearly $\mathbf{f}(\mu,\phi) \geq \mathbf{0}$ if and only if $\mu \in [m_1, m_3] = [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ and

$$0 \leq \phi \leq \phi_{\mathsf{mx}}(\mu) \equiv \min\left\{ \frac{m_3 - \mu}{m_3 - m_2}, \frac{\mu - m_1}{m_2 - m_1} \right\} .$$

This region is a triangle $\mathcal{T}_{\Lambda}$ in the $\mu\phi$-plane. Its base is the interval $[m_1, m_3]$ on the μ-axis and its peak is the point $(m_2, 1)$. It has height 1. Therefore the set Λ of long portfolios is given by

$$\Lambda = \left\{ \mathbf{f}(\mu,\phi) \; : \; (\mu,\phi) \in \mathcal{T}_{\Lambda} \right\} .$$

In other words, the triangle Λ in $\mathbb{R}^3$ is the image of the triangle $\mathcal{T}_{\Lambda}$ under the affine mapping $(\mu,\phi) \mapsto \mathbf{f}(\mu,\phi)$. Then for every $\mu \in [m_1, m_3]$ the set $\Lambda(\mu)$ is given by

$$\Lambda(\mu) = \left\{ \mathbf{f}(\mu,\phi) \; : \; 0 \leq \phi \leq \phi_{\mathsf{mx}}(\mu) \right\} .$$

In other words, the line segment $\Lambda(\mu)$ in $\mathbb{R}^3$ is the image of the interval $[0, \phi_{\mathsf{mx}}(\mu)]$ under the affine mapping $\phi \mapsto \mathbf{f}(\mu,\phi)$.

Hence, the point on the long frontier associated with $\mu \in [\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}]$ is $(\sigma_{\mathsf{lf}}(\mu), \mu)$ where $\sigma_{\mathsf{lf}}(\mu)$ solves the constrained minimization problem

$$\begin{aligned} \sigma_{\mathsf{lf}}(\mu)^2 &= \min\Big\{ \mathbf{f}^{\mathsf{T}}\mathbf{V}\mathbf{f} \ : \ \mathbf{f} \in \Lambda(\mu) \Big\} \\ &= \min\Big\{ \mathbf{f}(\mu,\phi)^{\mathsf{T}}\mathbf{V}\mathbf{f}(\mu,\phi) \ : \ 0 \leq \phi \leq \phi_{\mathsf{mx}}(\mu) \Big\} \, . \end{aligned}$$

Because the objective function

$$\mathbf{f}(\mu,\phi)^{\mathsf{T}}\mathbf{V}\mathbf{f}(\mu,\phi) = \mathbf{f}_{13}(\mu)^{\mathsf{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + 2\phi\, \mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{f}_{13}(\mu) + \phi^2\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}$$

is a quadratic in ϕ, we see that it has a unique global minimizer at

$$\phi = \phi_{\mathsf{f}}(\mu) = -\frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{f}_{13}(\mu)}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}} \, .$$

This global minimizer corresponds to the frontier. It will be the minimizer of our constrained minimization problem for the long frontier if and only if

$$0 \leq \phi_{\mathsf{f}}(\mu) \leq \phi_{\mathsf{mx}}(\mu) \, .$$

If $\phi_{\mathsf{f}}(\mu) < 0$ then the objective function is increasing over $[0, \phi_{\mathsf{mx}}(\mu)]$, whereby our constrained minimization problem has its minimizer at $\phi = 0$.

If $\phi_{\mathsf{mx}}(\mu) < \phi_{\mathsf{f}}(\mu)$ then the objective function is decreasing over $[0, \phi_{\mathsf{mx}}(\mu)]$, whereby our constrained minimization problem has its minimizer at $\phi = \phi_{\mathsf{mx}}(\mu)$.

Hence, the minimizer $\phi_{\mathsf{lf}}(\mu)$ of our constrained minimization problem is

$$\begin{aligned}\phi_{\mathsf{lf}}(\mu) &= \begin{cases} 0 & \text{if } \phi_{\mathsf{f}}(\mu) < 0 \\ \phi_{\mathsf{f}}(\mu) & \text{if } 0 \le \phi_{\mathsf{f}}(\mu) \le \phi_{\mathsf{mx}}(\mu) \\ \phi_{\mathsf{mx}}(\mu) & \text{if } \phi_{\mathsf{mx}}(\mu) < \phi_{\mathsf{f}}(\mu) \end{cases} \\ &= \max\{0\,,\, \min\{\phi_{\mathsf{f}}(\mu)\,,\, \phi_{\mathsf{mx}}(\mu)\}\} \\ &= \min\{\max\{0\,,\, \phi_{\mathsf{f}}(\mu)\}\,,\, \phi_{\mathsf{mx}}(\mu)\}\,.\end{aligned}$$

Therefore $\sigma_{\mathsf{lf}}(\mu)^2 = \mathbf{f}(\mu, \phi_{\mathsf{lf}}(\mu))^{\mathsf{T}} \mathbf{V} \mathbf{f}(\mu, \phi_{\mathsf{lf}}(\mu))$.

Understanding the long frontier thereby reduces to understanding $\phi_{\mathsf{lf}}(\mu)$. This can be done graphically in the $\mu\phi$-plane by considering the triangle $\mathcal{T}_{\Lambda}$ and the line $\mathcal{L}_{\mathsf{f}}$ given by

$$\phi = \phi_{\mathsf{f}}(\mu)\,.$$

Because

$$\mathbf{f}_{13}(m_1) = \mathbf{e}_1\,, \qquad \mathbf{f}_{13}(m_2) = -\mathbf{n} + \mathbf{e}_2\,, \quad \text{and} \quad \mathbf{f}_{13}(m_3) = \mathbf{e}_3\,,$$

we see that

$$\begin{aligned}
\phi_{\mathsf{f}}(m_1) &= -\frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{f}_{13}(m_1)}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}} = -\frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{e}_1}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}}\,,\\
\phi_{\mathsf{f}}(m_2) &= -\frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{f}_{13}(m_2)}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}} = 1 - \frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{e}_2}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}}\,,\\
\phi_{\mathsf{f}}(m_3) &= -\frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{f}_{13}(m_3)}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}} = -\frac{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{e}_3}{\mathbf{n}^{\mathsf{T}}\mathbf{V}\mathbf{n}}\,.
\end{aligned}$$

This shows we can read off from the entries of $\mathbf{Vn}$ that:

$$\mathcal{L}_\mathsf{f} \text{ lies below the vertex } (m_1, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \mathbf{e}_1^\top \mathbf{V} \mathbf{n} > 0\,;$$
$$\mathcal{L}_\mathsf{f} \text{ lies above the vertex } (m_1, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \mathbf{e}_1^\top \mathbf{V} \mathbf{n} < 0\,;$$
$$\mathcal{L}_\mathsf{f} \text{ lies below the vertex } (m_2, 1) \text{ of } \mathcal{T}_\Lambda \text{ iff } \mathbf{e}_2^\top \mathbf{V} \mathbf{n} > 0\,;$$
$$\mathcal{L}_\mathsf{f} \text{ lies above the vertex } (m_2, 1) \text{ of } \mathcal{T}_\Lambda \text{ iff } \mathbf{e}_2^\top \mathbf{V} \mathbf{n} < 0\,;$$
$$\mathcal{L}_\mathsf{f} \text{ lies below the vertex } (m_3, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \mathbf{e}_3^\top \mathbf{V} \mathbf{n} > 0\,;$$
$$\mathcal{L}_\mathsf{f} \text{ lies above the vertex } (m_3, 0) \text{ of } \mathcal{T}_\Lambda \text{ iff } \mathbf{e}_3^\top \mathbf{V} \mathbf{n} < 0\,.$$

Let us consider a few of the many different cases that can arise. For simplicity we will assume that $m_1 < m_2 < m_3$.

1. The line $\mathcal{L}_\mathsf{f}$ lies below the interior of $\mathcal{T}_\Lambda$ if and only if

$$\mathbf{e}_1^\top \mathbf{V} \mathbf{n} \geq 0\,, \quad \text{and} \quad \mathbf{e}_3^\top \mathbf{V} \mathbf{n} \geq 0\,.$$

Then $\phi_{\mathsf{lf}}(\mu) = 0$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\mathsf{lf}}(\mu) = \sqrt{\mathbf{f}_{13}(\mu)^{\mathsf{T}} \mathbf{V} \mathbf{f}_{13}(\mu)}\,,$$

which the long frontier built from assets 1 and 3.

2. The line $\mathcal{L}_{\mathsf{f}}$ lies above the interior of $\mathcal{T}_{\Lambda}$ if and only if

$$\mathbf{e}_1^{\mathsf{T}} \mathbf{V} \mathbf{n} \leq 0\,, \quad \mathbf{e}_2^{\mathsf{T}} \mathbf{V} \mathbf{n} \leq 0\,, \quad \text{and} \quad \mathbf{e}_3^{\mathsf{T}} \mathbf{V} \mathbf{n} \leq 0\,.$$

Then $\phi_{\mathsf{lf}}(\mu) = \phi_m x(\mu)$ for every $\mu \in [m_1, m_3]$ and the long frontier is

$$\sigma = \sigma_{\mathsf{lf}}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^{\mathsf{T}} \mathbf{V} \mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, m_2]\,, \\ \sqrt{\mathbf{f}_{23}(\mu)^{\mathsf{T}} \mathbf{V} \mathbf{f}_{23}(\mu)} & \text{for } \mu \in [m_2, m_3]\,. \end{cases}$$

This patches the long frontier built from assets 1 and 2 with the long frontier built from assets 2 and 3. There is generally a jump discontinuity in its first derivative at the node $\mu = m_2$.

3. The line $\mathcal{L}_{\mathsf{f}}$ lies above the base of $\mathcal{T}_{\wedge}$ but intersects the interior of $\mathcal{T}_{\wedge}$ if and only if

$$\mathbf{e}_1^{\mathsf{T}}\mathbf{V}\mathbf{n} < 0\,, \quad \mathbf{e}_2^{\mathsf{T}}\mathbf{V}\mathbf{n} > 0\,, \quad \text{and} \quad \mathbf{e}_3^{\mathsf{T}}\mathbf{V}\mathbf{n} < 0\,.$$

Then there exists $\mu_1 \in [m_1, m_2]$ and $\mu_2 \in [m_2, m_3]$ such that

$$\phi_{\mathsf{lf}}(\mu) = \begin{cases} \frac{\mu - m_1}{m_2 - m_1} & \text{for } \mu \in [m_1, \mu_1]\,, \\ \phi_{\mathsf{f}}(\mu) & \text{for } \mu \in (\mu_1, \mu_2)\,, \\ \frac{m_3 - \mu}{m_3 - m_2} & \text{for } \mu \in [\mu_2, m_3]\,. \end{cases}$$

The long frontier is

$$\sigma = \sigma_{\mathsf{lf}}(\mu) = \begin{cases} \sqrt{\mathbf{f}_{12}(\mu)^{\mathsf{T}}\mathbf{V}\mathbf{f}_{12}(\mu)} & \text{for } \mu \in [m_1, \mu_1]\,, \\ \sigma_{\mathsf{f}}(\mu) & \text{for } \mu \in (\mu_1, \mu_2)\,, \\ \sqrt{\mathbf{f}_{23}(\mu)^{\mathsf{T}}\mathbf{V}\mathbf{f}_{23}(\mu)} & \text{for } \mu \in [\mu_2, m_3]\,. \end{cases}$$

There are generally jump discontinuities in its second derivative at the nodes $\mu = \mu_1$ and $\mu = \mu_2$.

Simple Portfolio with Three Risky Assets. Recall the portfolio of three risky assets with mean vector $\mathbf{m}$ and covarience matrix $\mathbf{V}$ given by

$$\mathbf{m} = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m-d \\ m \\ m+d \end{pmatrix} , \qquad \mathbf{V} = s^2 \begin{pmatrix} 1 & r & r \\ r & 1 & r \\ r & r & 1 \end{pmatrix} .$$

Here $m \in \mathbb{R}$, $d, s \in \mathbb{R}_+$, and $r \in (-\frac{1}{2}, 1)$, where the last condition is equivalent to the condition that $\mathbf{V}$ is positive definite given $s > 0$. Its frontier parameters are

$$\sigma_{\mathsf{mv}} = \sqrt{\frac{1}{a}} = s\sqrt{\frac{1+2r}{3}} , \qquad \mu_{\mathsf{mv}} = \frac{b}{a} = m ,$$

$$\nu_{\mathsf{as}} = \sqrt{c - \frac{b^2}{a}} = \frac{d}{s}\sqrt{\frac{2}{1-r}} .$$

Its minimum volatility portfolio is $\mathbf{f}_{\mathsf{mv}} = \frac{1}{3}\mathbf{1}$, whereby we can take $\mu_0 = m$. Clearly $[\mu_{\mathsf{mn}}, \mu_{\mathsf{mx}}] = [m-d, m+d]$.

Its frontier is determined by

$$\sigma_{\mathsf{f}}(\mu) = s\sqrt{\frac{1+2r}{3} + \frac{1-r}{2}\left(\frac{\mu - m}{d}\right)^2} \qquad \text{for } \mu \in (-\infty, \infty)\,,$$

while the allocation of the frontier portfolio with return mean μ is

$$\mathbf{f}_{\mathsf{f}}(\mu) = \begin{pmatrix} \frac{1}{3} - \frac{\mu - m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu - m}{2d} \end{pmatrix} = \begin{pmatrix} \frac{m + \frac{2}{3}d - \mu}{2d} \\ \frac{1}{3} \\ \frac{\mu - m + \frac{2}{3}d}{2d} \end{pmatrix}.$$

The frontier portfolio holds long postitions when $\mu \in (m - \frac{2}{3}d, m + \frac{2}{3}d)$. Therefore $[\underline{\mu}_1, \overline{\mu}_1] = [m - \frac{2}{3}d, m + \frac{2}{3}d]$ and the long frontier satisfies

$$\sigma_{\mathsf{lf}}(\mu) = \sigma_{\mathsf{f}}(\mu) \qquad \text{for } \mu \in [m - \tfrac{2}{3}d, m + \tfrac{2}{3}d]\,.$$

The allocation of first asset vanishes at the right endpoint while that of the third vanishes at the left endpoint.

In order to extend the long frontier beyond the right endpoint $\overline{\mu}_1 = m + \frac{2}{3}d$ to $\mu_{\mathrm{mx}} = m + d$ we reduce the portfolio by removing the first asset and set

$$\overline{\mathbf{m}}_1 = \begin{pmatrix} m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} m \\ m + d \end{pmatrix}, \qquad \overline{\mathbf{V}}_1 = s^2 \begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix}.$$

Then

$$\overline{\mathbf{V}}_1^{-1} = \frac{1}{s^2(1 - r^2)} \begin{pmatrix} 1 & -r \\ -r & 1 \end{pmatrix}, \qquad \overline{\mathbf{V}}_1^{-1}\mathbf{1} = \frac{1}{s^2(1 + r)}\,\mathbf{1},$$

whereby

$$\overline{a}_1 = \mathbf{1}^{\mathsf{T}}\overline{\mathbf{V}}_1^{-1}\mathbf{1} = \frac{2}{s^2(1 + r)}, \qquad \overline{b}_1 = \mathbf{1}^{\mathsf{T}}\overline{\mathbf{V}}_1^{-1}\overline{\mathbf{m}}_1 = \frac{2m + d}{s^2(1 + r)},$$

$$\overline{c}_1 = \overline{\mathbf{m}}_1^{\mathsf{T}}\overline{\mathbf{V}}_1^{-1}\overline{\mathbf{m}}_1 = \frac{2m(m + d)}{s^2(1 + r)} + \frac{d^2}{s^2(1 - r^2)}.$$

The associated frontier parameters are

$$\sigma_{\mathsf{mv}_1} = \sqrt{\frac{1}{\bar{a}_1}} = s\sqrt{\frac{1+r}{2}}\,, \qquad \mu_{\mathsf{mv}_1} = \frac{\bar{b}_1}{\bar{a}_1} = m + \tfrac{1}{2}d\,,$$
$$\nu_{\mathsf{as}_1} = \sqrt{\bar{c}_1 - \frac{\bar{b}_1^{\,2}}{\bar{a}_1}} = \frac{d}{2s}\sqrt{\frac{2}{1-r}}\,,$$

whereby the frontier of the reduced portfolio is given by

$$\sigma_{\bar{\mathsf{f}}_1}(\mu) = s\sqrt{\frac{1+r}{2} + \frac{1-r}{2}\left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d}\right)^2}\,.$$

Similarly, to extend beyond the left endpoint we remove the third asset and find that the frontier of the reduced portfolio is given by

$$\sigma_{\underline{\mathsf{f}}_1}(\mu) = s\sqrt{\frac{1+r}{2} + \frac{1-r}{2}\left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d}\right)^2}\,.$$

By putting these pieces together we see that the long frontier is given by

$$\sigma_{\mathsf{lf}}(\mu) = \begin{cases} s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\frac{\mu - m + \frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m-d, m-\tfrac{2}{3}d]\,, \\ s\sqrt{\dfrac{1+2r}{3} + \dfrac{1-r}{2}\left(\frac{\mu - m}{d}\right)^2} & \text{for } \mu \in [m-\tfrac{2}{3}d, m+\tfrac{2}{3}d]\,, \\ s\sqrt{\dfrac{1+r}{2} + \dfrac{1-r}{2}\left(\frac{\mu - m - \frac{1}{2}d}{\frac{1}{2}d}\right)^2} & \text{for } \mu \in [m+\tfrac{2}{3}d, m+d]\,. \end{cases}$$

This is strictly convex and continuously differentiable over $[m-d, m+d]$. Its second derivative is defined and positive everywhere in $[m-d, m+d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where it has jump discontinuities. We have

$$\sigma_{\mathsf{lf}}(m \pm \tfrac{2}{3}d) = s\sqrt{\frac{5+4r}{9}}\,, \qquad \sigma_{\mathsf{lf}}(m \pm d) = s\,.$$

Finally, the long frontier allocations are given by

$$\mathrm{f}_{\mathsf{lf}}(\mu) = \begin{cases} \begin{pmatrix} \frac{m-\mu}{d} \\ \frac{\mu-m+d}{d} \\ 0 \end{pmatrix} & \text{for } \mu \in [m-d, m-\tfrac{2}{3}d]\,, \\ \begin{pmatrix} \frac{1}{3} - \frac{\mu-m}{2d} \\ \frac{1}{3} \\ \frac{1}{3} + \frac{\mu-m}{2d} \end{pmatrix} & \text{for } \mu \in [m-\tfrac{2}{3}d, m+\tfrac{2}{3}d]\,, \\ \begin{pmatrix} 0 \\ \frac{m+d-\mu}{d} \\ \frac{\mu-m}{d} \end{pmatrix} & \text{for } \mu \in [m+\tfrac{2}{3}d, m+d]\,. \end{cases}$$

Notice that the allocation weights do not depend on either s or r. They are continuous and piecewise linear over $[m-d, m+d]$. Their first derivatives are defined everywhere in $[m-d, m+d]$ except at the nodes $\mu = m \pm \frac{2}{3}d$ where they have jump discontinuities.